Lecture 1

Symmetries

Our modern understanding of particle physics rests on the fundamental concept of a "symmetry" i.e. invariance of the laws of physics under certain transformations. Every conserved quantity (e.g. angular momentum) can be thought of as arising through an underlying symmetry. There are two main categories — space-time symmetries and "internal" symmetries.

For 4D space-time, the symmetries form the so called the Poincaré group which includes Lorentz transformations (3 space rotations + 3 space-time boosts) and translations along each of the 4 axes. This means essentially that the laws of physics do not change if you move from one location to another, or if you choose to move with a constant velocity, or if you orient yourself in a different direction. As we shall see, requiring this invariance automatically implies conservation of momentum. Furthermore, what we call "spin" turns out to be a fundamental property of how a particle transforms under Lorentz transformations.

Aside from conserving energy-momentum etc., we also observe conservation of electric charge. This too also be understood as a result of an underlying *internal* symmetry called a U(1) symmetry. The full internal symmetry of the Standard Model is known to be SU(3)×SU(2)×U(1). The goal of this chapter (lectures 1–5) is to introduce what these symbols mean and how these symmetries are described mathematically to be able to write equations of motion for particles.

1 Lagrangian and equations of motion in classical mechanics

The Lagrangian L is a mathematical function defined (classically) as the difference of kinetic energy and potential energy. It is important because minimising the action, defined as

$$S = \int L dt \tag{1.1}$$

seems to return the equation of motion. In short, for a point particle with co-ordinates (x, y, z) moving under a potential $V(x, y, z)$ we have:

$$\begin{aligned} L &= T - V \\ &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \end{aligned} \tag{1.2}$$

Minimising the action gives the general form of the equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{1.3}$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{z}} &= m\dot{z} \\ \frac{\partial L}{\partial z} &= \frac{\partial V}{\partial z} \end{aligned}$$

Which recovers Newton's second law, i.e.

$$\Rightarrow m\ddot{\mathbf{x}} - \nabla V = 0 \tag{1.4}$$

The Lagrangian is preferred because one can derive the equations of motion irrespective of the co-ordinates used. We could just as well have written the lagrangian in terms of spherical co-ordinates. Furthermore, it is possible to read out conserved quantities easily. For example, assume that $V(x, y, z) = -mgz$ (i.e. it is independent of x,y). Then we automatically have

$$\begin{gathered} m\ddot{x} = 0 \;\;\&\;\; m\ddot{y} = 0 \\ \Rightarrow m\dot{x} = \text{const} \;\;\&\;\; m\dot{y} = \text{const} \end{gathered} \tag{1.5}$$

pointing to the conservation of momenta in x & y directions. In the z-direction, the equation of motion is our standard Newton's second law

$$m\ddot{z} = -mgz$$

Exercise 1.1

1. Write the general lagrangian for a single particle in cyclical co-ordinates in 2D and show that the angular momentum is a conserved quantity if the potential depends only on the radial co-ordinate.

2. Re-write the lagrangian in Cartesian co-ordinates. What does the conserved angular momentum correspond to?

Notice that if written in Cartesian co-ordinates, one has to combine terms in order to arrive at the conserved quantity. A clever thing to do would be to figure out the right co-ordinate system such that all conserved quantities would be clearly readable from the equations of motion (which were indeed studied extensively in the eighteenth century). However, an ingenious theorem was developed by Emmy Nœther to map out all conserved quantities (or "invariants") of a theory which we will study presently.

2 Lagrangian for a scalar field

We now generalise the Lagrangian for use with fields instead of particles. This simply amounts to writing the Lagrangian in terms of functions of space-time, e.g. real scalars $\phi(x)$ instead of position x or canonical co-ordinates q. We start with the classical description. In your quantum field theory course, you will learn how the field can be quantised.

The Lagrangian for a scalar particle in 4D is written as

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 \tag{1.6}$$

Functions of functions are called *functionals*. In day-to-day calculations, you can get away by treating them as ordinary co-ordinates except that "differentiation with respect to the function" is denoted by a δ rather than a d (i.e. like dx is the infinitesimal change in x, $\delta\phi$ is the infinetismal change in ϕ).

