Lecture 2

Groups & Representations

1 The rotation group: SO(N)

Consider rotations in $x - y$ plane. A rotation by angle θ can be written in the simple form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

where (x', y') are the new co-ordinates obtained from rotating (x, y) anti-clockwise by θ or $(x, y) \to (x', y') = R(\theta)(x, y)$. Here, the matrix R is the "representation" of the rotation in 2D space. The rotations themselves are a more abstract quantity and can be represented in different ways depending on the underlying space. For example, consider instead a close relative of the real 2D space — the complex plane. Then each $z = x + iy$ is a single complex number. The "rotation" in this case is given by $z \to z' = e^{-i\theta} z$. For each $R(\theta)$, there is a corresponding $e^{-i\theta}$. In both cases, rotation by θ_1 followed by a rotation by θ_2 is also a rotation (by $\theta_1 + \theta_2$). Obviously, multiplication by the unit matrix does not change the co-ordinates. And every rotation by θ can be undone by a rotation by $-\theta$. These are all self-evident properties for this simple case, but can be applicable in very different situations that may not even be rotations. Formally, rotations in 2D space or in the complex plane form a "group" (called SO(2) or U(1) group respectively).

Definition: Group

A group G is a collection of elements g with an operation $\cdot$ such that

1. If $g, g' \subset G$, then $g \cdot g' \subset G$
2. $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$
3. There is an "identity" element e such that $g \cdot e = e \cdot g = g$ for all $g \subset G$
4. for every g, there is an inverse g^{-1} such that $g \cdot g^{-1} = e = g^{-1} \cdot g$

Note that elements in the group *do not have to* commute (even though in the above example they did), i.e. in general $g_1 \cdot g_2 \neq g_2 \cdot g_1$. If they do commute, the group is called "Abelian".

1.1 Defining SO(N)

Consider real N-dimensional vector space with directions given by $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_N)$. Then each vector can be represented as a vector in N-space and a rotation can be written as real, $\mathrm{N} \times \mathrm{N}$ matrices.

$$\begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_N \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & ... & R_{1N} \\ R_{21} & R_{22} & ... & R_{2N} \\ \vdots & & \ddots & \\ R_{N1} & R_{N2} & ... & R_{NN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

Firstly, rotations do not change the length of the vector (given by $X^T X$). Thus, if $X \to RX$, then $X^T \to X^T R^T$.

$$X^T X = X'^T X' \Rightarrow R^T R = \mathbb{1} \tag{2}$$

This is called the orthogonality condition and corresponds to the 'O' in SO(N). Secondly, this also implies $R^{-1} = R^T$.

$$\det(R) = \det(R^T) \Rightarrow \det(R) = 1 \tag{3}$$

Which defines the 'S' standing for special in SO(N), meaning the determinant is unity.

Exercise 1.4

Find the number of independent entries in $R \subset \mathrm{SO}(N)$

1.2 Group Representations

The 3x3 matrices of SO(3) above act on the 3D real space co-ordinates. However, one can also ask for the action of the group on other kinds of spaces.

Definition: Group Representation

A representation is a mapping of the action of a group G on a vector space V where each element of g of group is mapped to a $V \times V$ matrix R with the following properties

1. $R(g_1 \cdot g_2) = R(g_1) \cdot R(g_2)$
2. $R(e) = \mathbb{1}$
3. $R^{-1}(g) = R(g^{-1})$

Note that a valid, but trivial representation can be made if all group elements are mapped to $\mathbb{1}$. If each element of G gets a distinct $V \times V$ matrix, the representation is called "faithful." This implies that the mapping is invertible i.e. it is possible to map each matrix to a unique group element.

1.3 Lie groups and algebras

The two examples of groups given above fall into a category of groups called "Lie groups" (after Norwegian mathematician Sophus Lie) and have the property that one element of the group can be got from another by a continuous change in parameters. For instance, in our SO(2) example above, each θ corresponds to a different matrix and one matrix can be obtained from another by continuously changing θ. Because the elements are continuous functions of parameters they are differentiable. The R in equation 1 can be differentiated at $\theta = 0$

$$\begin{aligned} \frac{dR(\theta)}{d\theta}|_{\theta=0} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\tau && (4) \\ \Rightarrow R(\theta) &= \exp\{-i\tau\theta\} && (5) \end{aligned}$$

The matrix τ is called the *generator* and has the property $\tau^\dagger = \tau$ (i.e. it is Hermitian).

1.3.1 SO(3) and its algebra so(3)

All elements of SO(3) can be generated by the following three matrices (corresponding to rotations around z, x, and y axes respectively).

$$R_1 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$
$$R_3 = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \qquad (6)$$

Differentiating with respect to θ give three generators of so(3) given by

$$\tau_1 = i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tau_2 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \tau_3 = i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad (7)$$

Exercise 1.5

Check that

1. the rotation matrices are recovered by exponentiating the so(3) generators.
2. The follow the commutation relation $[\tau_i, \tau_j] = i\epsilon_{ijk}\tau_k$

The generators form an "algebra" and satisfy

$$\begin{aligned} \mathrm{Tr}(\tau) &= 0 && (8) \\ \tau^\dagger &= \tau && (9) \\ [\tau_i, \tau_j] &= i\epsilon_{ijk}\tau_k && (10) \end{aligned}$$

which are generic conditions satisfied by all Lie algebras. The last of these conditions defines the Lie bracket ($[X,Y] = XY - YX$) which is known to us already from Quantum Mechanics as the commutator.

