Electrodynamics II: Lecture 9

Multipole radiation

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Outline

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Potential $\vec{\mathbf{A}}_\omega^{\rm rad}$ for monochromatic sources

- We are interested in calculating the radiative components of EM fields and related quantities (like radiated power) for a charge / current distribution that is oscillating with a frequency ω. The results for a general time dependence can be obtained by integrating over all frequencies (inverse Fourier transform), of course.
- We have already seen that it is enough to know about the current distribution (we are interested only in radiative parts), since the charge distribution is related to it by continuity. In such a case, we know that

$$\vec{\mathbf{A}}_\omega^{\rm rad}(\vec{\mathbf{x}}) = \frac{\mu_0}{4\pi}\int \vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')\frac{e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}d^3x' \tag{1}$$

- Given $\vec{\mathbf{A}}_\omega$, the rest of the quantities can be easily calculated in terms of it. We shall omit the “rad” label in this lecture, it is assumed to be everywhere except when specified.

$\vec{\mathbf{B}}^{\mathrm{rad}}_{\omega}$ and $\vec{\mathbf{E}}^{\mathrm{rad}}_{\omega}$ for monochromatic sources

- The radiative part of the magnetic field is then

$$\vec{\mathbf{B}}_{\omega} = \nabla \times \vec{\mathbf{A}}_{\omega} = ik\hat{\mathbf{r}} \times \vec{\mathbf{A}}_{\omega} \tag{2}$$

Note that here $\vec{\mathbf{r}} = \vec{\mathbf{x}}$, to be consistent with standard convention.

- The radiative part of the electric field can be obtained in this monochromatic case by using $\nabla \times \vec{\mathbf{B}}_{\omega} = \mu_0\epsilon_0(-i\omega)\vec{\mathbf{E}}_{\omega}$ (note that there is no current at large r):

$$\vec{\mathbf{E}}_{\omega} = \frac{ic^2}{\omega}\nabla \times \vec{\mathbf{B}}_{\omega} = c\vec{\mathbf{B}}_{\omega} \times \hat{\mathbf{r}} \tag{3}$$

- Thus, $\vec{\mathbf{E}}_{\omega}$ and $\vec{\mathbf{B}}_{\omega}$ fields are orthogonal to $\hat{\mathbf{r}}$, orthogonal to each other, and their magnitudes differ simply by a factor of c.

Long-distance approximation

- Since the sources are confined to a finite region, there will be some distance d such that $|\vec{\mathbf{x}}'| < d$. We shall work in the approximation $d \ll (1/k) \ll r$, where $r = |x'|$. In this approximation, we will be able to expand the radiation fields in a suitable form.
- Since $|\vec{\mathbf{x}}'| \ll |\vec{\mathbf{x}}|$, one can approximate

$$|\vec{\mathbf{x}} - \vec{\mathbf{x}}'| = r - \hat{\mathbf{r}} \cdot \vec{\mathbf{x}}' \tag{4}$$

- This allows us to expand

$$\frac{1}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} = \frac{1}{r - \hat{\mathbf{r}} \cdot \vec{\mathbf{x}}'} = \frac{1}{r} \sum_{\ell} \left(\frac{x'}{r} \right)^{\ell} P_{\ell}(\cos\theta') \tag{5}$$

  where θ' is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{x}}'$.
- Keeping only the leading term, the vector potential becomes

$$\vec{\mathbf{A}}_{\omega} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{\mathbf{J}}_{\omega}(\vec{\mathbf{x}}') e^{-i\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'} d^3x' \tag{6}$$

Long-distance approximation continued

- Since $|\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'| \ll 1$, we can expand the $e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}$ term:

$$\vec{\mathbf{A}}_\omega = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum\frac{(-ik)^n}{n!}\int\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')^n d^3x' \tag{7}$$

  Note that $\vec{\mathbf{k}} = k\hat{\mathbf{r}}$.

