STABLE CHROMOMAGNETIC QCD VACUUM AND CONFINEMENT

R.Parthasarathy
Chennai Mathematical Institute
H1, SIPCOT IT Park
Padur Post
Siruseri 603 103, India

Strong Interaction in the 21st century - TIFR - Feb. 10-12, 2010

The economical definition of confinement of quarks in QCD is the 'area law' for the Wilson loop. The gauge invariant Wilson loop is

$$W(C) \; = \; Tr \; P \; e^{-ig \oint \; dx^{\mu} \; A^{a}_{\mu}(x) t^{a}}, \qquad (1)$$

where P denotes the path ordering and t^a are the generators of the gauge group.

We shall consider $SU(2)$ Yang-Mills theory and choose the Savvidy classical backgroound

$$\bar{A}^{a}_{0} \; = \; 0 \; ; \; \bar{A}^{a}_{i} \; = \; \delta^{a3} \; (-\frac{Hy}{2}, \frac{Hx}{2}, 0). \qquad (2)$$

This choice solves the classical equation of motion $\bar{D}^{ab}_{\mu} \bar{F}^{\mu\nu b} = 0$, where $\bar{D}^{ab}_{\mu} = \partial_{\mu} \delta^{ab} + g \epsilon^{acb} \bar{A}^{c}_{\mu}$. The classical background corresponds to constant chromomagnetic field in the third color direction $\bar{F}^{3}_{12} = H$ and this comes from the derivative terms in $\bar{F}^{a}_{\mu\nu} = \partial_{\mu} \bar{A}^{a}_{\nu} - \partial_{\nu} \bar{A}^{a}_{\mu} + g \epsilon^{abc} \bar{A}^{b}_{\mu} \bar{A}^{c}_{\nu}$. For this reason, the Savvidy ansatz is called 'Abelian like'. So the classical background is esssentially Abelian-like, taking values in the Cartan subgroup of $SU(2)$.

The use of Abelian field strength can be understood from the idea of 't Hooft who proposed 'Abelian Projection'. This is a particular gauge fixing, breaking the gauge group $SU(N)$ to the maximal torus subgroup $H = U(1)^{N-1}$. For $SU(2)$, $H = U(1)$. This is realized in a specific gauge called the 'Maximal Abelian Gauge'. In the continuum formulation, this has the form

$$(\partial_\mu \delta^{ab} + g\epsilon^{a3b}\bar{A}^3_\mu)\bar{A}^b_\mu \;=\; 0, \tag{3}$$

and the classical Savvidy background satisfies this. Numerical simulations on the lattice have found that the Abelian projected Wilson loop defined by A^3_μ exhibits the 'area law' (T.Suzuki and I.Yotsuyanagi, Phys.Rev. **D42** (1990) 4257; K-I.Kondo and A.Shibata, hep-th/0801.4203). So (1) in this gauge (3)

$$\begin{aligned} W(c) &= \langle e^{-ig\oint dx^\mu\ A^3_\mu}\rangle, \\ &= \langle e^{-i\frac{g}{2}\int_S dS^{\mu\nu}F^3_{\mu\nu}}\rangle, \\ &= \langle e^{-i\frac{g}{2}H\times\ area}\rangle, \end{aligned} \tag{4}$$

where in (4), H should correspond to the *minimum value of the energy density*.

The classical energy density for the background (2), in the Euclidean formulation, is $\mathcal{E} = \frac{H^2}{2}$. This energy density has a minimum $\mathcal{E} = 0$ at $H = 0$ and so $W(C)$

in (4) does not give the area law. In order to realize the area law from (4), the minimum energy density should correspond to $H \neq 0$.

Savvidy has studied the quantum 1-loop effective energy density which has a minimum lower than the above classical minimum and for which $H \neq 0$.

However, Nielsen and Olesen pointed out that the 1-loop effective energy density in the background (2) had an imaginary part, stemming from the lowest Landau level and so the vacuum (ground state) of such a model is unstable. Various attempts were made to circumvent this sensitive issue which inhibited the progress.

All these calculations were performed in the Gaussian (keeping only the terms quadratic in quantum fluctuations) approximation.

We have reexamined this important issue by retaining *all* the terms in the quantum fluctuations. Besides quadratic terms, there are terms cubic and quartic in quantum fluctuations. Briefly, the Euclidean functional integral for $SU(2)$ pure Yang-Mills theory

$$\begin{aligned} Z &= \int [dA_\mu^a]\, e^S, \\ S &= \int d^4x \{-\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a\}, \\ F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c. \end{aligned} \tag{5}$$

The action above is expanded around the classical background $\bar{A}_\mu^a$ in (2) as

$$A_\mu^a = \bar{A}_\mu^a + a_\mu^a, \tag{6}$$

and the quantum fluctuations a_μ^a are taken to satisfy the 'background gauge'

$$\bar{D}_\mu^{ab} a_\mu^b = 0. \tag{7}$$

This gauge choice is important. First of all, there is no Gribov ambiguity in using this background gauge.

