Approach to phase transition in nuclear matter

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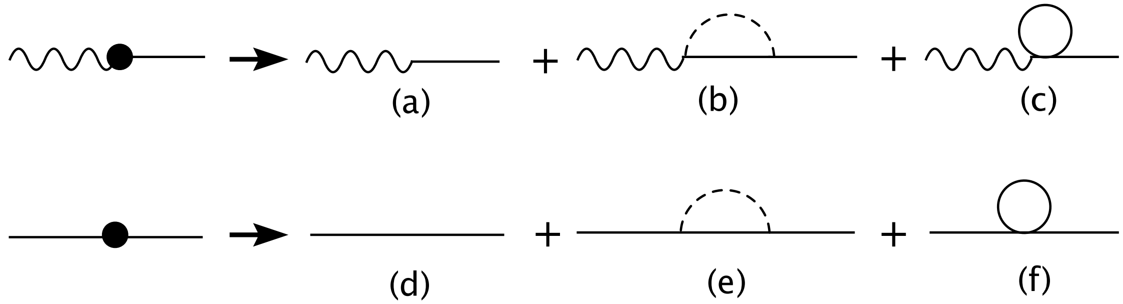
Introduction

We find indication of an approach to phase transition in nuclear matter by studying the two-point function

$$\Pi(E,\vec{p}) = i\int dtd^3x\, e^{i(Et-\vec{p}\cdot\vec{x})} Tr[e^{-\beta(H-\mu N)}T\eta(x)\bar{\eta}(0)]/\mathcal{Z}, \qquad \mathcal{Z} = Tr\, e^{-\beta(H-\mu N)} \tag{1}$$

at *non-zero nucleon chemical potential* (μ) and zero temperature. Here $\eta(x)$ is a three-quark current, having the quantum numbers of the nucleon. H and N are the Hamiltonian and the Number operator of the system. Clearly $\Pi(E,\vec{p})$, written in terms of quark fields, is closely related to the nucleon propagator.

Unfortunately, unlike the case at *finite temperature and zero nucleon chemical potential*, a straightforward calculation of $\Pi(E,\vec{p})$ based on Feynman graphs is *not* possible in this case. The difficulty is due to the appearance of new (unknown) and presumably large couplings.



Here the complete ηN vertex and the complete N self-energy are analysed in terms of low order perturbative vertices. Thus, up to terms linear in the nucleon field $N(x)$, the current $\eta(x)$ is

$$\eta(x) = \lambda \left(1 + \frac{i\boldsymbol{\phi}(x) \cdot \boldsymbol{\tau}}{2F_\pi} \gamma_5 + \cdots \right) N(x) , \tag{2}$$

where $\boldsymbol{\phi}(x)$ is the pion field and F_π is defined as usual by

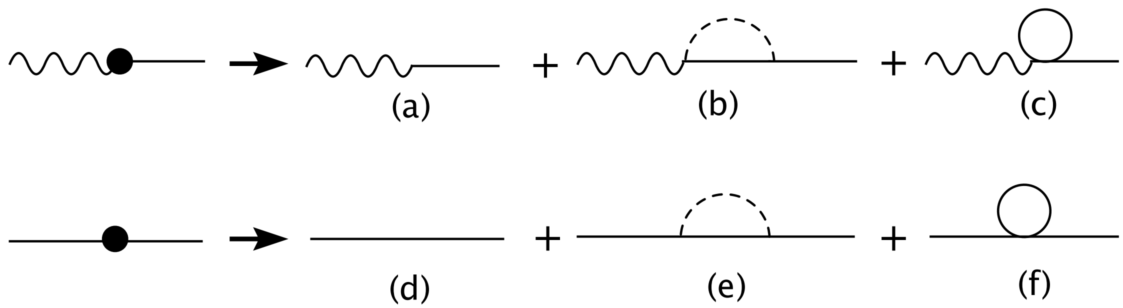
$$\langle 0 | A^i_\mu(x) | \pi^j(k) \rangle = i\delta^{ij} k_\mu F_\pi e^{ik\cdot x} , \quad F_\pi = 93\,\mathrm{MeV} , \tag{3}$$

