Propagation of decaying neutrinos in matter: an analytic treatment

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20 May 2022

The papers

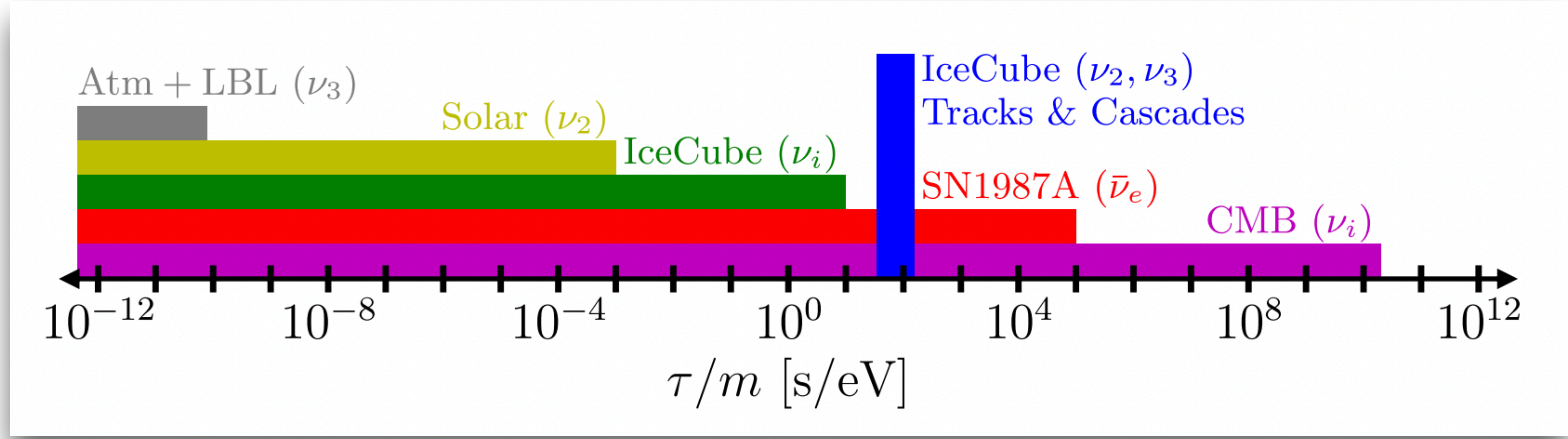
1. Dibya S. Chattopadhyay, Kaustav Chakraborty, Amol Dighe, Srubabati Goswami, S. M. Lakshmi, *"Neutrino propagation when mass eigenstates and decay eigenstates mismatch"*, arXiv:2111.13128 [hep-ph]

2. Dibya S. Chattopadhyay, Kaustav Chakraborty, Amol Dighe, Srubabati Goswami, *"Analytic treatment of 3-flavor neutrino oscillation and decay in matter"*, arXiv:2204.05803 [hep-ph]

Now, let us discuss neutrino oscillation + invisible decay + matter effects

Wait, what decay?

- Atm + LBL constraints on neutrino decay are **calculated independently**.



- Solar bounds on ν_2 decay in vacuum.
- Only applicable if all states decay with the same lifetime.
- At least one of the neutrino mass eigenstates survive.
- Typically a bound on N_{eff} , trickier to circumnavigate, but possible.

*A. Abdullahi and P. B. Denton, PRD 102, 023018 (2020)

Introduction

Setting up the problem of neutrino oscillation + invisible decay

- The inclusion of neutrino decay makes the effective Hamiltonian non-Hermitian.

$$\mathscr{H} = H - i\Gamma/2 \qquad \Gamma_{ij} = 2\pi \sum_k \langle \nu_i | \mathscr{H}' | \phi_k \rangle \langle \phi_k | \mathscr{H}' | \nu_j \rangle \, \delta(E_k - E_\nu)$$

- The Hermitian and the anti-Hermitian components may not commute:

$$[H, \Gamma] \neq 0$$

- A “**mismatch**” between the decay and the mass eigenstates in vacuum.

- Even if there’s no mismatch in vacuum, due to matter effects, the components will invariably become non-commuting.

Objective

- We derive **compact analytic** expressions for 2-flavor & 3-flavor neutrino probabilities with invisible **decay**, oscillation and **explicit matter effects included**.
- analytic results obtained provide physical understanding into possible effects of neutrino decay as it propagates through Earth matter
- Useful for: Long-Baseline, Atmospheric & Reactor Neutrino Experiments.
- We also point out an interesting non-intuitive feature of the neutrino propagation probability in the presence of decay, and explain them using our analytical approximations.

2 Flavor Treatment

First, let us solve the “toy” model problem

Based on “Neutrino propagation when mass eigenstates and decay eigenstates mismatch”, arXiv:2111.13128 [hep-ph]

Hamiltonian

- The 2 flavor Hamiltonian can be decomposed into:

$$\mathscr{H}_m = H_m - i\,\Gamma_m/2$$

- With, the flavor evolution of neutrinos:

$$\nu(t) = e^{-i\mathscr{H}_m t}\,\nu(0) \neq e^{-iH_m t}e^{-\Gamma_m t/2}\,\nu(0)$$

- The effective Hamiltonian takes the form:

$$\mathscr{H}_m = \begin{pmatrix} a_1 - i\,b_1 & -\frac{1}{2}i\,\gamma\,e^{i\chi} \\ -\frac{1}{2}i\,\gamma\,e^{-i\chi} & a_2 - i\,b_2 \end{pmatrix}$$

- where a_i, b_i, γ and χ are real. Since Γ_m needs to be positive semidefinite

$$b_i \geq 0 \quad \text{and} \quad \gamma^2 \leq 4\,b_1\,b_2\ .$$

Formalism

- Defining: $d_i \equiv a_i - ib_i$ $\qquad \Delta_a \equiv a_2 - a_1, \;\; \Delta_b \equiv b_2 - b_1, \;\; \Delta_d \equiv d_2 - d_1$

$$\bar{\gamma} \equiv \frac{\gamma}{|\Delta_d|}, \quad \bar{\Delta}_a \equiv \frac{\Delta_a}{|\Delta_d|}, \quad \bar{\Delta}_b \equiv \frac{\Delta_b}{|\Delta_d|}.$$

- One may write,

$$-i\mathcal{H}_m t = -\frac{it}{2}(d_1 + d_2)\,\mathbb{I} + \mathbb{X} + \mathbb{Y}.$$

$$\mathbb{X} \equiv -\frac{i\,\Delta_d t}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbb{Y} \equiv -\frac{\gamma t}{2}\begin{pmatrix} 0 & e^{i\chi} \\ e^{-i\chi} & 0 \end{pmatrix}$$

- The commutator of $\mathbb{X}$ and $\mathbb{Y}$ is

$$\mathscr{L}_{\mathbb{X}}\mathbb{Y} \equiv [\mathbb{X}, \mathbb{Y}] = i\frac{\gamma\,\Delta_d t^2}{2}\begin{pmatrix} 0 & -e^{i\chi} \\ e^{-i\chi} & 0 \end{pmatrix}.$$

Zassenhaus Expansion

Do not truncate arbitrarily

- The inverse Baker-Campbell-Hausdorff (BCH) or the Zassenhaus expansion:

$$e^{\mathbb{X}+\mathbb{Y}} = e^{\mathbb{X}} e^{\mathbb{Y}} e^{-\frac{1}{2}[\mathbb{X},\mathbb{Y}]} e^{\frac{1}{6}(2[\mathbb{Y},[\mathbb{X},\mathbb{Y}]]+[\mathbb{X},[\mathbb{X},\mathbb{Y}]])} \dots$$

- Note that, $|\mathbb{Y}| \sim \bar{\gamma}|\mathbb{X}|$ and $\mathscr{L}_{\mathbb{X}}\mathbb{Y} \sim \bar{\gamma}|\mathbb{X}|^2$. For higher-order commutators,

$$\mathscr{L}_{\mathbb{X}}^{k-1}\mathbb{Y} \sim \bar{\gamma}|\mathbb{X}|^k$$

- It is not possible to truncate the expansion at any fixed order of commutators

- One needs to collect $O(\gamma^k)$ terms[1]

$$e^{\mathbb{X}+\mathbb{Y}} = \left(1 + \sum_{p=1}^{\infty} \sum_{n_1,\dots,n_p=1}^{\infty} \frac{n_p \dots n_1}{n_p(n_p+n_{p-1})\dots(n_p+\dots+n_1)} \mathscr{Y}_{n_p} \dots \mathscr{Y}_{n_1}\right) e^{\mathbb{X}}$$

where $\mathscr{Y}_n = \frac{1}{n!}\mathscr{L}_{\mathbb{X}}^{n-1}\mathbb{Y}$.

[1] T. Kimura, *"Explicit Description of the Zassenhaus Formula"*, PTEP **2017**, no. 4, (2017) 041A03

- To obtain the expansion up to $O(\bar{\gamma})$, we need to sum upto $p = 1$.
- To obtain the expansion up to $O(\bar{\gamma}^2)$, we need to sum upto $p = 2$.

$$e^{\mathbb{X}+\mathbb{Y}} \approx \left(1 + \sum_{n_1=1}^{\infty} \mathcal{Y}_{n_1} + \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{n_1}{(n_1 + n_2)} \mathcal{Y}_{n_2} \mathcal{Y}_{n_1}\right) e^{\mathbb{X}},$$

- We calculate $\mathcal{Y}_n$ to be:

$$\mathcal{Y}_n = \frac{1}{n!} (i\Delta_d t)^{n-1} \sigma_3^{n-1} \mathbb{Y}$$

- Zassenhaus expansion technique is powerful: it can be generalized to 3 flavor formalism for neutrinos.

Now, we calculate the flavor amplitudes and probabilities to various degrees of accuracy.

