Wheeler-DeWitt and Loop quantum cosmology

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Introduction

Different approaches to quantize gravity-

- Secondary theories- Start from the quantum theory. Obtain gravity in certain limits.
  Examples- String theory, supergravity.
- Primary theories- Start with a classical theory and quantize on its own.
  - Covariant approach- Maintain 4-dimensional covariance throughout (example- Spin foam theory, path integral quantization)
  - **Canonical approach**- Separate space and time on classical level, and then canonical quantization procedure.
    Examples-
    - ★ Quantum geometrodynamics ⇒ **Wheeler-DeWitt quantum cosmology**
    - ★ Loop quantum gravity ⇒ **loop quantum cosmology**.

Wheeler-DeWitt quantum cosmology

- Our model- Spatially flat ($\boldsymbol{k = 0}$), FLRW (**homogeneous, isotropic**) universe coupled to a **massless scalar field** ϕ, which has zero potential.

$$ds^2 = -dt^2 + a^2 r^2 d\Omega^2$$

- **Lagrangian**- Calculate Ricci scalar. Plug it into Einstein-Hilbert action.

$$\mathcal{L} = -\frac{3V_0}{8\pi G} a\dot{a}^2 + \frac{a^3}{2}\dot{\phi}^2$$

(V_0 is related to the physical volume as $V = a^3 V_0$)

- **Canonical variables**- $(a,\, p_a)$ for geometry, and $(\phi,\, p_\phi)$ for the field.
- **Hamiltonian**-

$$\mathcal{H} = -\frac{2\pi G}{3V_0}\frac{p_a^2}{a} + \frac{p_\phi^2}{2a^3}$$

Wheeler-DeWitt quantum cosmology

- **Quantization**
  - Hamiltonian constraint- $\mathcal{H} \approx 0$
  - Promote canonical variables as operators. (a and ϕ act as multiplication, and p_a and p_ϕ as differentiation operators)

$$p_a \longrightarrow -i\hbar\partial_a$$
$$p_\phi \longrightarrow -i\hbar\partial_\phi$$

- Applying quantized Hamiltonian constraint to the wavefunction of universe gives **Wheeler-DeWitt equation**

$$\boxed{\left(\frac{4\pi G}{3V_0}(a\partial_a)^2 - \partial_\phi^2\right)\psi(a,\phi) = 0}$$

Wheeler-DeWitt quantum cosmology

- **Klein-Gordon equation**
  - Substitution

$$z = \sqrt{\frac{3V_0}{4\pi G}} \ln a$$

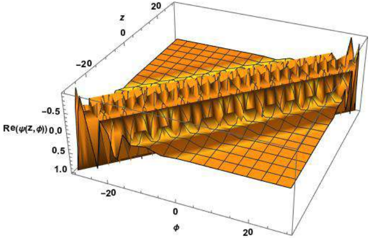
  -

$$\Rightarrow \boldsymbol{(\partial_z^2 - \partial_\phi^2)\psi(z,\phi) = 0}$$

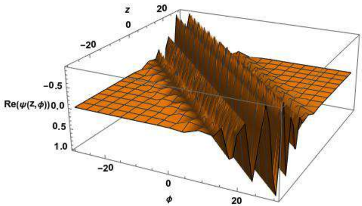
- Solution wave function $\Rightarrow \psi_k(z,\phi) = e^{ik(\phi \pm z)}$
- Wave packet $\Rightarrow \psi(z,\phi) = \int_{-\infty}^{\infty} dk \, A(k) \, e^{ik(\phi \pm z)}$

$$\psi(z,\phi) = e^{ik_0(\phi \pm z) - \frac{(\phi \pm z)^2 \sigma^2}{2}}$$

Wheeler-DeWitt quantum cosmology



$\phi - z$ case



$\phi + z$ case

Real parts of wavepackets

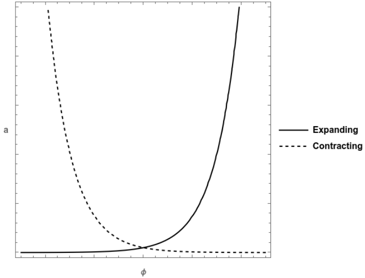
- Wavepackets **peak along** $z = \pm\phi$. Resubstituting and solving gives