- This is the "multipole expansion". Note that the subleading terms in $1/|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|$ are not included here, which is fine as long as $d/r << kd$, i.e. the expansion parameter in $1/|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|$ is much smaller than the expansion parameter in $e^{ik|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}$. At sufficiently large distances, this will always be true. However for practical situations, this needs to be checked.

- There is a general expression, valid even for intermediate distances, which we'll give on the next slide. For the purposes of this lecture, the approximation given above will suffice.

Radiation potential at intermediate distances

- An expansion for $e^{ik|\vec{\mathbf{r}}-\vec{\mathbf{x}}'|}/|\vec{\mathbf{r}}-\vec{\mathbf{x}}'|$ exists in terms of legendre polynomials, spherical Bessel functions and Hankel functions, which we give here without proof:

$$\frac{e^{ik|\vec{\mathbf{r}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{x}}'|} = ik\sum(2n+1)P_n(\cos\theta')j_n(k|\vec{\mathbf{x}}'|)h_n(kr) \tag{8}$$

- At $k|\vec{\mathbf{x}}'| << 1$, we have

$$j_n(k|\vec{\mathbf{x}}'|) = \frac{2^n n!}{(2n+1)!}(k|\vec{\mathbf{x}}'|)^n \tag{9}$$

- For $kr >> 1$, we have

$$h_n(kr) = (-i)^{n+1}\frac{e^{ikr}}{kr} \tag{10}$$

- Using these two, the long-distance approximation gives

$$\frac{e^{ik|\vec{\mathbf{r}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{x}}'|} = (-ik)^n\frac{e^{ikr}}{r}\sum\frac{2^n n!}{(2n)!}|\vec{\mathbf{x}}'|^n P_n(\cos\theta') \tag{11}$$

which should match our expansion (not checked explicitly yet).

Outline

The $n = 0$ term in the multipole expansion

- The leading ($n = 0$) term in the multipole expansion is

$$\vec{\mathbf{A}}_{\omega}^{(0)} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{\mathbf{J}}_{\omega}(\vec{\mathbf{x}}')d^3x' \tag{12}$$

- The integral may be written in a more familiar form by integrating $\int (J(\vec{\mathbf{x}}') \cdot 1)d^3x'$ by parts, and then using the continuity equation $\nabla' \cdot J(\vec{\mathbf{x}}') = -\partial\rho(\vec{\mathbf{x}}')/\partial t = -i\omega\rho(\vec{\mathbf{x}}')$:

$$\int \vec{\mathbf{J}}_{\omega}(\vec{\mathbf{x}}')d^3x' = -\int \nabla' \cdot \vec{\mathbf{J}}(\vec{\mathbf{x}}')\vec{\mathbf{x}}'d^3x \tag{13}$$

$$= -i\omega \int \vec{\mathbf{x}}'\rho(\vec{\mathbf{x}}')d^3\vec{\mathbf{x}}' = -i\omega\vec{\mathbf{p}} \tag{14}$$

where $\vec{\mathbf{p}}$ is the electric dipole moment.

- The $n = 0$ term thus represents the electric dipole radiation:

$$A_{\omega}^{ED} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\vec{\mathbf{p}} \tag{15}$$

Electric dipole radiation: $\vec{\mathbf{E}}_\omega$, $\vec{\mathbf{B}}_\omega$ and radiated power

- The magnetic and electric fields can immediately be written as

$$\vec{\mathbf{B}}^{ED}_\omega = ik\hat{\mathbf{r}} \times \vec{\mathbf{A}}^{ED}_\omega = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(ck^2)\hat{\mathbf{r}} \times \vec{\mathbf{p}} \quad (16)$$

$$\vec{\mathbf{E}}^{ED}_\omega = c\vec{\mathbf{B}}^{ED}_\omega \times \hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(c^2k^2)(\hat{\mathbf{r}} \times \vec{\mathbf{p}}) \times \hat{\mathbf{r}} \quad (17)$$