Neutrino flavor conversions up to $O(\bar{\gamma})$

- Using the Zassenhaus expansion

$$e^{\mathbb{X}+\mathbb{Y}} = \left(1 + \frac{\sin(\Delta_d t)}{\Delta_d t}\mathbb{Y} - \frac{\cos(\Delta_d t) - 1}{\Delta_d t} i\sigma_3 \mathbb{Y}\right) e^{\mathbb{X}}.$$

- The amplitude matrix in mass basis in matter is given by

$$\mathscr{A}_m \equiv e^{-i\mathscr{H}_m t} = \begin{pmatrix} e^{-id_1 t} & -i\frac{\gamma e^{i\chi} g_-(t)}{\Delta_d} \\ -i\frac{\gamma e^{-i\chi} g_-(t)}{\Delta_d} & e^{-id_2 t} \end{pmatrix},$$

- With $g_\pm(t) = \frac{1}{2}(e^{-id_2 t} \pm e^{-id_1 t})$

- The neutrino flavor conversion probability $P_{\beta\alpha}$ for $\nu_\beta \to \nu_\alpha$ conversion:

$$[\mathscr{A}_f]_{\alpha\beta} = \left[U_m \, e^{-i\mathscr{H}_m t} \, U_m^\dagger\right]_{\alpha\beta} \qquad P_{\beta\alpha} = |\mathscr{A}_{\alpha\beta}|^2$$

$$U_m = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix}$$

- Therefore, the flavor amplitude can be expressed as

$$\mathscr{A}_f = \begin{pmatrix} g_-(t)A(\chi) + g_+(t) & g_-(t)B(\chi) \\ g_-(t)B(-\chi) & -g_-(t)A(\chi) + g_+(t) \end{pmatrix}$$

- With $A(\chi)$ and $B(\chi)$ defined as:

$$A(\chi) \equiv A^{(0)} + \gamma A^{(1)} \equiv -\cos 2\theta_m - i\frac{\gamma}{\Delta_d}\sin 2\theta_m \cos\chi$$

$$B(\chi) \equiv B^{(0)} + \gamma B^{(1)} \equiv \sin 2\theta_m - i\frac{\gamma}{\Delta_d}\left(\cos 2\theta_m \cos\chi + i\sin\chi\right)$$

Neutrino probabilities up to $O(\bar{\gamma})$

- The survival probability of a neutrino flavor is

$$P_{\alpha\alpha} = \frac{e^{-(b_1+b_2)t}}{2}\Big[(1+|A|^2)\cosh(\Delta_b t) + (1-|A|^2)\cos(\Delta_a t) - 2\mathrm{Re}(A)\sinh(\Delta_b t) + 2\mathrm{Im}(A)\sin(\Delta_a t)\Big] \ .$$

- The conversion probability is given by

$$P_{\beta\alpha} = \frac{e^{-(b_1+b_2)t}}{2}|B(\chi)|^2\left[\cosh(\Delta_b t) - \cos(\Delta_a t)\right] \ .$$

- The probability $P_{\alpha\beta} \equiv P_{\beta\alpha}(\chi \to -\chi)$

Term	Expression
$\mathrm{Re}(A)$	$-\cos 2\theta_m + \bar{\gamma}\bar{\Delta}_b \sin 2\theta_m \cos\chi$
$\mathrm{Im}(A)$	$-\bar{\gamma}\bar{\Delta}_a \sin 2\theta_m \cos\chi$
$\|A\|^2$	$\cos^2 2\theta_m - 2\bar{\gamma}\bar{\Delta}_b \sin 2\theta_m \cos 2\theta_m \cos\chi$
$\|B\|^2$	$\sin^2 2\theta_m + 2\bar{\gamma}\sin 2\theta_m \left(\bar{\Delta}_a \sin\chi + \bar{\Delta}_b \cos 2\theta_m \cos\chi\right)$

Neutrino flavor conversions up to $O(\bar{\gamma}^2)$

- To derive this, we need to calculate the double summation:

$$\frac{1}{2}\sum_{n_1=1}^{\infty}\left(\sum_{n_2=1}^{\infty}\mathcal{Y}_{n_2}\mathcal{Y}_{n_1}+\sum_{n_2=n_1}^{\infty}\frac{n_1-n_2}{n_1+n_2}[\mathcal{Y}_{n_2},\mathcal{Y}_{n_1}]\right)$$

$$[\mathcal{Y}_{n_2},\mathcal{Y}_{n_1}]=\frac{(-1)^{n_2}-(-1)^{n_1}}{4n_1!n_2!}(i\Delta_d t)^{n_2+n_1-2}(\gamma t)^2\sigma_3$$

- Defining $\Delta_D=\sqrt{\Delta_d^2-\gamma^2}$, we may obtain the probabilities by replacing

$$\boldsymbol{\Delta_a \to \mathrm{Re}(\Delta_D)\ ,\quad \Delta_b \to -\,\mathrm{Im}(\Delta_D)\ ,}$$

$$\boldsymbol{A(\chi)\to A^{(0)}+\gamma A^{(1)}-\gamma^2\cos 2\theta_m/(2\Delta_d^2)}\qquad \boldsymbol{B(\chi)\to A^{(0)}+\gamma A^{(1)}+\gamma^2\sin 2\theta_m/(2\Delta_d^2)}$$

- The previous probability equations with these replacements will give results correct up to $O(\bar{\gamma}^2)$.

Exact results

- For any 2×2 matrix $\mathbb{K}$, one can write, using the Pauli matrices

$$e^{\mathbb{K}} = e^{k_0} \left[\mathbb{I} \cosh k + \frac{\vec{k} \cdot \vec{\sigma}}{k} \sinh k \right]$$

- where,

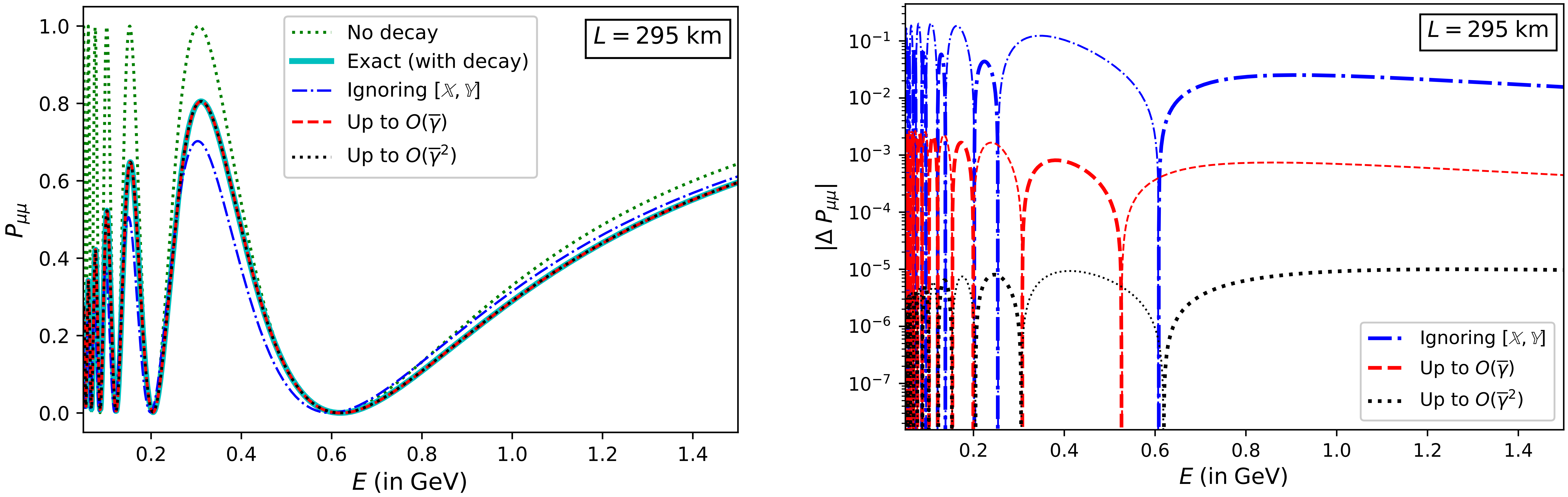
$$k_\mu \equiv \mathrm{Tr}(\mathbb{K} \cdot \sigma_\mu)/2 \qquad\qquad k \equiv \sqrt{k_1^2 + k_2^2 + k_3^2}$$

For the matrix $\mathbb{K} = -i\mathscr{H}_m t$, this gives: $k_0 = -\dfrac{it}{2}(d_1 + d_2)$, $k = \dfrac{it\Delta_D}{2}$.

- The exact probabilities are given by the replacement rule:

$$\boldsymbol{\Delta_a \to \mathrm{Re}(\Delta_D)\,, \quad \Delta_b \to -\,\mathrm{Im}(\Delta_D)\,, \quad A(\chi) \to \frac{\Delta_d}{\Delta_D} A(\chi) \quad , \quad B(\chi) \to \frac{\Delta_d}{\Delta_D} B(\chi)}$$

Numerical comparison



$$\Delta P_{\mu\mu} \equiv P_{\mu\mu}(\text{analytical}) - P_{\mu\mu}(\text{exact})$$

- $L = 295$ km, $E \sim 1$ GeV, $\Delta_a = 2.56 \times 10^{-3}$ eV$^2/(2E)$, $\theta_m = 45^\circ$, $(b_1, b_2, \gamma) = (3,6,8) \times 10^{-5}$ eV$^2/(2E)$, $\chi = \pi/4$.

Comparison with earlier results

- We find the most general form of the non-unitary matrix that would diagonalize a non-Hermitian $\mathscr{H}$ is:

$$N = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} 1 & -i\frac{\gamma e^{i\chi}}{\Delta_D + \Delta_d} \\ i\frac{\gamma e^{-i\chi}}{\Delta_D + \Delta_d} & 1 \end{pmatrix}$$

- since $(\Delta_D + \Delta_d)$ is **complex**, the off-diagonal elements of the second matrix are not complex conjugates of one another, an explicit assumption in[2].

- Even assuming $\bar{\Delta}_b \sim O(\bar{\gamma})$, there will be $O(\bar{\gamma}^2)$ corrections due to this.

[2] J. M. Berryman, A. de Gouvˆea, D. Herna´ndez, and R. L. N. Oliveira, *"Non-Unitary Neutrino Propagation From Neutrino Decay"* Phys. Lett. B 742 (2015) 74–79.

Special case: only one unstable neutrino

- If only ν_2 in vacuum decays, with $\alpha_2 = m_2/\tau_2$, in matter, we get:

$$a_{1,2} = \frac{\tilde{m}^2_{1,2}}{2E} \quad , \quad b_{1,2} = \frac{\alpha_2}{4E}[1 \mp \cos[2(\theta - \theta_m)] \ ,$$
$$\chi = 0 \quad , \quad \gamma = \frac{\alpha_2}{2E} \sin[2(\theta - \theta_m)] \ .$$

- The **off-diagonal term** γ is **generated**, even though it was absent in vacuum.
- **Inevitable "mismatch"** in matter. Taken care of by our analytic expressions.

Conclusion for the 2 flavor model

- Neutrino propagation probabilities for invisible decay and oscillation in matter.
- Mismatch is inevitable, and our analytic expressions can handle them.
- The Zassenhaus expansion can be extended to 3 flavors.
- May also be applicable to:
    1. Combined treatment of oscillations and absorption for high energy neutrinos
    2. Axion-photon oscillations in an optically semi-opaque medium
    3. The neutral meson mixing systems.