It has been shown by Amati and Rouet (Phys.Lett. **73B** 1978, 39) that the multiplicity of classical solutions satisfying the gauge condition is an irrelevant issue for quantizing non-Abelian Yang-Mills theories in the background gauge and an unambiguous generating functional is now possible. The correct treatment of the zero modes of the 1-loop operator gives the background gauge relative to the classical solution.

Further details can be seen in R.Parthasarathy, Lett. Math. Phys. **15** (1988) 179; Pramana **32** (1989) 563.

Second, with the background gauge (7), and using (3), we have $\bar{D}^{ab}_{\mu}(\bar{A}^{b}_{\mu} + a^{b}_{\mu}) = 0$ and so the 'Maximal Abelian Gauge' or Abelian projection is realized for the full gauge field $\bar{A}^{a}_{\mu} + a^{a}_{\mu}$.

Now using (6) and (7) in (5), the unambiguous Euclidean generating functional Z becomes,

$$Z \;=\; \int [da^a_\mu] e^{S'}, \tag{8}$$

with

$$\begin{aligned} S' \;=\; & \int d^4x \{ -\frac{1}{4} \bar{F}^a_{\mu\nu} \bar{F}^a_{\mu\nu} + \frac{1}{2} a^a_\mu \Theta^{ac}_{\mu\nu} a^c_\nu + g\epsilon^{acd} (\bar{D}^{ae}_\nu a^e_\mu) a^c_\mu a^d_\nu \\ & - \frac{g^2}{4} ((a^a_\mu a^a_\mu)^2 - a^a_\mu a^c_\mu a^a_\nu a^c_\nu) \} - \ell og \; det(-\bar{D}^{ab}_\mu \bar{D}^{bc}_\mu), \end{aligned} \tag{9}$$

where

$$\Theta^{ac}_{\mu\nu} \;=\; (\bar{D}^{ab}_\lambda \bar{D}^{bc}_\lambda) \delta_{\mu\nu} + 2g\epsilon^{aec} \bar{F}^e_{\mu\nu}. \tag{10}$$

In arriving at (9), we have introduced the gauge fixing and the Faddeev-Popov ghost Lagrangian for the background gauge (7) and integrated the ghost fields, resulting in the last term in (9). The expansion in (9) is exact. The purpose of writing S' in the form above is to isolate the stable and unstable modes of $\Theta^{ac}_{\mu\nu}$.

For the Savvidy background, $\Theta^{ac}_{44} = \Theta^{ac}_{33} = \bar{D}^{ab}_{\lambda}\bar{D}^{bc}_{\lambda}$ so that their contributions to Γ cancel the ghost contribution. Further the non-vanishing Θ are $\Theta_{ij}ac$ for $i, j = 1, 2$. The eigenmodes and eigenvalues are:

$$\begin{aligned}
a^3_1 \pm i a^3_2 \;&:\; {k_1}^2 + {k_2}^2 + {k_3}^2 + {k_4}^2 \;\; (plane\ waves),\\
(a^1_1 + i a^1_2) - i(a^2_1 + i a^2_2) \;&:\; (2n+1)gH + 2gH + k^2_3 + k^2_4, \;\; (stable)\\
(a^1_1 - i a^1_2) + i(a^2_1 - i a^2_2) \;&:\; (2n+1)gH + 2gH + k^2_3 + k^2_4, \;\; (stable)\\
(a^1_1 + i a^1_2) + i(a^2_1 + i a^2_2) \;&:\; (2n+1)gH - 2gH + k^2_3 + k^2_4, \;\; (unstable)\\
(a^1_1 - i a^1_2) - i(a^2_1 - i a^2_2) \;&:\; (2n+1)gH - 2gH + k^2_3 + k^2_4, \;\; (unstable)
\end{aligned}$$

The last two eigenvalues become negative when $n = 0$ and for low momenta. As we have logarithm of the eigenvalues, negative values make it imaginary and hence the effective energy density becomes complex indicating vacuum instability. This in the Gaussian approximation.

The contribution from the stable modes are found to be

$$\tfrac{10g^2H^2}{96\pi^2}\ \{\ell og(\tfrac{gH}{\mu^2}) + C\}, \qquad (11)$$

where C is a real (infinite) constant and μ^2 is a dimensioful constant introduced to render the argument of the logarithm dimensionless.