just as λ is defined by

$$\langle 0 | \eta(x) | N(p) \rangle = \lambda u(p) e^{ip\cdot x}, \tag{4}$$

where $u(p)$ is a positive energy Dirac spinor. The value of λ is obtained from QCD sum rules for nucleon *in vacuum*,

$$\lambda^2 = (1.2 \pm 0.6) \times 10^{-3} \mathrm{GeV}^6 . \tag{5}$$



Now terms in η proportional to $\overline{N}NN$ may also be obtained in the same way, but it brings in two more new coupling constants. Thus Unlike the vertex $\eta\overline{N}\phi$ in graph (b), which is related to the vertex $\eta\overline{N}$ of graph (a) itself, the vertex $\eta\overline{N}N\overline{N}$ in graph (c) is unknown and unlikely to be small.

For the self-energy graphs, we see that the graph (e), which can be calculated with the familiar pion-nucleon Lagrangian, the other graph (f) poses difficulty, even though chiral symmetry dictates the form of the four-nucleon effective Lagrangian,
S. Weinberg, Nucl. Phys. **B363**, 3 (1991)

Indeed, if one does calculate the graph (f) with this four-nucleon interaction, one gets an unacceptably large value for the nucleon self-energy
D. Montano, H.D. Politzer and M.B. Wise, Nucl. Phys. **B375**, 507 (1992).
The problem can be traced to the fact that there are bound and virtual states very close to threshold in the NN system.

In view of these difficulties, we give up calculating the nucleon self-energy and evaluate only the nucleon pole residue, by writing a QCD sum rule for an appropriate amplitude representing the two-point function.

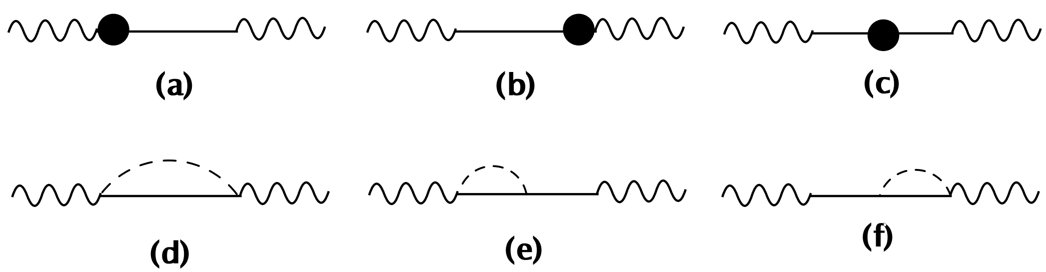
For the nucleon pole term in the medium, we take the nucleon self-energy from the variational and the Brueckner type calculations using the phenomenological NN interaction potentials
R. Brockmann and Machleidt, Phys. Rev. **C 42**, 1965 (1990).
B. tar Haar and R. Malfliet, Phys. Rep. **149**, 207 (1987)

The contributions from the remaining low energy singularities (branch cuts) are obtained by evaluating the relevant Feynman graphs.

QCD sum rules in medium are usually written for complete amplitudes, including their contributions from the vacuum. A sum rule of this type cannot be sensitive to the medium contributions. The reason is that in such sum rules the vacuum contributions dominate over the density dependent ones, at least at low densities, making the latter appear as *non-leading* terms.

In this work we *subtract* out the vacuum sum rule from the corresponding one in medium, that is, we exclude the vacuum contributions from both the spectral and the operator sides.

Spectral side of sum rule



For the low energy part of the spectral side of the sum rule, we take, besides the complete nucleon pole, also the branch cuts from πN exchange.