3 Flavor Treatment

Moving on to the "real world" version

Based on "*Analytic treatment of 3-flavor neutrino oscillation and decay in matter*", arXiv:2204.05803 [hep-ph]

Hamiltonian

- When only ν_3 mass eigenstate in vacuum decays

$$\mathscr{H}_f^{(\gamma_3)} = \frac{1}{2E_\nu} U \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix} - i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_3 \Delta m_{31}^2 \end{pmatrix} \right] U^\dagger + \begin{pmatrix} V_{cc} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- For the general decay matrix Γ we have

$$\mathscr{H}_f^{(\Gamma)} = U \left[\frac{1}{2E_\nu} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix} - \frac{i}{2}\Gamma \right] U^\dagger + \begin{pmatrix} V_{cc} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma = \frac{\Delta m_{31}^2}{E_\nu} \begin{pmatrix} \gamma_1 & \frac{1}{2}\gamma_{12} e^{i\chi_{12}} & \frac{1}{2}\gamma_{13} e^{i\chi_{13}} \\ \frac{1}{2}\gamma_{12} e^{-i\chi_{12}} & \gamma_2 & \frac{1}{2}\gamma_{23} e^{i\chi_{23}} \\ \frac{1}{2}\gamma_{13} e^{-i\chi_{13}} & \frac{1}{2}\gamma_{23} e^{-i\chi_{23}} & \gamma_3 \end{pmatrix}.$$

- Here, γ_i is defined such that $\gamma_i \Delta m_{31}^2 = m_i/\tau_i$.

Formalism and scales

- Let us define the "book-keeping" parameter $\lambda \equiv 0.2$.
- The "standard" parameters can then be expressed as

$$\alpha \approx 0.03 \simeq O(\lambda^2)\ , \qquad s_{13} \simeq 0.14 \simeq O(\lambda)\ .$$

- The decay parameters can be constrained from 2 considerations:
  1. Decay must be subleading to oscillation, i.e. decay length must be larger.
  2. Decay matrix should be positive definite.

$$\gamma_1\,,\gamma_2 \sim O(\lambda^3)\ , \qquad \gamma_3 \sim O(\lambda)\ , \qquad \gamma_{12} \sim O(\lambda^3)\ , \qquad \gamma_{13}\,,\gamma_{23} \sim O(\lambda^2)\ ,$$

- Dimensionless quantities:

$$A = \frac{2E_\nu V_{cc}}{\Delta m^2_{31}}\ , \qquad \Delta = \frac{\Delta m^2_{31} L}{4E_\nu}$$

The One Mass Scale Dominance Limit

- At the limit of $\alpha \to 0$, i.e. ignoring any effects from Δm^2_{21}.

$$\mathscr{H}_f^{(OMSD)} = \frac{\Delta m^2_{31}}{2E_\nu}\left[U\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1-i\gamma_3\end{pmatrix}U^\dagger + \begin{pmatrix}A & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right]$$

$$U = R_{23}(\theta_{23})\cdot R_{13}(\theta_{13})$$

- Using the symmetry in 23 sector in the potential term, we rotate to:

$$\widetilde{\mathscr{H}}_f^{(OMSD)} = \frac{\Delta m^2_{31}}{2E_\nu}\left[R_{13}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1-i\gamma_3\end{pmatrix}R_{13}^\dagger + \begin{pmatrix}A & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right].$$

With, “~” representing the R_{23} rotated basis

- The Hamiltonian can be represented as: $\widetilde{\mathscr{H}}_f^{(OMSD)} = \widetilde{H}_f^{(OMSD)} - \frac{i}{2}\widetilde{\Gamma}_f^{(OMSD)}.$

- Rotating away by $R_{13}^m \equiv R_{13}(\theta_{13}^m)$ to the basis where the Hermitian part is diagonal ("OMSD basis")

$$\tan 2\theta_{13}^m = \frac{\sin 2\theta_{13}}{\cos 2\theta_{13} - A} .$$

$$\widetilde{\mathscr{H}}_m^{\text{(OMSD)}} = \text{diag}[(\Lambda_1, \Lambda_2, \Lambda_3)] - \frac{i}{2}\widetilde{\Gamma}_m^{\text{(OMSD)}} .$$

$$\Lambda_{1,3} = \frac{\Delta m_{31}^2}{4E}\left[1 + A \mp C_{13}\right] , \qquad \Lambda_2 = 0 ,$$

$$C_{13} \equiv \sqrt{(\cos 2\theta_{13} - A)^2 + (\sin 2\theta_{13})^2}$$

$$\widetilde{\Gamma}_m^{\text{(OMSD)}} = R_{13}^{m\,\dagger}\,\widetilde{\Gamma}_f^{\text{(OMSD)}}\,R_{13}^m = \frac{\Delta m_{31}^2}{E_\nu}\begin{pmatrix} \gamma_1^m & 0 & \frac{1}{2}\gamma_{13}^m \\ 0 & 0 & 0 \\ \frac{1}{2}\gamma_{13}^m & 0 & \gamma_3^m \end{pmatrix},$$

$$\boldsymbol{\gamma_1^m \equiv \gamma_3 \sin^2 \delta\theta , \qquad \gamma_3^m \equiv \gamma_3 \cos^2 \delta\theta , \qquad \gamma_{13}^m \equiv -\gamma_3 \sin(2\,\delta\theta) ,}$$

where, $\delta\theta \equiv \theta_{13}^m - \theta_{13}$

- Even though we started with only ν_3 decaying, the decay matrix in the OMSD basis has multiple non-zero diagonal as well as off-diagonal elements.

OMSD Probabilities

- Neutrino survival/ conversion probabilities are given by $P_{\alpha\beta} = |A(\nu_\alpha \to \nu_\beta)|^2$

$$A(\nu_\alpha \to \nu_\beta) = [e^{-i\mathscr{H}_f^{(OMSD)}L}]_{\beta\alpha} = [R_{23}R_{13}^m \widetilde{\mathscr{A}}_m R_{13}^{m\,\dagger} R_{23}^\dagger]_{\beta\alpha} \,,$$

$$\widetilde{\mathscr{A}}_m = \exp[-i\,\widetilde{\mathscr{H}}_m^{(\mathrm{OMSD})}L]$$

- Defining:

$$D_{1,3} \equiv \left(1 + A - i(\gamma_1^m + \gamma_3^m) \mp \widetilde{C}_{13}^m\right)\frac{\Delta}{L}\,, \qquad \Delta_D \equiv D_3 - D_1 = 2\,\widetilde{C}_{13}^m \frac{\Delta}{L}\,,$$

$$\widetilde{C}_{13}^m = \sqrt{[C_{13} - i(\gamma_3^m - \gamma_1^m)]^2 - (\gamma_{13}^m)^2}\,. \qquad G_\pm(L) = \frac{1}{2}\left(e^{-iD_3L} \pm e^{-iD_1L}\right), \qquad a = -\frac{\Delta_d}{\Delta_D}\,, \qquad b = -2i\frac{\gamma_{13}^m}{\Delta_D}\frac{\Delta}{L}\,.$$

$$P_{\mu\mu} = \left|c_{23}^2 + s_{23}^2 G_+(t) - s_{23}^2\left(a\,\cos 2\theta_{13}^m + b\,\sin 2\theta_{13}^m\right)G_-(t)\right|^2,$$

$$P_{ee} = \left|G_+(t) + \left(a\,\cos 2\theta_{13}^m + b\,\sin 2\theta_{13}^m\right)G_-(t)\right|^2,$$

$$P_{e\mu} = \left|s_{23}\left(b\,\cos 2\theta_{13}^m - a\,\sin 2\theta_{13}^m\right)G_-(t)\right|^2.$$

Exact Results!

- Expressed explicitly, correct up to linear order in the off-diagonal decay term:

$$
\begin{aligned}
P_{\mu\mu} = {} & \left(c_{23}^2 + s_{23}^2\left[(s_{13}^m)^2 e^{-2\gamma_1^m\Delta} + (c_{13}^m)^2 e^{-2\gamma_3^m\Delta}\right]\right)^2 - s_{23}^4 \sin^2 2\theta_{13}^m\, e^{-2\gamma_+\Delta}\sin^2\Delta_m \\
& - \sin^2 2\theta_{23}\left[(s_{13}^m)^2 e^{-2\gamma_1^m\Delta}\sin^2\frac{\Delta_-}{2} + (c_{13}^m)^2 e^{-2\gamma_3^m\Delta}\sin^2\frac{\Delta_+}{2}\right] \\
& + \widetilde{\gamma} s_{23}^2 \sin 2\theta_{13}^m \Big\{ C_{13}\left[s_{23}^2 e^{-2\gamma_+\Delta}\sin 2\Delta_m + c_{23}^2\left(e^{-2\gamma_3^m\Delta}\sin\Delta_+ - e^{-2\gamma_1^m\Delta}\sin\Delta_-\right)\right] \\
& \qquad + \gamma_-\Big[s_{23}^2\left((c_{13}^m)^2 e^{-4\gamma_3^m\Delta} - (s_{13}^m)^2 e^{-4\gamma_1^m\Delta} - \cos 2\theta_{13}^m e^{-2\gamma_+\Delta}\cos 2\Delta_m\right) \\
& \qquad\qquad + c_{23}^2\left(e^{-2\gamma_3^m\Delta}\cos\Delta_+ - e^{-2\gamma_1^m\Delta}\cos\Delta_-\right)\Big]\Big\}\,, \\
P_{ee} = {} & \left[(c_{13}^m)^2 e^{-2\gamma_1^m\Delta} + (s_{13}^m)^2 e^{-2\gamma_3^m\Delta}\right]^2 - \sin^2 2\theta_{13}^m e^{-2\gamma_+\Delta}\sin^2\Delta_m \\
& - \widetilde{\gamma}\sin 2\theta_{13}^m\Big[C_{13} e^{-2\gamma_+\Delta}\sin 2\Delta_m - \gamma_-\Big(2\cos 2\theta_{13}^m e^{-2\gamma_+\Delta}\sin^2\Delta_m \\
& \qquad - \left[e^{-2\gamma_3^m\Delta} - e^{-2\gamma_1^m\Delta}\right]\left[(s_{13}^m)^2 e^{-2\gamma_3^m\Delta} + (c_{13}^m)^2 e^{-2\gamma_1^m\Delta}\right]\Big)\Big]\,, \\
P_{e\mu} = {} & s_{23}^2 \sin 2\theta_{13}^m\left[\sin 2\theta_{13}^m - 2\widetilde{\gamma}\gamma_-\cos 2\theta_{13}^m\right]\left[\tfrac{1}{4}\left(e^{-2\gamma_1^m\Delta} - e^{-2\gamma_3^m\Delta}\right)^2 + e^{-2\gamma_+\Delta}\sin^2\Delta_m\right]\,.
\end{aligned}
$$