For the unstable modes, we considered the full action in (9). The unstable modes involve the Lorentz indices 1 and 2 and the $SU(2)$ indices 1 and 2, because the classical background (2) is in the third color direction and so the cubic term in (9), namely, $\epsilon^{acd}(\bar{D}^{ae}_{\nu}a^{e}_{\mu})a^{c}_{\mu}a^{d}_{\nu}$ vanishes. The quartic term in (9) for the unstable modes is found to be $\frac{1}{8}(|a_u|^2)^2$ where a_u is the unstable mode. The functional integral Z for the unstable modes is evaluated and from this the finite part of the unstable mode contribution to the energy density is found to be

$$\frac{g^2H^2}{8\pi^2}\ell og(\frac{gH}{\mu^2}) - \frac{g^2H^2}{4\pi^2}\ell ogI, \tag{12}$$

where

$$I \;=\; \int dc' \; e^{-\{c'^2({k'_3}^2+{k'_4}^2-1)+\frac{g^2}{256\pi^2}c'^4\}}. \tag{13}$$

The integral I is convergent irrespective of whether ${k'_3}^2 + {k'_4}^2$ is $<$ or $>$ 1. Further I is real, finite and *independent* of H. Adding (12) to (11) and including the classical energy density, the effective energy density is found to be

$$\mathcal{E} \;=\; \frac{H^2}{2} + \frac{11g^2H^2}{48\pi^2}\{\ell og(\frac{gH}{\mu^2}) + C'\}. \tag{14}$$

The real constant C' is then fixed by Coleman-Weinberg normalization $\frac{\partial}{\partial H^2}\mathcal{E}|_{gH=\mu^2} = \frac{1}{2}$ as $-\frac{1}{2}$. Thus the effective energy density for $SU(2)$ Yang-Mills theory in Savvidy background becomes

$$\mathcal{E} \;=\; \frac{H^2}{2} + \frac{11g^2H^2}{48\pi^2}\{\ell og(\frac{gH}{\mu^2}) - \frac{1}{2}\}. \qquad (15)$$

This is real. The above result is genuine non-Abelian gauge theory effect.

Quarks (fermions) can be added by minimally coupling them with the background (2) and functionally integrating ψ and $\bar{\psi}$ in Z. The only change is the replacement of 11 in (15) by $(11 - N_f)$ for N_f quark flavors. The prefactor $\frac{11-N_f}{48\pi^2}$ can be obtained from group theory considerations. Extending to $SU(3)$, this factor becomes $\frac{33-2N_f}{96\pi^2}$.

In contrast to the classical energy density, the effective energy density in (15) has a minimum at non-zero H. The minimum occurs when

$$H \;=\; \frac{\mu^2}{g}\, e^{-\frac{24\pi^2}{11g^2}}, \tag{16}$$

with 11 appropriately replaced when quarks are included. Thus *vacuum expectation value* H (corresponding to the minimum of $\mathcal{E}$) is *not zero* which gives the Wilson loop the 'area law' and hence confinement. The minimum energy density is

$$\mathcal{E}_{min} \;=\; -\frac{11g^2H^2}{96\pi^2}, \tag{17}$$

which is lower than the classical minimum. This is the energy of the vacuum in the pure $SU(2)$ Yang-Mills theory.

The result that the minimum of the energy density occurs when $H \neq 0$ (17) and $\bar{F}^3_{12} = H$, imply that the vacuum expectation value $\langle \bar{F}^3_{12}\bar{F}^3_{12}\rangle \neq 0$. This indicates that $\langle g^2 F^a_{\mu\nu}F^a_{\mu\nu}\rangle \neq 0$, the occurence of 'gluon condensate'. In this case

$$\langle g^2 F^a_{\mu\nu}F^a_{\mu\nu}\rangle \;=\; 2g^2H^2_{min} \;=\; 2\mu^4\, e^{-\frac{24\pi^2}{11g^2}}. \tag{18}$$

Instead of using the strong coupling g which runs, we use the result from the Charmonium decay analysis, $\langle g^2 F^a_{\mu\nu} F^a_{\mu\nu} \rangle \sim 0.5 GeV^4$. Then, $gH \sim 0.5 GeV^2$. With this estimate

$$W(C) \sim e^{\frac{gH}{2} \, area} = e^{\sigma \, area}, \tag{19}$$

where $\sigma = \frac{gH}{2} = 0.25 GeV^2$. It is well known that the 'area law' corresponds to a linear potential and in the leading order $V = \sigma R$ where R is the separation of static quark and anti-quark.

For a linear potential $V = kr$, the non-relativistic variational calculations give the $c\bar{c}$ bound states for $k = 0.272 GeV^2$ which agrees with our estimate of σ as $0.25 GeV^2$.

Thus, the stable vacuum in the chromomagnetic background is very much indicative of confinement, gives in the leading order the linear potential whose parameter k is satisfactorily obtained.