We shall calculate amplitudes in the *real time version of the field theory in medium*, where a two-point function assumes the form of a 2×2 matrix. But the dynamics is given essentially by a single analytic function, obtained by diagonalising the original matrix. Thus if $\Pi_{11}(E, \vec{p})$ is the 11-component of the original matrix amplitude, the corresponding analytic function, to be denoted by the same symbol as in Eq. (1), has the spectral representation,

$$\Pi(E, \vec{p}) = \int_{-\infty}^{\infty} \frac{dE'}{2\pi} \frac{\sigma(E', \vec{p})}{E' - E - i\eta\epsilon(E')} \tag{6}$$

where the spectral function is related to the imaginary part of Π_{11} by

$$\sigma(E, \vec{p}) = 2\coth\{\beta(E - \mu)/2\}\, \mathrm{Im}\Pi_{11}(E, \vec{p}). \tag{7}$$

For generality we calculate the amplitudes retaining both μ and β and take the limit of zero

The two-point function due to the free propagation of nucleon, namely

$$-\frac{\lambda^2}{\not{p} - m + i\epsilon}$$

is modified by the vertex and the self-energy corrections of Figs. (a), (b) and (c) to

$$\Pi(E, \vec{p})|_{(a+b+c)} = -\frac{\lambda^{*2}}{\not{p} - m - \Sigma(p)} \tag{8}$$

where λ^* is the modified coupling and $\Sigma(p)$ is the nucleon self-energy in nuclear matter.

Here we restrict to $\vec{p} = 0$, when $\Sigma = \Sigma_S + \gamma^0 \Sigma_V$. Then Eq. (8) becomes

$$\Pi(E)|_{(a+b+c)} = -\lambda^{*2} \frac{\gamma^0 (E - \Sigma_V) + m^*}{(E - m_1)(E - m_2)} \, . \tag{9}$$

The scalar part Σ_S of the self-energy changes the mass m of the free particle to the effective mass m^* in the medium, $m^* = m + \Sigma_S$. The vector part Σ_V shifts the rest energies, $\pm m^*$ of the nucleon and the antinucleon, to $m_1 = m^* + \Sigma_V$ and $m_2 = -m^* + \Sigma_V$ respectively of the corresponding quasi-particles. Following our discussion earlier, we work with the *subtracted* nucleon pole term (the bar over a quantity indicates this subtraction),

$$\overline{\Pi}(E)|_{(a+b+c)} = \Pi(E)|_{(a+b+c)} + \lambda^2 \frac{\gamma^0 E + m}{E^2 - m^2} \tag{10}$$

The contribution of the graph (d) is given by

$$\frac{\mathrm{Im}\,\Pi(E)_{11}|_{(d)}}{\tanh[\beta(E-\mu)/2]} = -\left(\frac{3\lambda^2\pi}{4F_\pi^2}\right)\int \frac{d^3q}{(2\pi)^3 4\omega_1\omega_2} \times$$
$$[(-\gamma^0\omega_1 + m)\{(1-n_+ + n)\delta(E-\omega_1-\omega_2)$$
$$+(n_+ + n)\delta(E-\omega_1+\omega_2)\} - (\omega_{1,2} \to -\omega_{1,2},\, n_+ \to n_-)] \tag{11}$$

Here $\omega_1 = \sqrt{m^2 + \vec{q}^{\,2}}$, $\omega_2 = \sqrt{m_\pi^2 + \vec{q}^{\,2}}$ and $n_\pm$ and n are respectively the distribution functions for nucleons, antinucleons and pions,

$$n_\pm(\omega_1) = \frac{1}{e^{\beta(\omega_1 \mp \mu)} + 1}\,, \qquad n(\omega_2) = \frac{1}{e^{\beta\omega_2} - 1} \tag{12}$$

Restricting to zero temperature, $n_+ \to \theta(\mu - \omega_1)$, we get the spectral function as

$$\overline{\sigma}(E)|_{(d)} = \frac{3\lambda^2}{16\pi F_\pi^2 E}\sqrt{\omega^2 - m^2}(\gamma^0\omega - m) \tag{13}$$

on the Landau cut $(0 \le E \le m - m_\pi)$ and the negative of the same quantity on the unitary cut $(m + m_\pi \le E \le \infty)$. Here $\omega = (E^2 + m^2 - m_\pi^2)/(2E)$. Because of the θ-function in the integrand, the two cuts shrink respectively to

$$\mu - \sqrt{\mu^2 - m^2 + m_\pi^2} \le E \le m - m_\pi \tag{14}$$

and

$$m + m_\pi \le E \le \mu + \sqrt{\mu^2 - m^2 + m_\pi^2}. \tag{15}$$

We now adopt a choice of the amplitude suggested in
X. Jin, et al, Phys. Rev. **C49**, 464 (1994)
to split $\overline{\Pi}$ into even and odd parts, $\overline{\Pi}(E) = \overline{\Pi}^{(e)}(E^2) + E\overline{\Pi}^{(o)}(E^2)$,
and deal with the combination, $\widetilde{\Pi}(E^2) = \overline{\Pi}^{(e)}(E^2) - m_2\overline{\Pi}^{(o)}(E^2)$. It removes the quasi anti-nucleon pole and becomes proportional to $\frac{1}{2}(1+\gamma^0)$