Correct up to linear order in γ_{13}^m

$$
\Delta_m \equiv C_{13}\Delta\,, \qquad \Delta_\pm \equiv (1 + A \pm C_{13})\,\Delta
$$

$$
\gamma_\pm \equiv \gamma_1^m \pm \gamma_3^m\,, \qquad \widetilde{\gamma} \equiv \frac{\gamma_{13}^m}{C_{13}^2 + \gamma_-^2}\,.
$$

The 3-flavor Zassenhaus expansion

- In vacuum, all $\gamma_{ij} < O(\lambda^2)$, so we can now truncate the Zassenhaus expansion at the first summation.

$$e^{\mathbb{X}+\mathbb{Y}} \approx \left(1 + \sum_{n_1=1}^{\infty} \mathcal{Y}_{n_1} + \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{n_1}{(n_1+n_2)} \mathcal{Y}_{n_2} \mathcal{Y}_{n_1}\right) e^{\mathbb{X}},$$

- For the general decay scenario in the presence of matter:

$$\mathcal{H}_m = \frac{\Delta m^2_{31,m}}{2E_\nu} \begin{pmatrix} -i\gamma_1^m & -\frac{i}{2}\gamma_{12}^m e^{i\chi_{12}^m} & -\frac{i}{2}\gamma_{13}^m e^{i\chi_{13}^m} \\ -\frac{i}{2}\gamma_{12}^m e^{-i\chi_{12}^m} & \alpha_m - i\gamma_2^m & -\frac{i}{2}\gamma_{23}^m e^{i\chi_{23}^m} \\ -\frac{i}{2}\gamma_{13}^m e^{-i\chi_{13}^m} & -\frac{i}{2}\gamma_{23}^m e^{-i\chi_{23}^m} & 1 - i\gamma_3^m \end{pmatrix}$$

- All the quantities are to be taken in the presence of matter:

$$\Delta m^2_{31,\,m},\ \Delta_m,\ \alpha_m,\ U_m,\ \theta_{ij}^m,\ \gamma_i^m, \gamma_{ij}^m, \chi_{ij}^m$$

- Let us decompose: $\mathbb{A} \equiv -i\mathscr{H}_m L \equiv \mathbb{X} + \mathbb{Y}$, and we then get

$$\mathbb{X} = -2i \begin{pmatrix} -i\gamma_1^m & 0 & 0 \\ 0 & \alpha_m - i\gamma_2^m & 0 \\ 0 & 0 & 1 - i\gamma_3^m \end{pmatrix} \Delta_m$$

$$\mathbb{Y} = -\begin{pmatrix} 0 & \gamma_{12}^m e^{i\chi_{12}^m} & \gamma_{13}^m e^{i\chi_{13}^m} \\ \gamma_{12}^m e^{-i\chi_{12}^m} & 0 & \gamma_{23}^m e^{i\chi_{23}^m} \\ \gamma_{13}^m e^{-i\chi_{13}^m} & \gamma_{23}^m e^{-i\chi_{23}^m} & 0 \end{pmatrix} \Delta_m$$

$$\mathbb{Y} = \mathbb{Y}_{12} + \mathbb{Y}_{13} + \mathbb{Y}_{23}$$

$$\Delta_{ij} \equiv \mathbb{X}_{ii} - \mathbb{X}_{jj}$$

$$\mathscr{Y}_n = \frac{1}{n!}\left(\Delta_{12}^{n-1}\Sigma_{12}^{n-1}\mathbb{Y}_{12} + \Delta_{13}^{n-1}\Sigma_{13}^{n-1}\mathbb{Y}_{13} + \Delta_{23}^{n-1}\Sigma_{23}^{n-1}\mathbb{Y}_{23}\right),$$

$$\Sigma_{12} = \mathrm{diag}[(1,-1,0)]\,, \quad \Sigma_{13} = \mathrm{diag}[(1,0,-1)]\,, \quad \Sigma_{23} = \mathrm{diag}[(0,1,-1)]$$

$$\mathscr{A}_m = \begin{pmatrix} e^{-2\gamma_1^m\Delta_m} & 2\Delta_m \frac{\gamma_{12}^m e^{i\chi_{12}^m}}{\Delta_{12}} g_{12}(L) & 2\Delta_m \frac{\gamma_{13}^m e^{i\chi_{13}^m}}{\Delta_{13}} g_{13}(L) \\ [7pt] 2\Delta_m \frac{\gamma_{12}^m e^{-i\chi_{12}^m}}{\Delta_{12}} g_{12}(L) & e^{-2i\alpha_m\Delta_m - 2\gamma_2^m\Delta_m} & 2\Delta_m \frac{\gamma_{23}^m e^{i\chi_{23}^m}}{\Delta_{23}} g_{23}(L) \\ [7pt] 2\Delta_m \frac{\gamma_{13}^m e^{-i\chi_{13}^m}}{\Delta_{13}} g_{13}(L) & 2\Delta_m \frac{\gamma_{23}^m e^{-i\chi_{23}^m}}{\Delta_{23}} g_{23}(L) & e^{-2i\Delta_m - 2\gamma_3^m\Delta_m} \\ [7pt] \end{pmatrix}$$

With, $g_{ij}(L) \equiv \frac{1}{2}\left(\exp[\mathbb{X}_{jj}] - \exp[\mathbb{X}_{ii}]\right)$

Probabilities in vacuum, expanded in α, s_{13}, and γ_3

Let us express the probabilities in the form $P_{\alpha\beta} = P_{\alpha\beta}^{(0)} + P_{\alpha\beta}^{(\gamma_3)}$, we get:

$$
\begin{aligned}
P_{\mu\mu}^{(0)} &= 1 - \sin^2 2\theta_{23} \sin^2 \Delta \\
&\quad +4 s_{13}^2 s_{23}^2 \cos 2\theta_{23} \sin^2 \Delta + \\
&\quad \alpha \sin^2 2\theta_{23}\, c_{12}^2\, \Delta \sin 2\Delta + O(\lambda^3) \\
P_{\mu\mu}^{(\gamma_3)} &= -\gamma_3 \Delta \left(\sin^2 2\theta_{23} \cos 2\Delta + 4 s_{23}^4\right) \\
&\quad +\gamma_3^2 \Delta^2 \left(\sin^2 2\theta_{23} \cos 2\Delta + 8 s_{23}^4\right) + O(\lambda^3)
\end{aligned}
$$

$$
\begin{aligned}
P_{ee}^{(0)} &= 1 - 4 s_{13}^2 \sin^2 \Delta + O(\lambda^4) \\
P_{ee}^{(\gamma_3)} &= -4\,\gamma_3\, s_{13}^2\, \Delta \cos 2\Delta + O(\lambda^4) \\
P_{e\mu}^{(0)} &= 4 s_{13}^2 s_{23}^2 \sin^2 \Delta \\
&\quad +2\alpha\, s_{13}\, \sin 2\theta_{12} \sin 2\theta_{23}\, \Delta\, \cos\left(\Delta - \delta_{CP}\right) \sin \Delta + O(\lambda^4) \\
P_{e\mu}^{(\gamma_3)} &= -8 \gamma_3\, s_{13}^2 s_{23}^2\, \Delta \sin^2 \Delta + O(\lambda^4)
\end{aligned}
$$

- The effects of decay is most prominent in $P_{\mu\mu}$.
- The modifications due to γ_3 only manifest at the third order

$$P_{\mu e} \equiv P_{e\mu}(\delta_{CP} \to -\delta_{CP}) \qquad\qquad P_{\bar{\alpha}\bar{\beta}} = P_{\alpha\beta}(\delta_{CP} \to -\delta_{CP})$$

Probabilities in matter

Expanded in α_m , s_{13}^m , γ_i^m and γ_{ij}^m

- In vacuum, no effect of General decay matrix need to be considered.
- In matter (whether in matter/vacuum basis), all eigenstates may decay.

$$P_{\alpha\beta} = P_{\alpha\beta}^{(0)} + P_{\alpha\beta}^{(\gamma_3)} + P_{\alpha\beta}^{(\Gamma)}$$

- The decay matrix in matter basis is, $\Gamma_m = U_m^\dagger\, U\, \Gamma\, U^\dagger\, U_m$.
- The first 2 probability terms may be obtained by:

$$\theta_{ij} \to \theta_{ij}^m, \quad \gamma_3 \to \gamma_3^m, \quad \chi_{ij} \to \chi_{ij}^m, \quad \Delta \to \Delta_m, \quad \alpha \to \alpha_m$$

- The modification due to General decay matrix Γ is given by

$$P^{(\Gamma)}_{\mu\mu} = \sin 2\theta^m_{23} \left(\gamma^m_{13} s^m_{12} \cos\chi^m_{13} - \gamma^m_{23} c^m_{12} \cos\chi^m_{23}\right) \sin 2\Delta_m$$

$$P^{(\Gamma)}_{ee} = -4\gamma^m_1 \Delta_m (c^m_{12})^2 - 4\gamma^m_2 \Delta_m (s^m_{12})^2 - 2\gamma^m_{12}\Delta_m \sin 2\theta^m_{12} \cos\chi^m_{12}$$
$$-2s^m_{13}\left[\gamma^m_{13} c^m_{12} \cos\left(\delta_{CP} + \chi^m_{13}\right) + \gamma^m_{23} s^m_{12} \cos\left(\delta_{CP} + \chi^m_{23}\right)\right] \sin 2\Delta_m$$

$$P^{(\Gamma)}_{e\mu} = -4s^m_{13}(s^m_{23})^2 \left[\gamma^m_{13} c^m_{12} \sin\left(\delta_{CP} + \chi^m_{13}\right) + \gamma^m_{23} s^m_{12} \sin\left(\delta_{CP} + \chi^m_{23}\right)\right] \sin^2 \Delta_m$$

$$P^{(\Gamma)}_{\mu e} = P^{(\Gamma)}_{e\mu}(\delta^m_{CP} \to -\delta^m_{CP}, \chi^m_{ij} \to -\chi^m_{ij})$$

$$P_{\bar\alpha\bar\beta} = P_{\alpha\beta}(\delta^m_{CP} \to -\delta^m_{CP}, \chi^m_{ij} \to -\chi^m_{ij})$$

- Since the amplitude derived above is only accurate up to $O(\gamma^m_{ij})$, our probability expression will only be correct up to $O(\lambda)$, even when s^m_{13} is not too large.