Finite Temperature Studies

In the studies of the Savvidy vacuum at finite temperature in the *Gaussian Approximation*, the effective energy density involved a temperature dependent imaginary part. We have extended our zero-temperature studies (including the cubic and quartic terms) to finite temperature with chemical potential.

A chemical potential for massless non-Abelian bosons can be introduced (Anishetty, 1984), by observing that there are conserved color charges $Q^a = \int d^3x j_0^a;\ j_\mu^a = f^{abc} A_\nu^b F_{\nu\mu}^c$. For $SU(2)$,one chooses Q^3. The grand canonical partition function will now have μQ^3 in the hamiltonian. This leads to the result of using $A_0^a = -i\mu\delta^{a3}$. This is not possible for Abelian theories.

The role of chemical potential as a constant term in A_0^a is similar to the use of Polyakov loop specified by a constant A_0^a field in the third color direction. Now the Savvidy background becomes

$$\bar{\mathcal{A}}_\mu^a \ = \ \delta^{a3}\{\frac{i\mu}{g}, -\frac{Hy}{2}, \frac{Hx}{2}, 0\}, \tag{20}$$

which gives $\bar{F}_{12}^3 = H$ which solves the classical equation of motion.

The background covariant derivative (Euclidean) now is

$$\bar{D}^{ab}_{\lambda} \;=\; \partial_{\lambda}\delta^{ab} + g\epsilon^{a3b}\bar{A}^{3}_{\lambda} + \mu\epsilon^{a3b}v_{\lambda}, \qquad (21)$$

where $v_{\lambda} = (1, 0, 0, 0)$.

The inclusion of the cubic and quartic terms makes the calculations involved. The result is a *real* effective energy density which including the zero temperature contribution is

$$\begin{aligned}\mathcal{E} \;&=\; \frac{H^2}{2} + \frac{11(gH)^2}{48\pi^2}(\ell og(\frac{gH}{\Lambda^2}) - \frac{1}{2}) + \frac{\pi^2}{45\beta^4} \\ &+\; \frac{(gH)^{\frac{3}{2}}}{\beta\pi^2}\sum_{\ell=1}^{\infty}\frac{\cos(\mu\beta\ell)}{\ell}\{-\frac{\pi}{2}Y_1(\beta\ell\sqrt{gH}) + K_1(\beta\ell\sqrt{gH}) \\ &+\; 2\sum_{n=1}^{\infty}\sqrt{2n+1}K_1(\sqrt{2n+1}\beta\ell\sqrt{gH})\}, \qquad (22)\end{aligned}$$

where Y_1, K_1 are modified Bessel functions. Setting $\beta = \frac{a}{\sqrt{gH}}$ and $\mu = b\sqrt{gH}$, the temperature dependence is written as

$$\begin{aligned}\frac{\mathcal{E}_T}{(gH)^2} \;&=\; \frac{\pi^2}{45a^4} + \frac{1}{\pi^2 a}\sum_{\ell=1}^{\infty}\frac{\cos(ab\ell)}{\ell}\{-\frac{\pi}{2}Y_1(a\ell) + K_1(a\ell) \\ &+\; 2\sum_{n=1}^{\infty}\sqrt{2n+1}K_1(a\ell\sqrt{2n+1})\}. \qquad (23)\end{aligned}$$

We have plotted $\frac{\mathcal{E}_T}{(gH)^2}$ with T in units of $\sqrt{gH}$. b involves the chemical potential. For $b = 0$, zero chemical potential, the variation is smooth. At high temperatures, the behaviour is like that of non-interacting relativistic gas.

For $b = 2, 3$ the variation shows a minimum and then smooth rise. A non-zero chemical potential triggers deconfinement phase transition.

Deconfinement occurs for $b = 1$ around $T/\sqrt{gH} \sim 0.4$ and for $b = 2$ around $T/\sqrt{gH} \sim 0.7$. We have identified the string tension σ as $\frac{gH}{2}$ and so our results give for the deconfining temperature $\frac{T}{\sqrt{\sigma}} \sim 0.5656$ for $b = 1$ and 0.9899 for $b = 2$. It is interesting to compare with the lattice studies. For $SU(2)$, Lucini, M.Teper and Wenger (hep-lat/0307017; 0502003) find $\frac{T}{\sqrt{\sigma}} \sim 0.709$, the agreement is satisfactory for $1 > b < 2$. As the chemical potential $\mu = b\sqrt{gH}$, using $gH = 0.5\ GeV^2$, the lowest value for the chemical potential triggering deconfinement is $0.7\ GeV < \mu < 1.41\ GeV$.

The details of the calculations presented are in

D.Kay, A.Kumar and R.Parthasarathy, Mod.Phys.Lett. **A20** (2005) 1655;
R.Parthasarathy and A.Kumar, Phys.Rev. **D75** (2007) 0805007.