We thus get the amplitudes of the different graphs of as

$$\widetilde{\Pi}(E^2)|_{(a+b+c)} = -\frac{2\lambda^{*2}m^*}{E^2 - m_1^2} + \frac{\lambda^2(m - m_2)}{E^2 - m^2} \tag{16}$$

$$\widetilde{\Pi}(E^2)|_{(d)} = \frac{3\lambda^2}{32\pi^2 F_\pi^2}\int_C \frac{dE'}{E'}\frac{f(E')}{E'^2 - E^2} \tag{17}$$

$$\widetilde{\Pi}(E^2)|_{(e+f)} = -\frac{3\lambda^2 g_A}{16\pi^2 F_\pi^2}\int_C \frac{dE'}{E'}\frac{E' + m}{E' - m}\frac{f(E')}{E'^2 - E^2} \ . \tag{18}$$

with $f(E) = (E - m_2)\sqrt{\omega^2 - m^2}(\omega - m)$ and the subscript C on integrals denotes difference of contributions over the unitary and the Landau cuts.

Operator side of sum rule

We now need the explicit form of the nucleon current $\eta(x)_{D,i}$ with spin and isospin indices D annd i in terms of the quark fields, which for proton $(i=1)$ is

$$\eta(x)_{D,1} = \epsilon^{abc}(u^{aT}(x)C\gamma^{\mu}u^{b}(x))(\gamma_5\gamma_{\mu}d^{c}(x))_D\,,$$

where C is the charge conjugation matrix and a, b, c are the colour indices.

The unit operator does not contribute to our sum rule. The other operators form two sets: the old set, appearing already to the vacuum sum sum rule and the new set, involving u^{μ}, the four-velocity of the medium.

We denote by q any of the u and d quark flavours. Then the contributing operators of lowest dimension are $\overline{q}q$, $\overline{q}\not{u}q$ (=$q^{\dagger}q$ in the rest frame of matter, where $u^{\mu}=(1,\vec{0})$). Next, there are the operators of dimension four, namely $G^2=(\alpha_s/\pi)G^{a}_{\mu\nu}G^{\mu\nu a}$ and $\Theta^{f,g}\equiv u^{\mu}u^{\nu}\Theta^{f,g}_{\mu\nu}$, where $\Theta^{f,g}_{\mu\nu}$ are the usual (traceless) energy-momentum tensors of quarks and gluons respectively. Of the remaining operators we retain only the four-quark operators.

In terms of the above operators, the operator expansion gives

$$\Pi(E,\vec{0}) \quad \xrightarrow{OPE} \frac{1}{4\pi^2}(\langle\bar{u}u\rangle + 4\langle u^\dagger u\rangle\gamma^0)E^2\ln(-E^2/\mu^2)$$
$$-\frac{1}{6\pi^2}\left(\frac{3}{16}\langle G^2\rangle + 5\langle\Theta^f\rangle\right)\gamma^0 E\ln(-E^2/\mu^2)$$
$$-\frac{2E}{3E^2}(\gamma^0\langle\bar{u}u\rangle^2 + 2\langle\bar{u}u\rangle\langle u^\dagger u\rangle) \qquad (19)$$

with coefficients to zeroth order in α_s. The renormalization scale μ is taken at 1 GeV.

We now estimate the operator matrix elements. Ignore renormalisation group improvrments. Factorisation of four-quark matrix element with correction

$$\langle\bar{u}u\rangle^2 \to (1-f)\langle 0|\bar{u}u|0\rangle^2 + f\langle\bar{u}u\rangle^2 \qquad (20)$$

where f is a real parameter in the range $0 \leq f \leq 1$.