- No explicit matter dependence either!

We need explicit matter dependence

Explicit matter dependence, with decay of ν_3 only

- We employ the Cayley-Hamilton theorem.
- And calculate relevant probabilities with explicit dependence on the "normalized" matter potential $A \equiv 2E_\nu V_{cc}/\Delta m^2_{31}$.
- Any function $g(\mathbb{M})$ of a matrix $\mathbb{M}$ can be expressed in terms of its eigenvalues

$$g(\mathbb{M}) = \sum_{i=1}^{k} M_i \, g\left(\lambda_i\right) \ , \quad \text{with} \quad M_i \equiv \prod_{j=1, j\neq i}^{k} \frac{1}{\lambda_i - \lambda_j}(\mathbb{M} - \lambda_j \mathbb{I})$$

Taking $\mathbb{M} = -i\mathscr{H}_f^{(\gamma_3)} L$:

$$\begin{aligned}\mathcal{A}_f = \exp[-i\mathcal{H}_f^{(\gamma_3)} L] = & \frac{e^{-iE_1 L}}{(E_1 - E_2)(E_1 - E_3)} \left[\mathcal{H}_f^{(\gamma_3)} - E_2 \mathbb{I}\right] \left[\mathcal{H}_f^{(\gamma_3)} - E_3 \mathbb{I}\right] \\ & + \frac{e^{-iE_2 L}}{(E_2 - E_1)(E_2 - E_3)} \left[\mathcal{H}_f^{(\gamma_3)} - E_1 \mathbb{I}\right] \left[\mathcal{H}_f^{(\gamma_3)} - E_3 \mathbb{I}\right] \\ & + \frac{e^{-iE_3 L}}{(E_3 - E_1)(E_3 - E_2)} \left[\mathcal{H}_f^{(\gamma_3)} - E_1 \mathbb{I}\right] \left[\mathcal{H}_f^{(\gamma_3)} - E_2 \mathbb{I}\right] .\end{aligned}$$

Probabilities expanded in s_{13}, α and γ_3

$$P_{\mu\mu}^{(0)} = 1 - \sin^2 2\theta_{23} \sin^2\Delta - \frac{2}{A-1} s_{13}^2 \sin^2 2\theta_{23} \left(\sin\Delta \cos A\Delta \frac{\sin[(A-1)\Delta]}{A-1} - \frac{A}{2}\Delta \sin 2\Delta\right) - 4s_{13}^2 s_{23}^2 \frac{\sin^2[(A-1)\Delta]}{(A-1)^2} + \alpha\, c_{12}^2 \sin^2 2\theta_{23}\, \Delta \sin 2\Delta + O(\lambda^3)\,,$$

$$P_{\mu\mu}^{(\gamma_3)} = -\gamma_3\Delta\left(\sin^2 2\theta_{23}\cos 2\Delta + 4s_{23}^4\right) + \gamma_3^2\Delta^2\left(\sin^2 2\theta_{23}\cos 2\Delta + 8s_{23}^4\right) + O(\lambda^3)\,.$$

$$E_1^{(0)} + E_1^{(\gamma_3)} \equiv E_1 = \frac{\Delta m_{31}^2}{2E_\nu}\left(A + \alpha\, s_{12}^2 + s_{13}^2 \frac{A}{A-1} - i\gamma_3\, s_{13}^2 \frac{A^2}{(A-1)^2}\right) + O(\lambda^4)\,,$$

$$E_2^{(0)} + E_2^{(\gamma_3)} \equiv E_2 = \frac{\Delta m_{31}^2}{2E_\nu}\left(\alpha\, c_{12}^2\right) + O(\lambda^4)\,,$$

$$E_3^{(0)} + E_3^{(\gamma_3)} \equiv E_3 = \frac{\Delta m_{31}^2}{2E_\nu}\left(1 - i\gamma_3 - s_{13}^2 \frac{A}{A-1} + i\gamma_3\, s_{13}^2 \frac{A^2}{(A-1)^2}\right) + O(\lambda^4)\,.$$

- Valid as long as $\alpha\Delta \lesssim 1$ and $\gamma_3\Delta \lesssim 1$.
- We name it: “Full-expansion”

$$P_{\mu e} = P_{e\mu}(\delta_{CP} \to -\delta_{CP})$$

$$P_{\bar{\alpha}\bar{\beta}} = P_{\alpha\beta}(\delta_{CP} \to -\delta_{CP}\,,\, A \to -A)$$

$$P_{ee}^{(0)} = 1 - 4s_{13}^2 \frac{\sin^2[(A-1)\Delta]}{(A-1)^2} + O(\lambda^4)\,,$$

$$P_{ee}^{(\gamma_3)} = \gamma_3\, s_{13}^2 \left(4A \frac{\sin[2(A-1)\Delta]}{(A-1)^3} - 4\Delta \frac{1+A^2}{(A-1)^2} + 8\Delta \frac{\sin^2[(A-1)\Delta]}{(A-1)^2}\right) + O(\lambda^4)\,,$$

$$P_{e\mu}^{(0)} = 4s_{13}^2 s_{23}^2 \frac{\sin^2[(A-1)\Delta]}{(A-1)^2} + 2\alpha\, s_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos(\Delta - \delta_{\rm CP}) \frac{\sin[(A-1)\Delta]}{A-1} \frac{\sin A\Delta}{A} + O(\lambda^4)\,,$$

$$P_{e\mu}^{(\gamma_3)} = -8\gamma_3\, s_{13}^2 s_{23}^2\, \Delta \frac{\sin^2[(A-1)\Delta]}{(A-1)^2} + O(\lambda^4)\,.$$

Probabilities expanded in s_{13} and α, exact in γ_3

$$P_{\mu\mu} = \left| c_{23}^2 + s_{23}^2 e^{-2i(1-i\gamma_3)\Delta} - 2i\alpha\, c_{12}^2 c_{23}^2\, \Delta + s_{13}^2 s_{23}^2 \left(e^{-2iA\Delta} \frac{(1-i\gamma_3)^2}{[A-(1-i\gamma_3)]^2} \right.\right.$$
$$\left.\left. + e^{-2i(1-i\gamma_3)\Delta} \left[2iA\Delta\,[A-(1-i\gamma_3)] - (1-i\gamma_3) \right] \frac{1-i\gamma_3}{[A-(1-i\gamma_3)]^2} \right) \right|^2 + O(\lambda^3)\,.$$

$$P_{\mu\mu}^{\text{leading}} = c_{23}^4 + s_{23}^4\, e^{-4\gamma_3\Delta} + 2 s_{23}^2 c_{23}^2 \cos(2\Delta) e^{-2\gamma_3\Delta}$$
$$= 1 - \sin^2 2\theta_{23} \sin^2\Delta - s_{23}^4 \left(1 - e^{-4\gamma_3\Delta}\right) - 2 s_{23}^2 c_{23}^2 \cos(2\Delta) \left(1 - e^{-2\gamma_3\Delta}\right)\,.$$

$$E_1 \simeq \frac{\Delta m_{31}^2}{2E_\nu} \left(A + \alpha\, s_{12}^2 + s_{13}^2 \frac{A\,(1-i\gamma_3)}{A-(1-i\gamma_3)} \right) + O(\lambda^4)\,,$$
$$E_2 \simeq \frac{\Delta m_{31}^2}{2E_\nu} \left(\alpha\, c_{12}^2 \right) + O(\lambda^4)\,,$$
$$E_3 \simeq \frac{\Delta m_{31}^2}{2E_\nu} \left(1 - i\gamma_3 - s_{13}^2 \frac{A\,(1-i\gamma_3)}{A-(1-i\gamma_3)} \right) + O(\lambda^4)\,.$$

- Valid as long as $\alpha\Delta \lesssim 1$.
- With exact dependence on γ_3, **valid at lower energies**.
- We name it: "Expansion"

$$P_{ee} = 1 - 2s_{13}^2 \left[\left(1 - e^{-2\gamma_3\Delta}\cos[2(A-1)\Delta]\right) \left[\frac{1+\gamma_3^2}{(A-1)^2+\gamma_3^2} - \frac{2A^2\gamma_3^2}{[(A-1)^2+\gamma_3^2]^2} \right] \right.$$
$$\left. + e^{-2\gamma_3\Delta} \frac{2A\gamma_3\,(1-A+\gamma_3^2)}{[(A-1)^2+\gamma_3^2]^2} \sin[2(A-1)\Delta] + \frac{2A^2\gamma_3\Delta}{(A-1)^2+\gamma_3^2} \right] + O(\lambda^4)\,,$$

$$P_{e\mu} = s_{13}^2 s_{23}^2 \left(1 + e^{-4\gamma_3\Delta} - 2e^{-2\gamma_3\Delta}\cos[2(A-1)\Delta]\right) \frac{\gamma_3^2+1}{(A-1)^2+\gamma_3^2}$$
$$+ \alpha\, s_{13}\, \sin 2\theta_{12} \sin 2\theta_{23} \frac{\sin A\Delta}{A}$$
$$\times \left[\left(\sin\left[(A-2)\Delta + \delta_{\text{CP}}\right] e^{-2\gamma_3\Delta} + \sin\left[A\Delta - \delta_{\text{CP}}\right] \right) \frac{(A-1)-\gamma_3^2}{(A-1)^2+\gamma_3^2} \right.$$
$$\left. + \gamma_3 \left(\cos\left[A\Delta - \delta_{\text{CP}}\right] - \cos\left[(A-2)\Delta + \delta_{\text{CP}}\right] e^{-2\gamma_3\Delta} \right) \frac{A}{(A-1)^2+\gamma_3^2} \right] + O(\lambda^4)\,.$$

Explicit matter dependence, with Γ

- The modification to the eigenvalue due to γ_1 , γ_2 and γ_{ij} :

$$E_i = E_i^{(0)} + E_i^{(\gamma_3)} + E_i^{(\Gamma)}$$

$$E_1^{(\Gamma)} = \frac{\Delta m_{31}^2}{2E_\nu}\Big[- i\gamma_1\, c_{12}^2 - i\gamma_2\, s_{12}^2 - i\gamma_{12}\, s_{12}c_{12}\cos\chi_{12} \quad (6$$

$$- i s_{13}\Big(\gamma_{13}\, c_{12}\cos[\delta_{\rm CP}+\chi_{13}] + \gamma_{23}\, s_{12}\cos[\delta_{\rm CP}+\chi_{23}]\Big)\frac{A}{A-1}\Big] + O(\lambda^4)\,,$$

$$E_2^{(\Gamma)} = \frac{\Delta m_{31}^2}{2E_\nu}\Big[- i\gamma_1\, s_{12}^2 - i\gamma_2\, c_{12}^2 - i\gamma_{12}\, s_{12}c_{12}\cos\chi_{12}\Big] + O(\lambda^4)\,, \quad (6$$

$$E_3^{(\Gamma)} = \frac{\Delta m_{31}^2}{2E_\nu}\Big[i\, s_{13}\Big(\gamma_{13}\, c_{12}\cos[\delta_{\rm CP}+\chi_{13}] + \gamma_{23}\, s_{12}\cos[\delta_{\rm CP}+\chi_{23}]\Big)\frac{A}{A-1}\Big] + O(\lambda^4)\,.$$