$$a = e^{\pm\sqrt{\frac{4\pi G}{3V_0}}\phi}$$

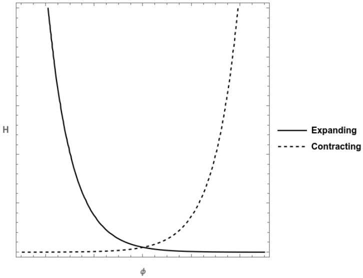
- Solving for Hubble parameter $H = \frac{\dot{a}}{a}$,

$$H = \frac{1}{3} e^{\mp\sqrt{\frac{12\pi G}{V_0}}\phi}$$

Wheeler-DeWitt quantum cosmology



Scale factor a vs. ϕ



Hubble parameter vs. ϕ

Trajectories followed by the peaks of wavepackets

These match with the classical trajectories.

Loop quantum gravity

- **The idea-**
  - Gravity is the curvature of spacetime. (Thanks to GR)
  - Quantizing gravity means quantizing spacetime itself.
  - This picture is realized in Loop quantum gravity. In 1994, Lee Smolin and Carlo Rovelli showed that the **quantum operators** for **area** and **volume** have **discrete spectra**.
- What is so 'loopy'?- Abhay Ashtekar (1986) rewrote GR with new variables. One of them was called Ashtekar connection (related to spin connection).

  Trace of **holonomy** of Ashtekar connection around closed loops gives a quantity- Wilson loops. It form a basis of loop representation.

Loop quantum gravity

- New variables for LQG
  - **First variable-** Denistized triad $(\mathbf{E_i^a})$. An object with a spatial index (a) and one internal index (i), both ranging from 1 to 3.
  - **Second variable**-Ashtekar connection $(\mathbf{A_a^i})$. It is related to spin connection.
- The Hamiltonian is

$$\mathcal{H}_{grav} = -\gamma^{-2} \int_C d^3x q^{-\frac{1}{2}} \epsilon^{ij}{}_k E_i^a E_j^b F_{ab}^k$$

γ is the Barbero-Immirzi parameter. Appears in the definition of Ashtekar connection. Can take any non-zero complex values.

F_{ab}^k is known as the field strength tensor of the connection A_a^i.

Loop quantum cosmology

- **Canonical variables**- (b, v) for geometry and (ϕ, p_ϕ) for the field, as before.

$$b := \gamma \frac{\dot{a}}{a} \qquad\qquad v := \frac{a^3 V_0}{2\pi G}$$

- **Hamiltonian (LQC)**-

$$\mathcal{H} = -\frac{3\pi G}{\gamma^2} v \frac{\sin^2 \lambda b}{\lambda^2} + \frac{p_\phi^2 V_0}{v}$$

λ^2 is the minimum eigenvalue of the area operator.

  - WDW case-

$$\mathcal{H} = -\frac{3\pi G}{\gamma^2} v b^2 + \frac{p_\phi^2 V_0}{v}$$

Loop quantum cosmology

- **Quantization-**
  - Hamiltonian constraint: $\mathcal{H} \approx 0$
  - And promote canonical variables as operators.
- Applying quantized Hamiltonian constraint to the wavefunction of the universe gives

$$\left(\frac{12\pi G}{V_0} \left(\frac{\sin \lambda b}{\lambda} \partial_b \right)^2 - \partial_\phi^2 \right) \psi(b, \phi) = 0$$

  - Similar to the Wheeler-DeWitt equation-

$$\left(\frac{12\pi G}{V_0} \left(b\partial_b \right)^2 - \partial_\phi^2 \right) \psi(b, \phi) = 0$$

Loop quantum cosmology

- **Klein-Gordon equation**
  - Substitution

$$y = \sqrt{\frac{V_0}{12\pi G}} \ln\left(\tan\frac{\lambda b}{2}\right)$$

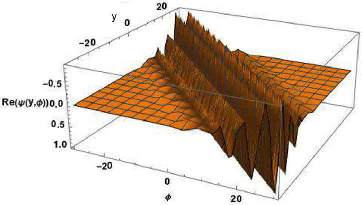
  -

$$\Rightarrow \boldsymbol{(\partial_y^2 - \partial_\phi^2)\psi(y,\phi) = 0}$$

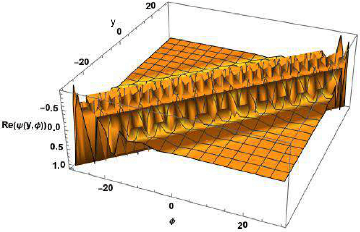
- Following the exact same procedure as in WDW quantum cosmology.
- Solution wave function $\Rightarrow \psi_k(y,\phi) = e^{ik(\phi \pm y)}$
- Wave packet $\Rightarrow \psi(y,\phi) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik(\phi\pm y)}$