The nucleon number density $\overline{n}$ is related to the Fermi momentum p_F by $\overline{n} = 2p_F^3/(3\pi^2)$. In normal nuclear matter, it is given by $\overline{n}_0 = (110\,\mathrm{MeV})^3$ corresponding to $p_F = 270\,\mathrm{MeV}$. To first order in $\overline{n}$, the change in the expectation value of an operator O in nuclear matter relative to that in vacuum is given by its nucleon matrix element as

$$\langle O\rangle = \langle 0|O|0\rangle + \frac{\langle p|O|p\rangle}{2m}\overline{n} \qquad (21)$$

We now apply this equation to the different operators.

For $\bar{u}u$ and $u^\dagger u$, we get

$$\langle \bar{u}u \rangle = \langle 0|\bar{u}u|0\rangle + \frac{\sigma}{2\hat{m}}\overline{n}, \qquad \langle u^\dagger u\rangle = \frac{3}{2}\overline{n} \tag{22}$$

where σ is the so-called nucleon σ-term,

$$\sigma = \hat{m}\langle p|\bar{u}u|p\rangle/m = 45 \pm 8\,\mathrm{MeV}\,. \tag{23}$$

The quark mass and the vacuum condensate are related by the Gell-Mann, Oakes and Renner formula,

$$F_\pi^2 m_\pi^2 = -2\hat{m}\langle 0|\bar{u}u|0\rangle\,. \tag{24}$$

Two determinations of these quantities exist in the literature,

$$\hat{m} = 7.2\,\mathrm{MeV}\,, \quad \langle 0|\bar{u}u|0\rangle = -(225\,\mathrm{MeV})^3 \tag{25}$$

$$\hat{m} = 5.5\,\mathrm{MeV}\,, \quad \langle 0|\bar{u}u|0\rangle = -(245\,\mathrm{MeV})^3 \tag{26}$$

Wwe may write the condensate in nuclear matter as

$$\langle \bar{u}u\rangle = \langle 0|\bar{u}u|0\rangle \left(1 - \frac{\sigma\,\overline{n}}{m_\pi^2 F_\pi^2}\right)\,, \tag{27}$$

which vanishes at $\overline{n} = 2.8\overline{n}_0$.

Next, for Θ^f we can write the nucleon matrix element as

$$\langle p|\Theta^f_{\mu\nu}|p\rangle = 2A^f(p_\mu p_\nu - g_{\mu\nu}m^2/4) \quad A^f = 0.62, \tag{28}$$

Then noting the normalization condition, $\langle 0|\Theta^f_{\mu\nu}|0\rangle = 0$, we get,

$$\langle \Theta^f \rangle = \frac{3}{4} m A^f \overline{n} \tag{29}$$

Finally for the operator G^2, we use the trace anomaly to relate it to the trace of the full energy momentum tensor $\Theta_{\mu\nu}$,

$$\Theta^\mu_\mu = -\frac{9}{8} G^2 + 2\hat{m}\bar{u}u + c \cdot 1 \tag{30}$$

where we add the c-number term to fix again its vacuum normalization, $\langle 0|\Theta_{\mu\nu}|0\rangle = 0$. Taking the vacuum and the ensemble expectation values, we get

$$\langle G^2 \rangle = \langle 0|G^2|0\rangle - \frac{8}{9}(m - \sigma)\overline{n} \tag{31}$$

With the above results, we can subtract out the vacuum contributions from Eq.(29) for $\Pi(E,0)$ and write the result for the amplitude combination (22) as

$$\widetilde{\Pi}(Q^2) \xrightarrow{OPE} \left[-\frac{A}{8\pi^2}Q^2 \ln\left(\frac{Q^2}{\mu^2}\right) + \frac{Bm_2}{8\pi^2}\ln\left(\frac{Q^2}{\mu^2}\right) - \frac{2Cm_2}{3Q^2}\right]\overline{n} \tag{32}$$

where A, B and C stand for the constants,

$$\begin{aligned} A &= \frac{\sigma}{\hat{m}} + 12 \\ B &= 5mA^f - \frac{2}{9}(m-\sigma) \\ C &= \langle 0|\bar{u}u|0\rangle \left(f\frac{\sigma}{\hat{m}} + 3\right) \end{aligned} \tag{33}$$

