# Gauge symmetries and covariant derivative

## Part I: Covariant derivative for U(1)

So far, we have looked at

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^* \partial^\mu \phi - \frac{m^2}{2} \phi^* \phi$$

which is invariant under $\phi \to e^{i\alpha} \phi$.

A gauge symmetry (or a "local" symmetry) is when $\alpha$ depends on space-time i.e. $\alpha(x)$

$$\partial_\mu \phi \to \partial_\mu \left( e^{i\alpha(x)} \phi \right)$$
$$= e^{i\alpha(x)} \partial_\mu \phi + i \left( \partial_\mu \alpha(x) \right) e^{i\alpha(x)} \phi$$

We need a way to absorb the extra $i\partial_\mu \alpha(x)$ piece. $\partial_\mu \alpha(x)$ is a Lorentz vector.

$\therefore$ Define a co-variant derivative with an extra vector field $A_\mu$

$$D_\mu \phi(x) \equiv \partial_\mu \phi(x) - i A_\mu(x)$$

where $A_\mu \to A_\mu + \partial_\mu \alpha(x)$

Then we have under $\phi \to e^{i\alpha(x)}\phi$

$$D_\mu \phi(x) \to e^{i\alpha(x)} D_\mu \phi(x)$$

i.e. $(D_\mu \phi)$ transforms in the same way as $\phi$ $\Rightarrow$ "Covariant derivative".

$$\begin{aligned}(D_\mu \phi)^* D_\mu \phi = {} & \partial_\mu \phi^* \partial_\mu \phi \\ & - i A_\mu \left(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*\right) \\ & + A_\mu A^\mu \phi^* \phi\end{aligned}$$

We will also need to add an extra kinetic term to describe the behaviour of $A_\mu$.

From classical electrodynamics, we know that the electro-magnetic field can be described by Maxwell's equations.

Consider the field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

If we start out with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

You can show that (exercise!)

$$\partial_\mu \left[ \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} \right] - \frac{\delta \mathcal{L}}{\delta A_\mu} = 0$$

gives the right equations of motion.

We therefore can write the full Lagrangian as

$$\mathcal{L} = \frac{1}{2} (D_\mu \phi)^* (D^\mu \phi) - \frac{m^2}{2} \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

~~Case~~ Also note that in this simple case $D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

This is ~~not~~ only true for the $U(1)$ case (i.e. when $\phi \to U\phi$ with $U = e^{i\alpha(x)}$)

We can use the same trick to describe local symmetries of fermions.

Recall that fermion lagrangian is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi$$

(Exercise: check this gives the Dirac equation as equation of motion)

If $\psi \to e^{i\alpha(x)}\psi$

and $A_{\mu} \to A_{\mu} + i\partial_{\mu}\alpha(x)$

Then $D_{\mu}\psi \to e^{i\alpha(x)} D_{\mu}\psi$

where $D_{\mu} \equiv \partial_{\mu} - iA_{\mu}$

We can define the "charge" of the fermion as the multiplier $Q$ in $e^{iQ\alpha(x)}$

i.e. $\psi \to e^{iQ\alpha(x)}$ will need to have $D_{\mu} = \partial_{\mu} - iQA_{\mu}$ to correctly cancel the extra term. This gives

$$\bar{\psi}\gamma^{\mu} D_{\mu}\psi = \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - iQ\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

Two fermion with different Q-charges will couple differently to the same field $A_\mu$.

The full gauge-invariant Lagrangian is then

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where $D_\mu \equiv \partial_\mu - i\,Q_\psi A_\mu$

charge of fermion $\psi$.

When the transformation is $e^{i\alpha(x)}$, we have

$$e^{i\alpha_1(x)}\,e^{i\alpha_2(x)} = e^{i\alpha_2(x)}\,e^{i\alpha_1(x)}$$

i.e. $\psi \xrightarrow{e^{i\alpha_1}} \psi_1 \xrightarrow{e^{i\alpha_2}} \psi_2$ is the same as

$\psi \xrightarrow{e^{i\alpha_2}} \psi_1' \xrightarrow{e^{i\alpha_1}} \psi_2'$

Since the transformations commute, we call $U(1)$ an Abelian group.