- The Poynting vector $\vec{\mathbf{N}}_\omega = \vec{\mathbf{E}}^*_\omega \times \vec{\mathbf{H}}^\omega$ is normal to both, $(\hat{\mathbf{r}} \times \vec{\mathbf{p}})$ and $[(\hat{\mathbf{r}} \times p) \times \hat{\mathbf{r}}]$, i.e. along $\hat{\mathbf{r}}$, as expected. The average rate of energy radiated is

$$\langle\vec{\mathbf{N}}\rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}k^4c^3|\hat{\mathbf{r}} \times \vec{\mathbf{p}}|^2\hat{\mathbf{r}} \quad (18)$$

$$= \frac{\mu_0}{32\pi^2 r^2}k^4c^3|\vec{\mathbf{p}}|^2 \sin^2\theta\, \hat{\mathbf{r}} \quad (19)$$

- The average power radiated per solid angle is then

$$\frac{dP}{d\Omega} = \langle\vec{\mathbf{N}}\rangle \cdot r^2\hat{\mathbf{r}} = \frac{\mu_0}{32\pi^2}k^4c^3|\vec{\mathbf{p}}|^2 \sin^2\theta \quad (20)$$

Electric dipole radiation: salient features

- The radiated power is proportional to the fourth power of frequency. This results in the blue colour of the sky: the sunlight induces dipoles in the air molecules, which then radiate, giving out more light at high frequencies, i.e. near the blue end of the spectrum.
- The angular dependance is $\sin^2\theta$, i.e. there is no radiation in the direction of the dipole, most of the radiation is in the equatorial plane.
- At large wavelengths ($\lambda > L$), antennas (discussed in the last class) also emit dipole radiation.

Outline

$n = 1$ term in the multipole expansion

- The $n = 1$ term in the expansion is

$$A_\omega^{(1)} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int \vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')d^3x' \tag{21}$$

- Using $(\vec{\mathbf{x}}' \times \vec{\mathbf{J}}) \times \hat{\mathbf{r}} = (\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\vec{\mathbf{J}} - (\hat{\mathbf{r}}\cdot\vec{\mathbf{J}})\vec{\mathbf{x}}'$, the integral may be separated into two parts:

$$\int \vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')d^3x' = I_{MD} + I_{EQ} \tag{22}$$

where

$$I_{MD} = \int \frac{1}{2}[\vec{\mathbf{x}}' \times J(\vec{\mathbf{x}}')] \times \hat{\mathbf{r}}\ d^3x' \tag{23}$$

$$I_{EQ} = \int \frac{1}{2}[(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\vec{\mathbf{J}}(\vec{\mathbf{x}}') + (\hat{\mathbf{r}}\cdot\vec{\mathbf{J}}(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x' \tag{24}$$

- These two terms correspond to the magnetic dipole and the electric quadrupole components, respectively, as we shall see.

Magnetic dipole radiation

- Since the magnetic dipole moment is defined as

$$\vec{\mathbf{m}} = \int \frac{1}{2}[\vec{\mathbf{x}}' \times \vec{\mathbf{J}}(\vec{\mathbf{x}}')]d^3x' \tag{25}$$

the component of $\vec{\mathbf{A}}_\omega$ corresponding to I_{MD} becomes

$$\vec{\mathbf{A}}_\omega^{MD} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)(\vec{\mathbf{m}} \times \hat{\mathbf{r}}) \tag{26}$$

- This immediately leads to

$$\vec{\mathbf{B}}_\omega^{MD} = (ik)\hat{\mathbf{r}} \times \vec{\mathbf{A}}_\omega^{MD} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}k^2\, \hat{\mathbf{r}} \times (\vec{\mathbf{m}} \times \hat{\mathbf{r}}) \tag{27}$$

$$\vec{\mathbf{E}}_\omega^{MD} = c\vec{\mathbf{B}}_\omega^{MD} \times \hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}k^2\, [\hat{\mathbf{r}} \times (\vec{\mathbf{m}} \times \hat{\mathbf{r}})] \times \hat{\mathbf{r}} \tag{28}$$

- And the average power radiated per unit area is

$$\langle\vec{\mathbf{N}}\rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}k^4c^3|\vec{\mathbf{m}}|^2 \sin^2\theta\, \hat{\mathbf{r}} \tag{29}$$

where θ is the angle between $\vec{\mathbf{m}}$ and $\hat{\mathbf{r}}$.