Sum rule

It is now simple to take the Borel transform of the spectral and the operator sides and get the desired sum rule

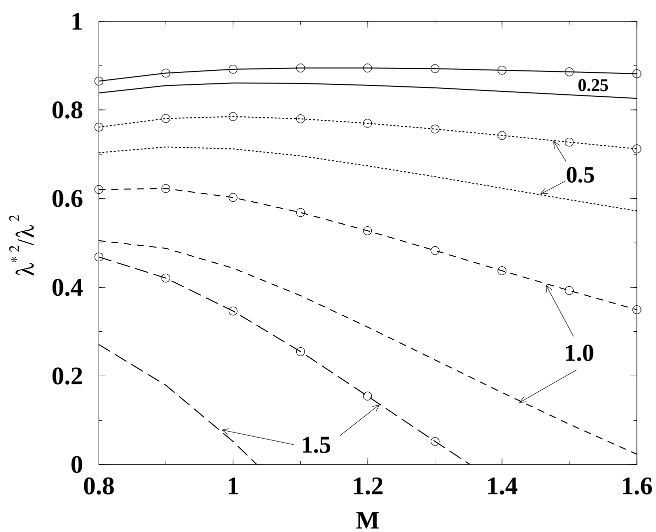
$$
\begin{aligned}
\lambda^{*2} \quad &= \lambda^2 e^{m_1^2/M^2} \left[\frac{m - m_2}{2m^*} e^{-m^2/M^2} - \frac{3}{64\pi^2 F_\pi^2 m^*} \times \right. \\
& \int_C \frac{dE}{E} f(E) \left\{ 1 - 2g_A \left(\frac{E+m}{E-m} \right) \right\} e^{-E^2/M^2} \\
& \left. - \frac{M^2}{2\lambda^2 m^*} \left(\frac{M^2}{8\pi^2} A V_2 + \frac{m_2}{8\pi^2} B V_1 + \frac{2 C m_2}{3M^2} \right) \overline{n} \right]
\end{aligned} \tag{34}
$$

where $f(E)$ is given earlier and $V_1 = 1 - e^{-W^2/M^2}, \quad V_2 = 1 - (1 + W^2/M^2) e^{-W^2/M^2}$. The deviation of $V_{1,2}$ from unity represents the contribution from the high energy region on the spectral side, obtained by continuing the result for operator expansion to the time-like region. Here W is a parameter determining the onset of this continuum contribution. We take $W = 2\,\text{GeV}$, as assumed for the vacuum sum rules.

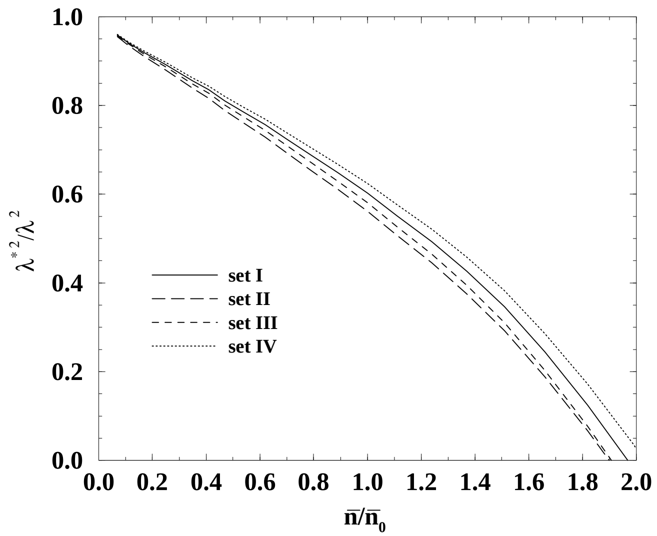
Distant singularities on the spectral side ?
Higher dimension operators on the operator side ?