Other members of $SU(N)$ are non-abelian

$N>1$.

# Part II: Covariant derivative for SU(N)

Let us start with the fermionic

Lagrangian

$$\mathcal{L} = \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi$$

$\mathcal{L}_\psi$ has an internal SU(N) symmetry

under $\psi \to U\psi$ where $U = e^{i\sum_a t^a \alpha^a(x)}$

$t^a$: SU(N) generators

$$\bar{\psi} \to \bar{\psi}\, U^+$$

( Note that $t^a$s and $\gamma^\mu$'s commute

because the operate on different indices

$$\psi_{\alpha i}$$

$i$ — SU(N) index : 1,.., N

$\alpha$ — spinor index : 0−3

$$\bar{\psi}\,\gamma^\mu \partial_\mu \psi \;\to\; \bar{\psi}\, U^+ \partial_\mu \left[ U\psi \right]$$

$$\bar\psi\, U^\dagger \gamma^\mu \partial_\mu (U\psi) = \bar\psi\gamma^\mu\partial_\mu\psi + \bar\psi\gamma^\mu(U^\dagger\partial_\mu U)\psi$$

$$U = \exp\{i\, t^a \alpha^a(x)\}$$ (sum over repeated indices)

$$\partial_\mu U = \exp\{i\, t^a\alpha^a(x)\} \times i\, t^b \partial_\mu \alpha^b(x)$$

$$= i U\, t^a \partial_\mu \alpha^a(x).$$

$$\bar\psi\, U^\dagger\gamma^\mu\partial_\mu(U\psi) = \bar\psi\gamma^\mu\partial_\mu\psi + i\, \underbrace{\bar\psi\gamma^\mu t^a\psi}_{\text{fermionic current}}\, \partial_\mu\alpha^a$$

Coupling to gauge field.

$$\sim g\,(\bar\psi\gamma^\mu t^a\psi)\, A^a_\mu$$

Let us define covariant derivative

$$D_\mu = \mathbb{1}\partial_\mu - ig\,(t^a) A^a_\mu$$ ← matrix $N \times N$

or $$D_\mu = \mathbb{1}\partial_\mu - ig\,\bar{A}_\mu$$ → matrix $(t^a A^a_\mu)$

Requirement for a co-variant derivative
[to leave $\bar{\Psi}\gamma^\mu D_\mu \psi$ invariant]