The algebra name is generally written with lower-case letters i.e. SO(3) has the algebra so(3), and so on. As you will notice, each so(3) generator τ_k can also be thought of as $i\epsilon_{ijk}$ where (i,j) are the row and column of the τ_k matrix. This is particularly important to keep in mind as we will see from the next example of su(2).

1.4 Another so(3) algebra representation

A second representation, also in terms of 3x3 matrices can be constructed to describe the "internal" angular momentum (or spin) of a spin-1 particle. We will see further examples of describing different spins when we work with Lorentz transformations.

$$L_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = \frac{i}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{11}$$

This is a representation in the so-called "spherical basis". The basis vectors of the underlying space correspond to the cartesian co-ordinates as $(\frac{x+iy}{\sqrt{2}}, \quad z, \quad \frac{x-iy}{\sqrt{2}})^T$. The three possible spin states of the particle are represented by $(1,0,0)^T$, $(0,1,0)^T$ and $(0,0,1)^T$. This representation is useful in describing the polarisation of a spin-1 particle. For example, note that for longitudinal polarisation along the z-axis (x,y components are zero), the eigenvalue of L_z operator is zero. Whereas for circular polarisation, the eigenvalues are ± 1 based on whether it is polarised anti-clockwise or clockwise.

2 Representations of SU(N)

2.1 The special unitary group: SU(N)

As the SO(N) group describes transformations that do not change the length of the vector in a *real* vector space, the SU(N) describes transformations that do not change the length of a *complex* vector given by $X^\dagger X$. As such it is particularly important in describing transformations of amplitudes in quantum mechanics and fields in quantum field theory which are complex. This requirement of unchanged length corresponds to the requirement that the probability remains unchanged under a change of co-ordinates.

The $N = 1$ group is simply called $U(1)$ and we have already encountered this in our previous example as rotations in complex plane. Its elements can be written as $e^{i\theta}$. The SU(N) group (the special unitary group) is simply the group of $N \times N$ complex matrices with the following requirements

$$U^\dagger U = UU^\dagger = \mathbb{1} \Rightarrow U^{-1} = U^\dagger \tag{12}$$

$$\det U = 1 \tag{13}$$

In general, the first of these equations implies $|\det U| = 1$ which allows $\det U$ to take all values of the form $e^{i\theta}$ (which is the U(1) group). The choice of $\det U = +1$ is "special". In general, for unitary matrices, the group is denoted by $U(N) = SU(N) \times U(1)$.

Exercise 1.6

Show that the number of independent entries in U is $N^2 - 1$. This means one needs $N^2 - 1$ generators to get all possible elements. This is the dimension of the group.

2.2 The Lie Algebra su(N)

Similar to SO(N), SU(N) can also be described by exponentiating generators of its algebra. In general, let us define t^a as the generators of su(N) where a runs from 1 to $N^2 - 1$. The actual form of t^a of course will depend on the vector space on which it acts. Then we can write each element $X \in G$ as

$$X = \exp\{it^a x^a\}$$

where x^a are continuous parameters similar to θ in the U(1) element $e^{i\theta}$. The generators t^a satisfy the following conditions:

- $Tr(t^a) = 0$
- $[t^a, t^b] = f_{abc}t^c$, where f_{abc} are called structure constants and depend on the N in SU(N)
- $[[t^a, t^b], t^c]+$ cyclic permutations $= 0$. (This is called the Jacobi identity)

2.3 SU(2) representations and su(2) algebra

As you may have already encountered in describing spin angular momenta in Quantum Mechanics, generators of su(2) are given by

$$\tau_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \tau_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$

which means that any element of SU(2) can be represented as $\exp\{-i \sum_k \tau_k \theta_k\}$.

2.4 Fundamental and Adjoint representations

When SU(N) is represented in terms of $N \times N$ complex, unitary matrices, acting on N-dimensional vector space, it is called the "fundamental representation". This is sort of a circular definition, but serves to distinguish it from other representations.

A particularly important representation, called the *adjoint* representation, is obtained when the vector space on which the group acts is the one spanned by the generators of the algebra. That means, each element in the vector space can be written as

$$c_1 t^1 + c_2 t^2 + ... + c_{N^2-1} t^{N^2-1}$$

for real numbers c_i. In the case of SU(2), this means that the basis vectors $\{t^a\}$ are the three Pauli matrices.

In the adjoint representation, the mapping is from

$$Ad : G \to \mathbf{R}_{(N^2-1)} \times \mathbf{R}_{(N^2-1)}$$

i.e. to an $(N^2 - 1) \times (N^2 - 1)$ real matrix (the $\mathbf{R}$ here corresponds to the real number line) and is denoted by $Ad(U)$ for each element $U \in G$. The action of the group is defined by

$$Ad(U)X = U^\dagger X U \tag{15}$$