Evaluation

Among all the parameters, it is the λ, which enters most sensitively in the sum rule and also suffers from the largest uncertainty in its value ($\lambda^2 = (1.2 \pm 0.6) \times 10^{-3}\mathrm{GeV}^6$) Thus we fix λ by requiring maximal stability of the results against variation with respect to the Borel mass. Numerical evaluation shows that bigger the value of λ, the more stable is the result. Thus taking $\lambda^2 = .0018\,\mathrm{GeV}^6$, the largest in the allowed range, we see that there is a reasonable plateau up to about normal nuclear density.



Finally we vary the values for the sigma term and for the pair, the quark mass and the quark condensate. As seen from Fig. 5, the uncertainty in these parameters again does not give rise to any significant spread in the values of λ^*. Also the term with the parameter f arising from the approximation to the four-quark condensate is relatively too small to change the results appreciably.



We thus show unambiguously a decreasing trend for λ^* with the rise of density at least up to normal nuclear density.

Different results

We now bring together some similar, known results for current-particle couplings and the quark condensate. Consider first the *pionic medium at low temperature*. The coupling parameter F_π in vacuum changes to F_π^T,

$$F_\pi^T = F_\pi \left(1 - \frac{T^2}{12F_\pi^2}\right) . \tag{35}$$

The coupling of the baryonic current with nucleon, that we are considering here, also changes from the vacuum value λ to λ^T ,

$$\lambda^T = \lambda \left\{1 - \frac{(g_A^2 + 1)}{32}\frac{T^2}{F_\pi^2}\right\} , \tag{36}$$

For the quark condensate, we have

$$\langle \bar{u}u \rangle^T = \langle 0|\bar{q}q|0\rangle \left(1 - \frac{T^2}{8F_\pi^2}\right) , \tag{37}$$

where we keep only the leading term, though it has been calculated up to $O(T^6)$.

Considering *nuclear matter (at zero temperature)*, the Lorentz invariance already breaks at leading order for the axial-vector current coupling to pion,

$$k_\mu F_\pi \to k_0 F_\pi^t \delta_{\mu 0} + k_i F_\pi^s \delta \mu i \tag{38}$$

giving rise to two decay parameters. They change with nuclear density as

$$F_\pi^t = F_\pi \left\{ 1 - (0.26 \pm 0.04) \frac{\overline{n}}{\overline{n}_0} \right\} \tag{39}$$

$$F_\pi^s = F_\pi \left(1 - (1.23 \pm 0.07) \frac{\overline{n}}{\overline{n}_0} \right\} \tag{40}$$

Observe that the two parameters have quite different density dependence. But one can argue [**?**] that it is the temporal component that reflects the spontaneous breaking of chiral symmetry. To these pion decay parameters, we add the result of the present work,

$$\lambda^* = \lambda \left\{ 1 - (0.20 \pm 0.04) \frac{\overline{n}}{\overline{n}_0} \right\} , \tag{41}$$

obtained as a linear fit to the curves of the last Fig. up to normal nuclear density. Also the quark condensate in this medium was stated earlier,

$$\langle \bar{u} u \rangle = \langle 0 | \bar{u} u | 0 \rangle \left(1 - \frac{\sigma \, \overline{n}}{m_\pi^2 F_\pi^2} \right) , \tag{42}$$

Conclusion

All these results for the current particle couplings and quark condensate are *low-density expansions* in pionic and nuclear matter. We know that any such density expansion of the parameters, even if carried to arbitrarily high order, would not be valid close to phase transition.

It is, however, the case that often the above first order formulae give critical values in qualitative agreement with lattice and exact model calculations. The best example is $\langle\bar{q}q\rangle^T$, for which the value given by the linear formula is of the same order as the one from the lattice.

Thus at finite temperature, the same value approximately for the coefficients of T^2 in F_π^T, λ^T and $\langle\bar{q}q\rangle^T$ tends to support the expectation that they all go to zero at the same (critical) temperature. In the same way, at finite nucleon chemical potential, we find that the coefficients of $\overline{n}$ are again approximately the same for all the three quantities, allowing us to expect that they all disappear together at the same critical density, which is several times away from the normal nuclear density.

Our calculation at finite nuclear density together with this speculation has an added importance in that quantitative calculation on the lattice proves difficult at finite chemical potential, as summarised in a recent review,
R.S. Bhalerao and R. Gavai, arXiv: 0812.1619v1 [hep-ph]