- The modification to the probabilities, $P_{\alpha\beta}^{(\Gamma)}$:

$$P_{\mu\mu}^{(\Gamma)} = \sin 2\theta_{23}\,(\gamma_{13}\, s_{12}\cos\chi_{13} - \gamma_{23}\, c_{12}\cos\chi_{23})\sin 2\Delta + O(\lambda^3)\,,$$

$$P_{ee}^{(\Gamma)} = -4\gamma_1\, c_{12}^2\Delta - 4\gamma_2\, s_{12}^2\Delta - 2\gamma_{12}\,\Delta\sin 2\theta_{12}\cos\chi_{12}$$

$$+ 2s_{13}\Big(\gamma_{13}\, c_{12}\cos[\delta_{\rm CP}+\chi_{13}] + \gamma_{23}\, s_{12}\cos[\delta_{\rm CP}+\chi_{23}]\Big)\left(\frac{\sin[2(A-1)\Delta]}{(A-1)^2} - \frac{2A\Delta}{A-1}\right)$$

$$+ O(\lambda^4)\,,$$

$$P_{e\mu}^{(\Gamma)} = -4s_{13}s_{23}^2\,(\gamma_{23}\, s_{12}\sin[\delta_{\rm CP}+\chi_{23}] + \gamma_{13}\, c_{12}\sin[\delta_{\rm CP}+\chi_{13}])\frac{\sin^2[(A-1)\Delta]}{(A-1)^2} + O(\lambda^4)\,,$$

With $P_{\mu e} = P_{e\mu}(\delta_{CP} \to -\delta_{CP}, \chi_{ij} \to -\chi_{ij})$, and
$P_{\bar{\alpha}\bar{\beta}} = P_{\alpha\beta}(\delta_{CP} \to -\delta_{CP}\,, \chi_{ij} \to -\chi_{ij}\,, A \to -A)$

- The $P_{\mu\mu}^{(\Gamma)}$ contributions do not have any matter dependence.
- The probability modifications $P_{ee}^{(\Gamma)}$, $P_{e\mu}^{(\Gamma)}$ are $\sim O(\lambda^3)$

Comparing analytic with numeric

- How accurate are our analytic expressions?

- We compare against the numerically calculated result (in constant matter density approximation)

$$\theta_{12} = 33^\circ\,, \qquad \theta_{23} \simeq 45^\circ\,, \qquad \theta_{13} \simeq 8.5^\circ\,, \qquad \delta_{\rm CP} = 0^\circ\,,$$
$$\Delta m^2_{21} = 7.37\times 10^{-5}\ {\rm eV}^2\,, \qquad \Delta m^2_{31} = 2.56\times 10^{-3}\ {\rm eV}^2\,.$$

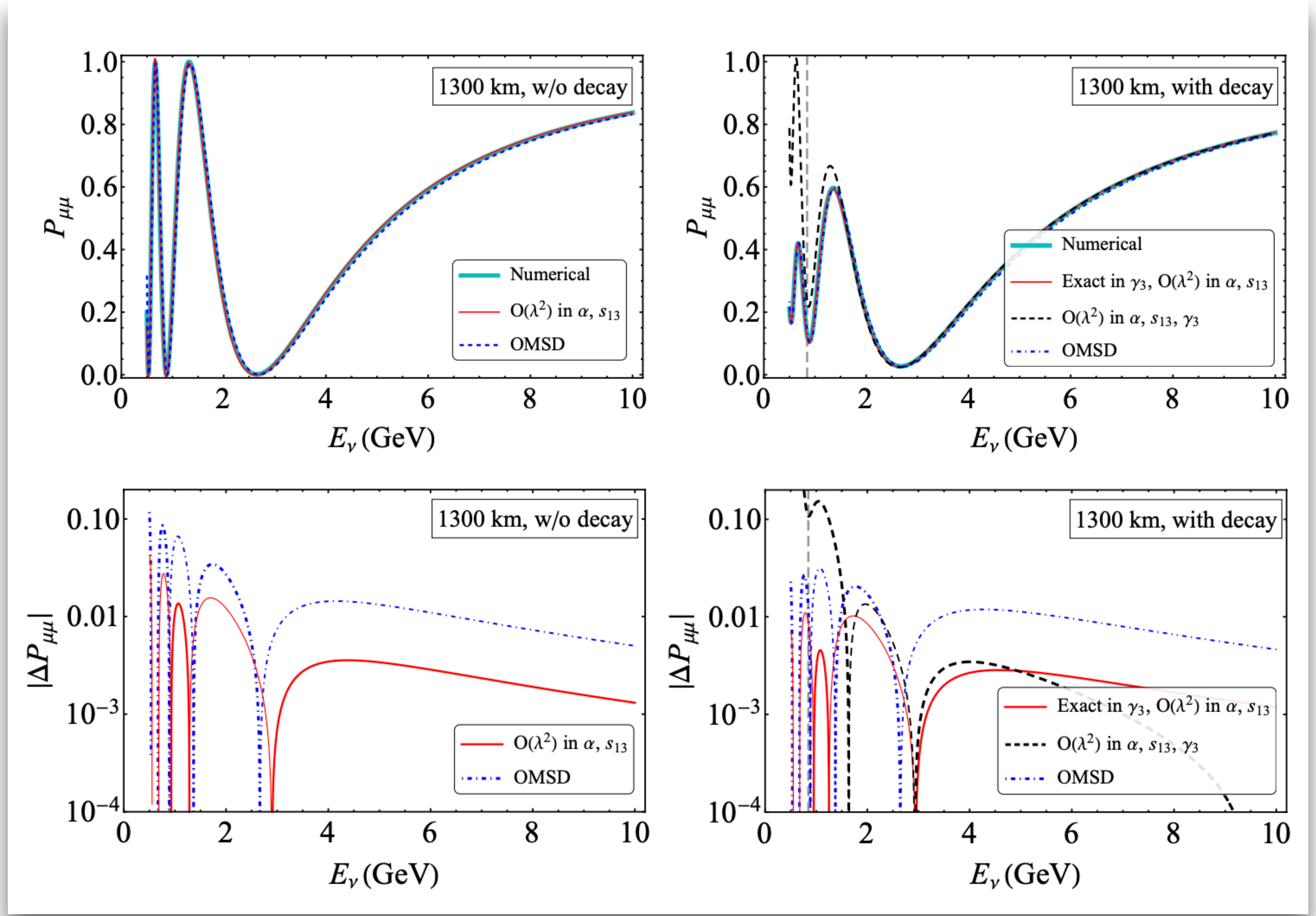
$$\boldsymbol{\gamma_3 = 0.1}$$

- We quantify the accuracy of the analytic approximations:

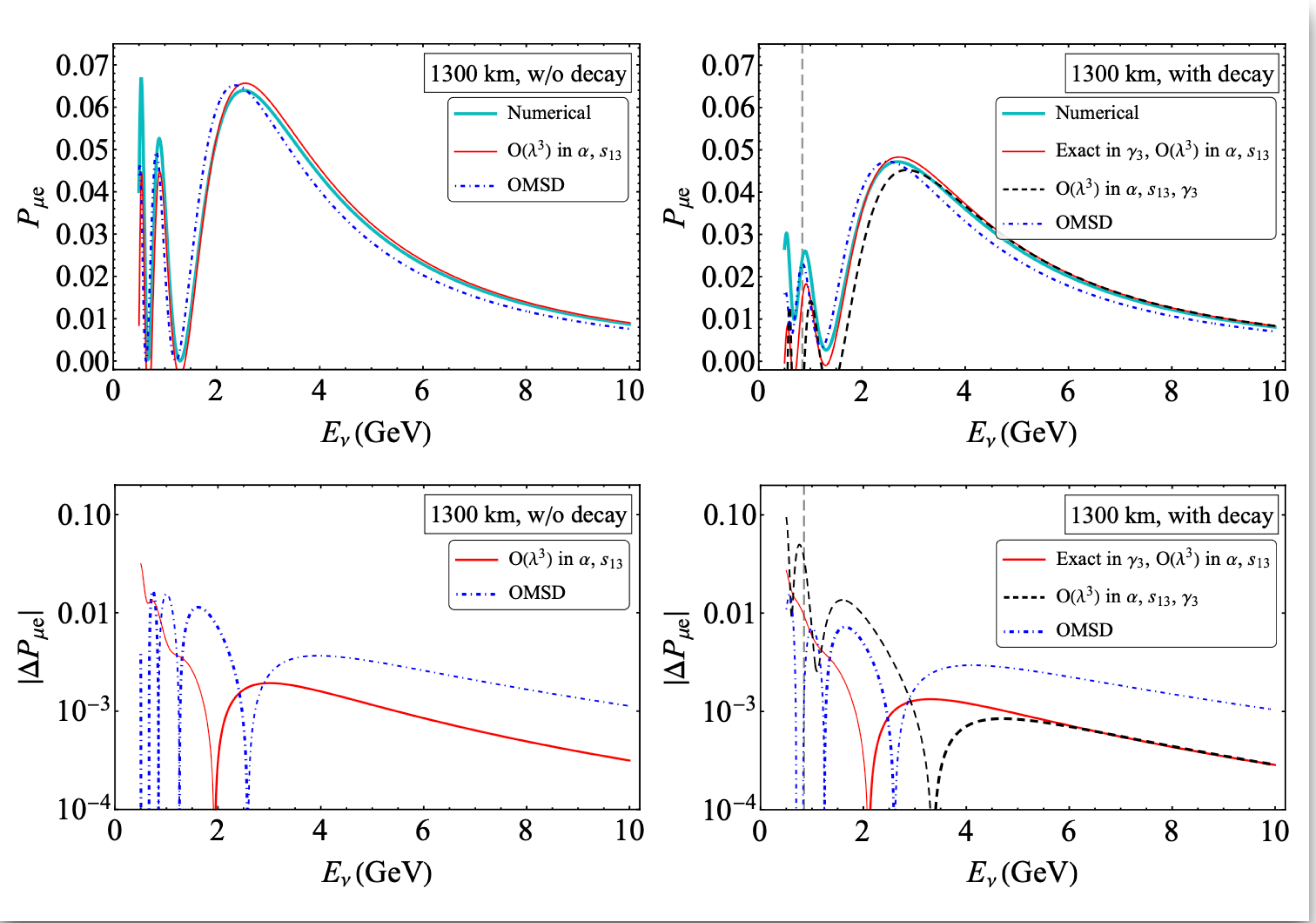
$$\Delta P_{\alpha\beta} = P_{\alpha\beta}(\text{analytic}) - P_{\alpha\beta}(\text{numerical})$$

- For $L = 1300$ km, $L = 7000$ km, and finally, over a wide range of baselines and energies.