Electric quadrupole radiation

- The remaining component of $A_\omega^{(1)}$ is the electric quadrupole part (as will be clear soon):

$$A_\omega^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int \frac{1}{2}[(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\vec{\mathbf{J}}(\vec{\mathbf{x}}') + (\hat{\mathbf{r}}\cdot\vec{\mathbf{J}}(\vec{\mathbf{x}}'))\vec{\mathbf{x}}']d^3x' \quad (30)$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)}{2}\int \vec{\mathbf{x}}'(\hat{\mathbf{r}}\cdot\vec{\mathbf{x}}')\rho(\vec{\mathbf{x}}')d^3x' \quad (31)$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{(-k^2c)}{2}\frac{1}{3}\vec{\mathbf{Q}}(\hat{\mathbf{r}}) \quad (32)$$

- Here, $\vec{\mathbf{Q}}(\hat{\mathbf{r}})$ is the component of the electric quadrupole moment along $\hat{\mathbf{r}}$, i.e.

$$\vec{\mathbf{Q}}_\alpha = \sum Q_{\alpha\beta} r_\beta \ , \quad (33)$$

with

$$Q_{\alpha\beta} \equiv \int (3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta})\rho(\vec{\mathbf{x}}')d^3x' \ , \quad (34)$$

the electric quadrupole moment.

Electric quadrupole: $\vec{\mathbf{B}}_\omega$, $\vec{\mathbf{E}}_\omega$ and power radiated

- Now we can calculate $\vec{\mathbf{B}}_\omega$, $\vec{\mathbf{E}}_\omega$:

$$B_\omega^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{-ik^3c}{6}\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}}) \tag{35}$$

$$E_\omega^{EQ} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{-ik^3c^2}{6}(\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}})) \times \hat{\mathbf{r}} \tag{36}$$

- The average Poynting vector is

$$\langle\vec{\mathbf{N}}\rangle = \frac{1}{2}\frac{\mu_0}{(4\pi)^2}\frac{1}{r^2}\frac{k^6c^3}{36}\left|\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}})\right|^2 \hat{\mathbf{r}} \tag{37}$$

- The average power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{4\pi}\frac{k^6c^3}{288}\left|\hat{\mathbf{r}} \times \vec{\mathbf{Q}}(\hat{\mathbf{r}})\right|^2 \tag{38}$$

Comment on Electric quadrupole radiation

- If the charge distribution is azimuthally symmetric, and has a reflection symmetry about z axis (spheroidal distribution is a special case of this), then

$$Q_{xy} = Q_{yz} = Q_{xz} = 0 , \quad Q_{xx} = Q_{yy} = Q_0 \quad Q_{zz} = -2Q_0 \tag{39}$$

In such a case, it can be shown that the power radiated is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{4\pi} \frac{k^6 c^3}{32} |Q_0|^2 \sin^2\theta \cos^2\theta \tag{40}$$

where θ is the angle between $\hat{\mathbf{r}}$ and $\vec{\mathbf{Q}}(\hat{\mathbf{r}})$.

- The gravitational radiation has a similar form to the electric quadrupole radiation, except one has to deal with time-dependent mass distribution rather than time-dependent charge distribution.

Recap of topics covered in this lecture

- Calculating $\vec{\mathbf{B}}_\omega$ and $\vec{\mathbf{E}}_\omega$ from $\vec{\mathbf{A}}_\omega$ (for their radiative components)
- Multipole expansion when $|\vec{\mathbf{x}}'| < \lambda < |\vec{\mathbf{x}}|$
- Electric dipole radiation as the leading term in multipole expansion
- Separating magnetic dipole moment and electric quadrupole moment contributions from the subleading term
- $\vec{\mathbf{E}}_\omega$, $\vec{\mathbf{B}}_\omega$, Poynting vector, average rate of radiated power, and the angular distribution of radiated power