$$\psi(y,\phi) = e^{ik_0(\phi\pm y) - \frac{(\phi\pm y)^2\sigma^2}{2}}$$

Loop quantum cosmology



$\phi - y$ case



$\phi + y$ case

Real parts of wavepackets

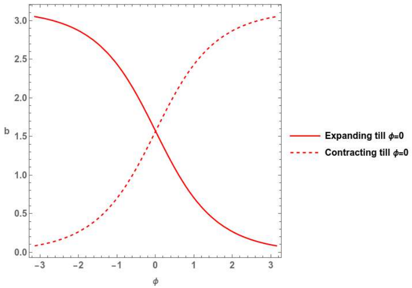
- Wavepackets **peak along $y = \pm\phi$**. Resubstituting and solving gives

$$b = \frac{2}{\lambda} \tan^{-1}\left(e^{\mp\sqrt{\frac{12\pi G}{V_0}}\phi}\right)$$

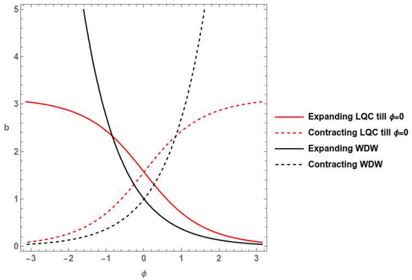
  - Wheeler-DeWitt case-

$$b = \frac{\gamma}{3} e^{\mp\sqrt{\frac{12\pi G}{V_0}}\phi}$$

Loop quantum cosmology



LQC



LQC and WDW

Hubble parameter

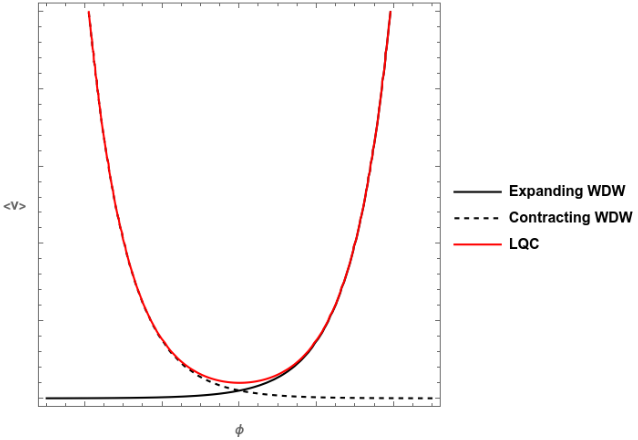
Volume operator expectation values

WDW

- $\langle \hat{V} \rangle = V_* e^{\pm\sqrt{\frac{12\pi G}{V_0}}\phi}$

LQC

- $\langle \hat{V} \rangle \equiv V_- e^{\sqrt{\frac{12\pi G}{V_0}}\phi} + V_+ e^{-\sqrt{\frac{12\pi G}{V_0}}\phi}$



Criticism on LQC

- Criticism on common claims in LQC (M. Bojowald) -
  - The claimed bounce is not necessarily generic.
  - For isotropy and homogeneity to be valid on a smaller scale, some changes need to be made.
  - Those changes would lead to huge quantum fluctuations as $\phi \longrightarrow 0$.
  - The bouncing solution is not generic.

Conclusion

- For Wheeler-DeWitt and Loop quantum comsologies we compared
  - Hamiltonian
  - Hubble parameter
  - Expectation values of the volume operator
- Criticism on the 'Bounce' claim of LQC by M. Bojowald.

My current work

- Pursuing PhD at the University of Canterbury, New Zealand, under the supervision of Dr. Chris Gordon.
- Gravitational wave background- Data from Pulsar Timing Array (PTA)
- Dark matter model- Self interacting dark matter (SIDM) spike around Supermassive black hole (SMBH)
- Testing this DM model against PTA data using Bayesian statistics.

Thank you!