$L = 1300$ km

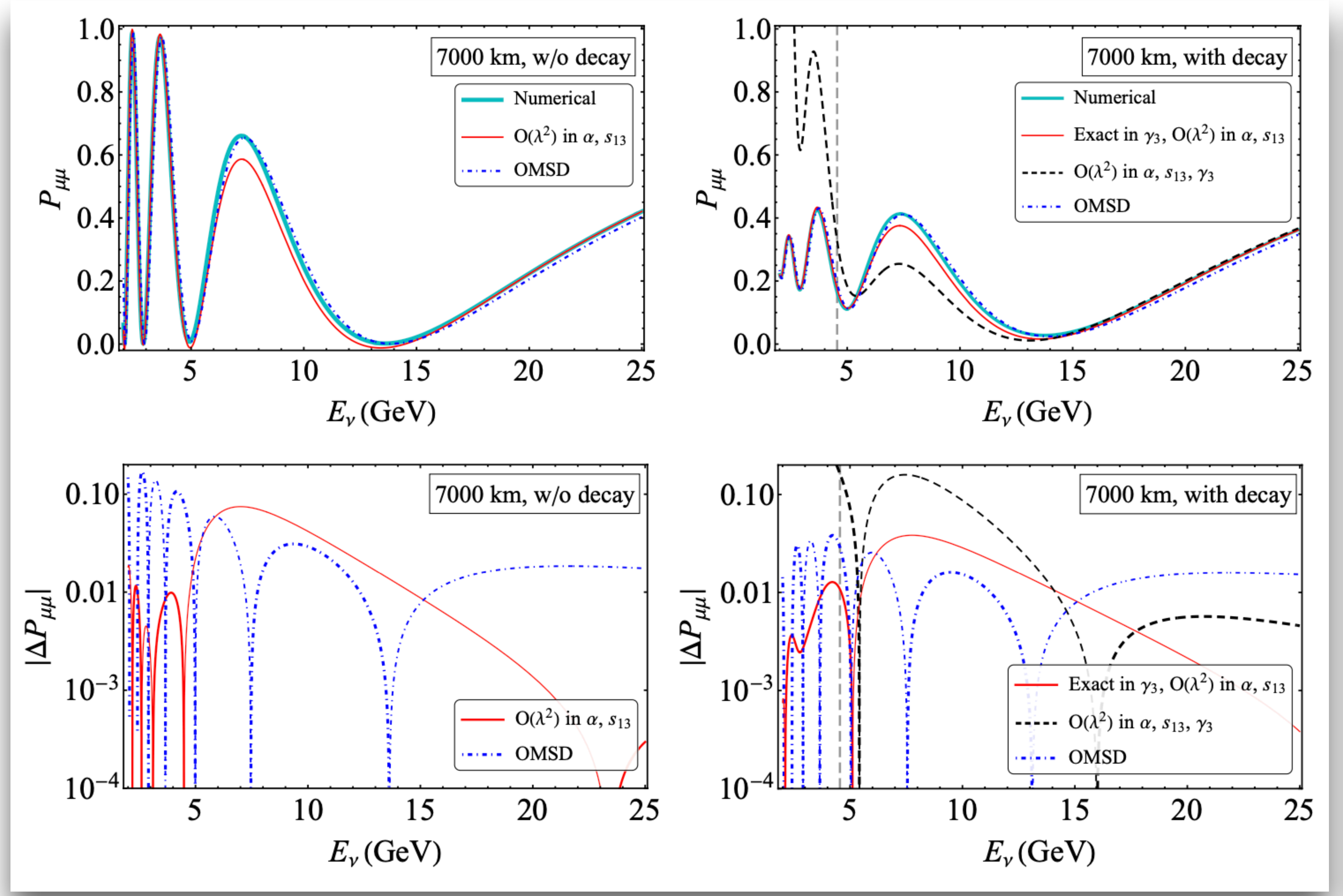


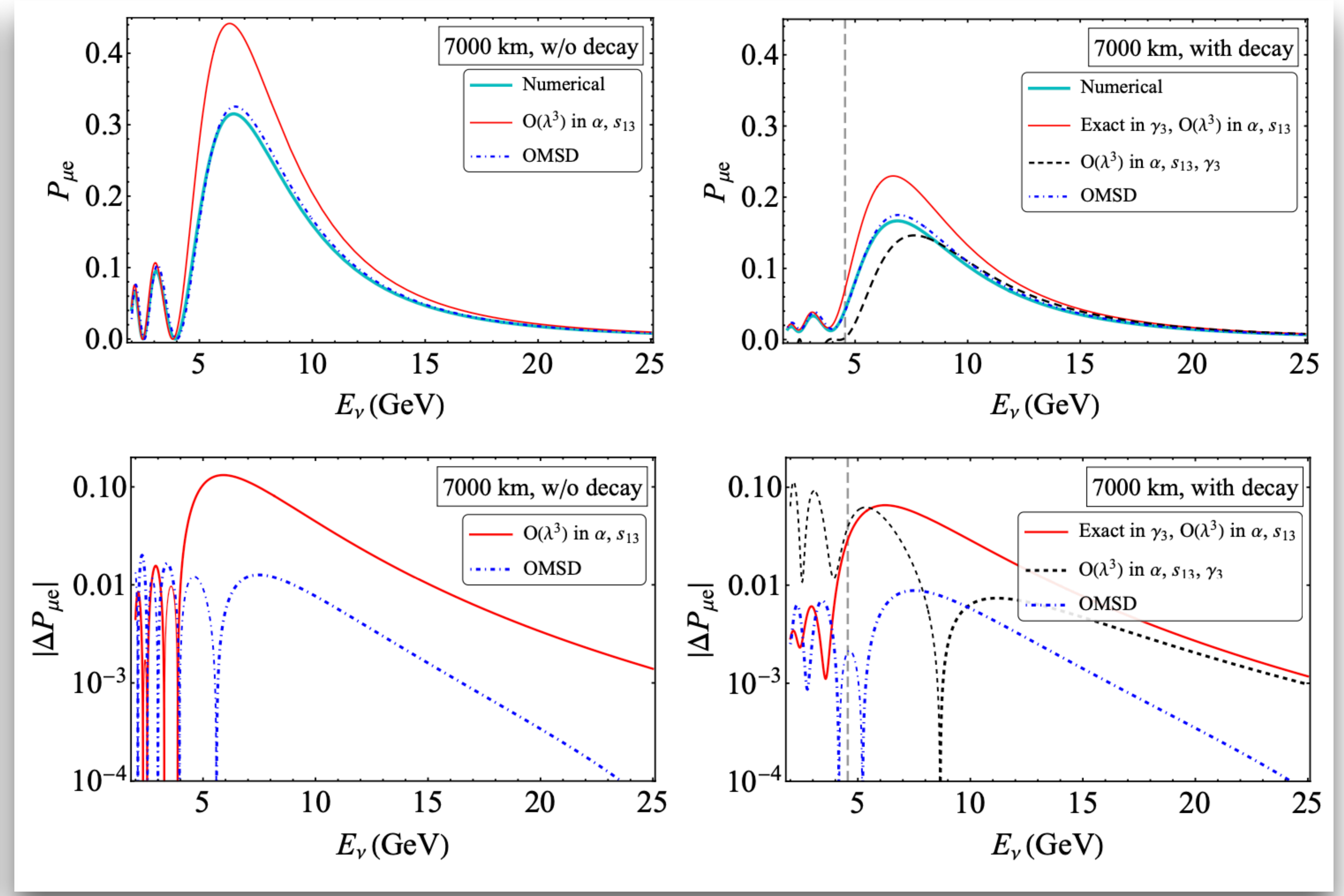
- Plotting $P_{\mu\mu}$
- Dip and peak position reproduced accurately.
- $|\Delta P_{\mu\mu}| < 0.01$ for almost all energies
- Decrease in 1st osc. peak
- Increase at osc. dips.



- Plotting $P_{\mu e}$
- Dip and peak position reproduced accurately.
- $|\Delta P_{\mu e}| < 0.01$ for almost all energies
- Decrease in 1st osc. peak
- Increase at osc. dips.