if $\psi \to U\psi$ $\quad D_\mu \psi \to U(D_\mu \psi)$

i.e. $\partial_\mu (U\psi) - ig(\bar{A}_\mu)\, U\psi$

$$= U(\partial_\mu \psi - ig\, \bar{A}'_\mu \psi)$$

LHS $= U\partial_\mu \psi + (\partial_\mu U)\psi - ig\bar{A}_\mu U\psi$

RHS $= U\partial_\mu \psi - igU\bar{A}'_\mu \psi$

$$\Rightarrow -igU\bar{A}'_\mu = \partial_\mu U - ig\bar{A}_\mu U$$

$$\Rightarrow -ig\,\bar{A}'_\mu = U^+\partial_\mu U - igU^+\bar{A}_\mu U$$

$$\bar{A}'_\mu = U^+\bar{A}_\mu U - \frac{1}{ig}U^+\partial_\mu U$$

$$U^+ \bar{A}_\mu U = e^{-it^a\alpha^a} \bar{A}_\mu e^{it^b\alpha^b}$$

call $t^a \alpha^a \equiv \bar{\alpha}$

$$U^+ \bar{A}_\mu U = e^{-i\bar{\alpha}} \bar{A}_\mu e^{i\bar{\alpha}}$$

consider $e^{-X} Y e^{X} \cong (1-X) Y (1+X)$

$$= Y - XY + YX$$

$$= Y - [X, Y]$$

$$= Y + [Y, X]$$

$$U^+ \bar{A}_\mu U = \bar{A}_\mu + [\bar{A}_\mu, \bar{\alpha}]$$

$$= A_\mu^a t^a + i f^{abc} A_\mu^a \alpha^b t^c$$

$$U^+ \partial_\mu U = U^+ \partial_\mu \left(e^{i\alpha^a t^a}\right)$$

$$= U^+ e^{i\alpha^a t^a} (i t^a \partial_\mu \alpha^a)$$

$$= i U^+ U \, t^a \partial_\mu \alpha^a$$

$$A'_\mu = \bar{A}_\mu - \frac{1}{g}\partial_\mu \bar{\alpha} + i f^{abc} A_\mu^a \alpha^b t^c$$

OR

$$A'^{c}_\mu = \bar{A}_\mu^{c} - \frac{1}{g}\partial_\mu \alpha^c + i f^{abc} A_\mu^a \alpha^b$$

Field strength tensor

$$\bar{F}_{\mu\nu} = D_\mu \bar{A}_\nu - D_\nu \bar{A}_\mu$$

$$= \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu - ig[A_\nu, A_\mu]$$

$$= \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu + ig(if^{abc}) A_\mu^a A_\nu^b t^c$$

$$= \left(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c - g f^{abc} A_\mu^a A_\nu^b\right) t^c$$

Note that $g$ appears inside $\bar{F}_{\mu\nu}$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

gives terms like $g\, \partial_\mu A_\nu^c f^{abc} A_\mu^a A_\nu^b$

3 gauge-boson ~~term~~ vertex

$\psi \to e^{ig\alpha^a t^a}\psi$ (what we did in class)

$$\Rightarrow \quad \cancel{\bar{A}'_\mu = U^\dagger \partial_\mu U - (ig)^{-1} U^\dagger \bar{A}_\mu U}$$

$$A'_\mu = U^\dagger \bar{A}_\mu U - \frac{1}{ig} U^\dagger \partial_\mu U$$

$$U^\dagger \partial_\mu U = ig\, t^a \partial_\mu \alpha^a$$

$$\Rightarrow \quad \bar{A}'_\mu = \bar{A}_\mu - \partial_\mu \alpha^a t^a + i f^{abc} A^a_\mu \alpha^b t^c$$

Expressions for fermionic current & interaction same!

These are equivalent formulations.

Another common notation in literature is to use $A_\mu \Rightarrow g A_\mu$

$$\Rightarrow \quad D_\mu = \partial_\mu - iA_\mu \quad , \quad \psi \to e^{i\alpha^a t^a}\psi$$

$$F_{\mu\nu}F^{\mu\nu} \to \frac{1}{g^2} F_{\mu\nu}F^{\mu\nu}$$

$$[\bar{A}_\mu, \bar{A}_\nu] = ig f^{abc}$$

↑ g becomes part of algebra

# Covariant derivative for SU(2)

Generators of SU(2) are Pauli matrices $\left(\frac{\sigma^i}{2}\right)$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$t^a = \frac{\sigma^a}{2}$$

Gauge bosons of SU(2) are denoted by $W_\mu$

$$D_\mu = \partial_\mu - ig\, t^a W_\mu^a$$

$$= \partial_\mu - \frac{ig}{2}\left[\sigma^1 W_\mu^1 + \sigma^2 W_\mu^2 + \sigma^3 W_3^\mu\right]$$

$$= \partial_\mu - \frac{ig}{2}\begin{bmatrix} W_\mu^3 & W_\mu^1 - iW_2^\mu \\ W_\mu^1 + iW_2^\mu & -W_\mu^3 \end{bmatrix}$$

$$= \partial_\mu - \frac{ig}{2}\begin{bmatrix} W_\mu^3 & W_\mu^+ \\ W_\mu^- & -W_\mu^3 \end{bmatrix}$$

where $W_\mu^\pm = W_\mu^1 \mp W_\mu^2$