If the field ϕ changes by an infinetesimal amount $\delta\phi$, we have (retaining terms up to order $\delta\phi$)

$$\mathcal{L}(\phi + \delta\phi) = \mathcal{L}(\phi) - \partial_\mu\phi\partial^\mu\delta\phi - M^2\phi\delta\phi \tag{1.7}$$

Notice that

$$\frac{\delta \mathcal{L}}{\delta \phi} = -M^2 \phi \text{ and } \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} = \partial^\mu \phi$$

This confirms that the functional follows the same differential rule as ordinary differentiation. We can write (using $\delta \partial_\mu \phi = \partial_\mu \delta \phi$)

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \partial_\mu \phi + \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi \tag{1.8}$$

The equations of motion are obtained by a corresponding formula

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \right) - \frac{\delta \mathcal{L}}{\delta \phi} = 0 \tag{1.9}$$

You can check that this results in the right equation of motion, viz. the Kein-Gordon equation.

$$(\partial_\mu \partial^\mu + M^2)\phi = 0$$

Exercise 1.2

The Lagrangian for a complex scalar field ϕ is given by

$$\mathcal{L}(\phi) = \frac{1}{2} \partial_\mu \phi^* \partial^\mu \phi - \frac{1}{2} M^2 \phi^* \phi \tag{1.10}$$

1. Rewrite ϕ in terms of component fields $\phi_1 + i\phi_2$. What does this Lagrangian correspond to? Write it in terms of column matrix

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

2. Check that the Lagrangian in (1.10) is invariant under the transformation $\phi \to e^{i\alpha}\phi$.

3. What does this transformation correspond to in terms of Φ?

3 Nœther theorem: conserved currents & charges

In the above exercise, we saw an example of what is called a "symmetry" of the theory — a transformation under which the Lagrangian remains unchanged. Consider the expression

$$j^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

it is easy to check that

$$\partial_\mu j^\mu = 0$$

once we use the equations of motion $(-\partial_\mu \partial^\mu + M^2)\phi^{(*)} = 0$. We expect this quantity to be conserved. Notice also its similarity to angular momentum $i(x\partial_y - y\partial_x)$. We will clarify this correspondence in the next few lectures. But how does one arrive at this expression?

Let us start again with (1.8)

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \partial_\mu \phi + \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi$$

Using $\delta \partial_\mu \phi = \partial_\mu \delta \phi$, we can write this as

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\mu \delta \phi + \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi$$

Further using the equation of motion (1.9) and substituting for

$$\frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \right)$$

we have,

$$\begin{aligned} \delta \mathcal{L} &= \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\mu \delta \phi + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \right) \delta \phi \\ &= \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \phi \right) \equiv \partial_\mu j^\mu \end{aligned} \tag{1.11}$$

If the Lagrangian dependes on many fields $\{\phi_i\}$, the equation is modified to a sum over all fields

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_i} \delta \phi_i \right) \equiv \partial_\mu j^\mu \tag{1.12}$$

The equations of motion are unchanged if Lagrangian changes by a total derivative $\partial_\mu \Lambda^\mu$. The most general definition of j^μ is then

$$j^\mu \equiv \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_i} \delta \phi_i \right) - \Lambda^\mu \tag{1.13}$$

Let us now go back to the symmetry of the Lagrangian in (1.10), i.e. $\phi \to e^{i\alpha}\phi$. Under this transformation, we have

$$\delta\phi = e^{i\alpha}\phi - \phi = i\phi$$

$$\delta\phi^* = e^{-i\alpha}\phi - \phi = -i\phi^*$$

Therefore, considering ϕ and ϕ^* as the two degrees of freedom, we arrive at the expression for j^μ

$$j^\mu = i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) \tag{1.14}$$

The form of j^μ is determined entirely by the inherent symmetry of the Lagrangian. In the next lecture we will formalise the notion of symmetry using group theory.

Exercise 1.3

Verify that the expression for j^μ remains the same even when written in terms of real components $(\phi_1 + i\phi_2)$.

1. Assuming α is small, find the transformations of ϕ_1 and ϕ_2
2. Calculate j^μ using equations (1.12) and (1.14).

Take home points

1. Lagrangian formulation is a way to describe physics that makes it easy to determine conserved quantities
2. Equations of motion describe how a particle moves or a field evolves given initial conditions. Minimising the Lagrangian gives equations of motion.
3. The Lagrangian for a single complex scalar of mass M is invariant under the transformation $\phi \to e^{i\alpha}\phi$
4. For any transformation under which the Lagrangian remains invariant (up to total derivative), there is a unique conserved current. Finding all possible transformations under which the Lagrangian is invariant will give all conserved currents.