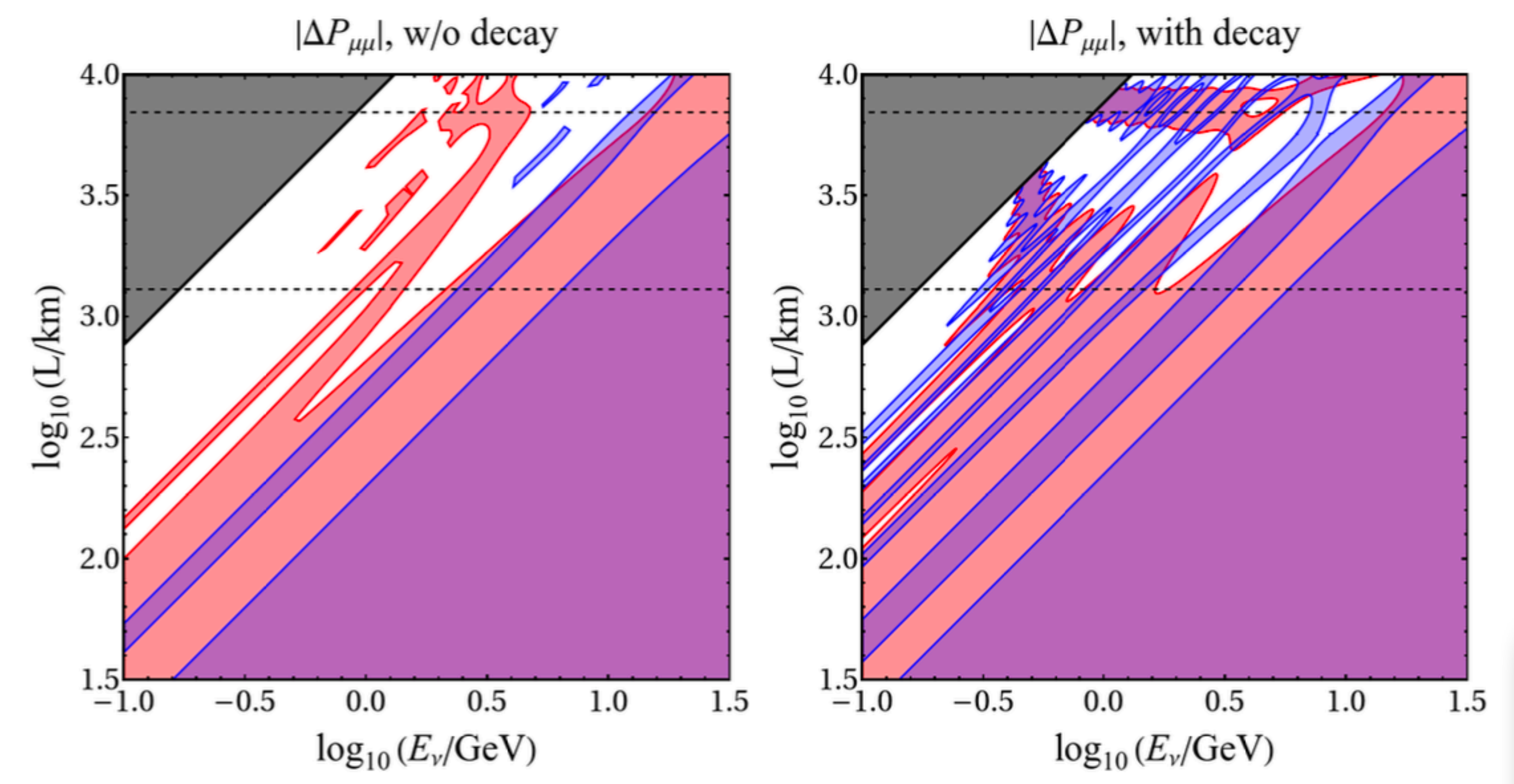
$L = 7000$ km





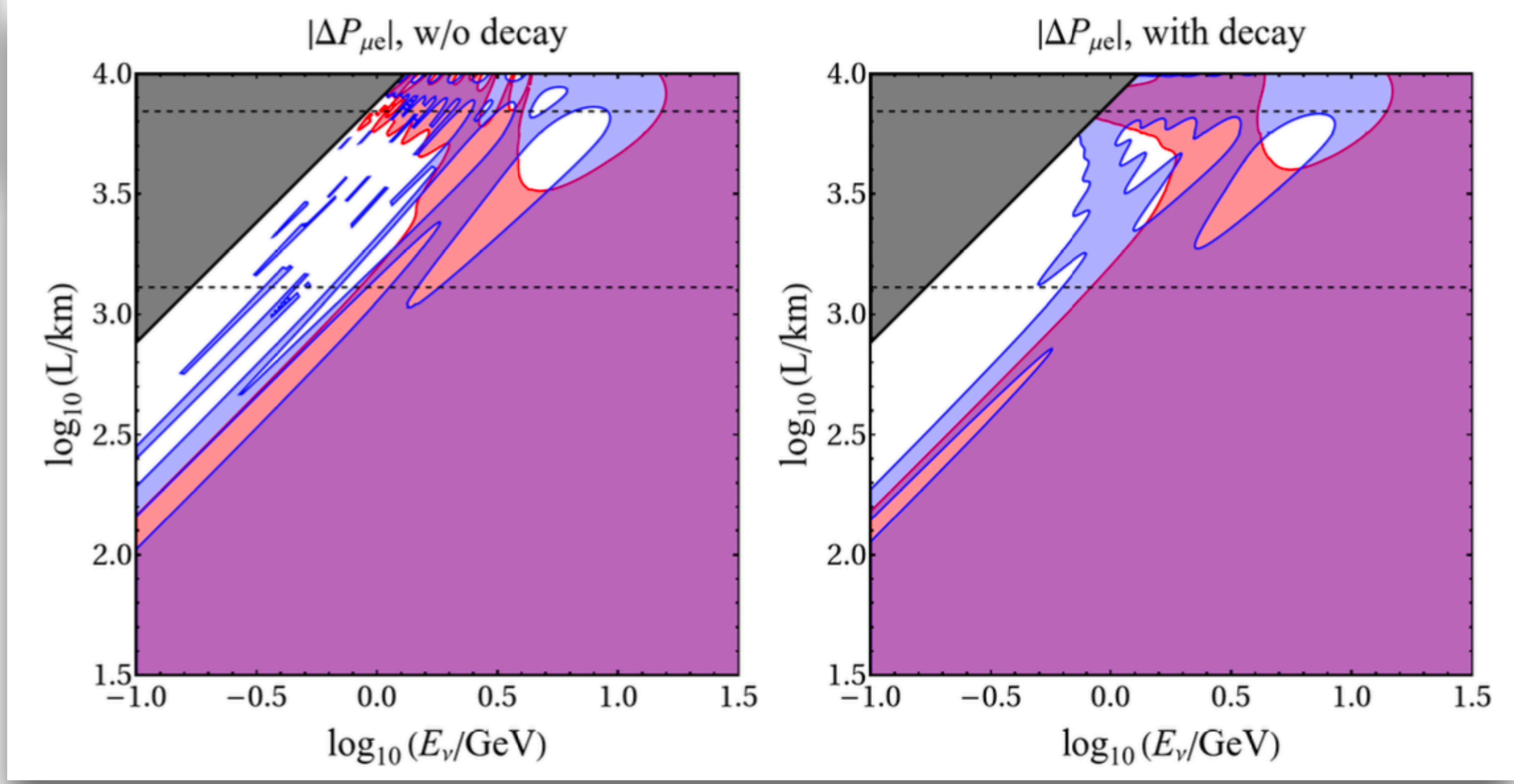
- Peak and dips positions are reproduced correctly by all expansions.
- But **OMSD** gives **very accurate** height also, especially for $P_{\mu e}$.

Over a wide range of baselines

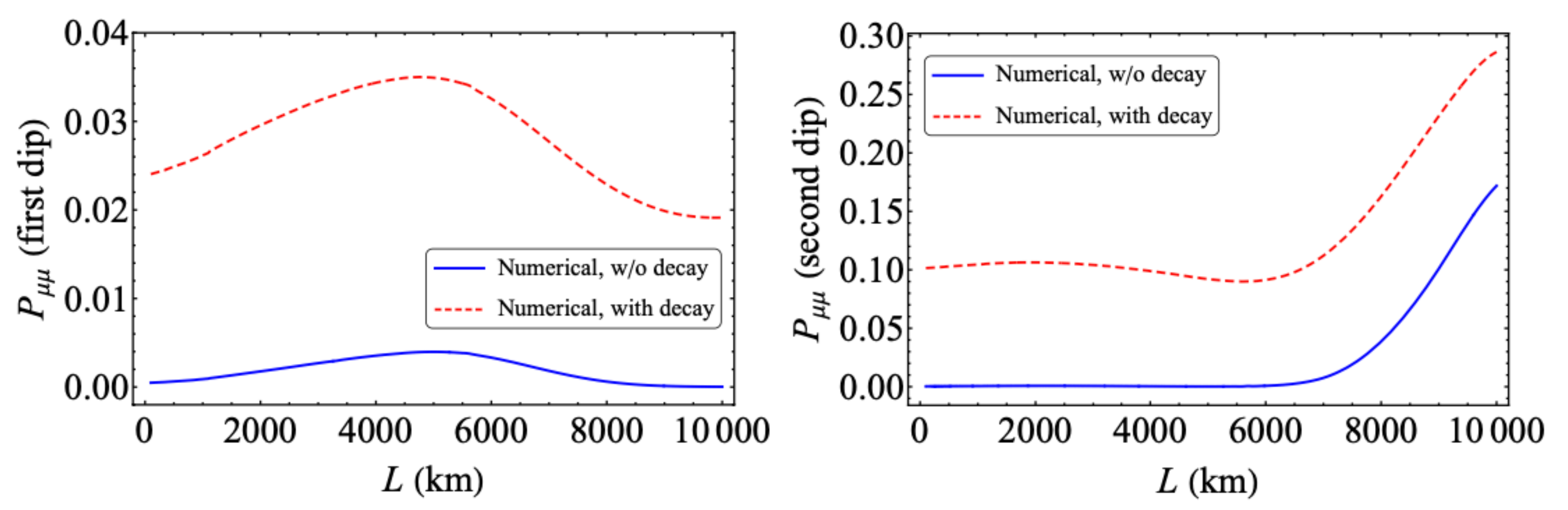


- The regions in the (E_ν, L) parameter space where $|\Delta P_{\alpha\beta}| < 1\,\%$

- OMSD approximation: Blue
- Expansion: Red
- Both: Purple



The first two oscillation dips in $P_{\mu\mu}$



$$P_{\mu\mu}^{\text{leading}}(\text{dip}) = 1 - \sin^2 2\theta_{23} - s_{23}^4\left(1 - e^{-4\gamma_3\Delta}\right) + 2s_{23}^2 c_{23}^2\left(1 - e^{-2\gamma_3\Delta}\right)$$

$$P_{\mu\mu}(\text{first dip}) \simeq P_{\mu\mu}^{\text{leading}}(\Delta \simeq \pi/2) = \frac{1}{4}\left(1 - e^{-\pi\gamma_3}\right)^2 \geq 0$$

$$P_{\mu\mu}(\text{second dip}) \simeq P_{\mu\mu}^{\text{leading}}(\Delta \simeq 3\pi/2) \simeq \frac{1}{4}\left(1 - e^{-3\pi\gamma_3}\right)^2 \geq 0$$

- **Increase in probability** at first and second osc. dips due to ν_3 decay.
- For $\gamma_3 = 0.1$, increase of ~0.02 at 1st and ~0.1 at 2nd osc. dip.
- At longer baselines, one may look at the 2nd osc. dip at $E_\nu \simeq 0.69\,(L/1000$ km) GeV.

Conclusion for the 3 flavor model

- We have considered 2 scenarios:
    - Decay of ν_3 only
    - General decay matrix Γ.
- Used 3 different methods:
    - OMSD+ Pauli exponentiation/ 2-flavor Zassenhaus
    - 3 Flavor Zassenhaus expansion
    - Cayley-Hamilton theorem
- Derived analytic approximations which are extremely accurate.
- Was able to explain even the rise in the dip, a non-intuitive signal due to ν_3 decay.