mion Couplings to $SU(2) \times U(1)$ gauge bosons

Recall in the Higgs mechanism we had

$$SU(2) \times U(1) \longrightarrow U(1)$$

4 massless gauge bosons W_μ^i, B_μ $\qquad$ 1 massless A_μ + 3 massive $W_\mu^\pm$, Z_μ

$$B_\mu = \cos\theta\, A_\mu - \sin\theta\, Z_\mu$$

$$W_{3\mu} = +\sin\theta\, A_\mu + \cos\theta\, Z_\mu$$

$$\sin\theta = \frac{g'}{\sqrt{g^2+g'^2}} \qquad \cos\theta = \frac{g}{\sqrt{g^2+g'^2}}$$

Gauge states: B_μ, W_μ^i

Mass states: A_μ, $W_\mu^\pm$, Z_μ.

Mass of $W^\pm = \frac{1}{2} g v^2$

$$Z = \frac{1}{2}\sqrt{g^2+g'^2}\, v^2 .$$

$$\Rightarrow \frac{M_W}{M_Z} = \frac{g}{\sqrt{g^2+g'^2}} = \cos\theta .$$

Covariant derivative for left-handed fermions

$$D_\mu^L = \partial_\mu - ig\left(\frac{\sigma^a}{2}\right) W_\mu^a - ig'\frac{Y}{2} B_\mu$$

$\frac{Y}{2}$ is "charge" under B_μ $\quad \psi \to e^{i\frac{Y}{2}\alpha(x)}\psi$

Let us work out the couplings.

Remember:

$$\frac{\sigma^a}{2} W_\mu^a = \frac{1}{2}\begin{bmatrix} W_\mu^3 & W_\mu^+ \\ W_\mu^- & -W_\mu^3 \end{bmatrix}$$

$$\mathcal{L} = \bar{\psi}\gamma^\mu D_\mu \psi .$$

$\Rightarrow$ couplings with W_μ are

$$\frac{-ig}{2}\bar{\psi}\begin{bmatrix} W_\mu^3 & W_\mu^+ \\ W_\mu^- & -W_\mu^3 \end{bmatrix}\psi$$

coupling with B_μ

$$\bar{\psi}\left(\frac{-ig'}{2}Y\right)\begin{bmatrix} B_\mu & 0 \\ 0 & B_\mu \end{bmatrix}\psi$$

Fermion content of the SM

3 left-handed leptons doublets + 3 right handed singlets

$$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \quad e_R,\ \mu_R,\ \tau_R.$$

3 left-handed quark doublets + 6 right handed singlets

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}, \quad u_R,\ d_R,\ s_R,\ c_R,\ b_R,\ t_R$$

We denote left-handed lepton doublet $L = \begin{pmatrix} \nu \\ e \end{pmatrix}$

left handed quark doublet $Q = \begin{pmatrix} u \\ d \end{pmatrix}$

Fermions occur in generations (we don't know why)

1st	2nd	3rd
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$ e_R	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$ μ_R	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$ τ_R
$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$ u_R d_R	$\begin{pmatrix} c \\ s \end{pmatrix}_L$ c_R s_R	$\begin{pmatrix} t \\ b \end{pmatrix}_L$ t_R, b_R

(I) Charged current part (i.e. $W_\mu^\pm$ couplings)

$$-\frac{ig}{2}\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\gamma^\mu\begin{bmatrix}0 & W_\mu^+ \\ W_\mu^- & \end{bmatrix}\begin{bmatrix}\nu_L \\ e_L\end{bmatrix}$$

$$\mathcal{L} \supset -\frac{ig}{2}\left[\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \bar{e}_L\gamma^\mu \nu_L W_\mu^-\right]$$

similarly for quark-sector

$$\mathcal{L} \supset -\frac{ig}{2}\left[\bar{u}_L\gamma^\mu d_L W_\mu^+ + \bar{d}_L\gamma^\mu u_L W_\mu^-\right]$$

W couples the same way to quarks & leptons
$\Rightarrow$ universal coupling.

(II) Neutral current part (i.e. Z_μ & A_μ)

$$-\frac{i}{2}\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\gamma^\mu\begin{bmatrix}+gW_\mu^3 + Yg'B_\mu & 0 \\ 0 & -gW_3^\mu + Yg'B_\mu\end{bmatrix}\begin{bmatrix}\nu_L \\ e_L\end{bmatrix}$$

consider the term $gW_\mu^3 + Yg'B_\mu$

Substituting expressions for W_μ^3 & B_μ in terms of A_μ & Z_μ

Also $\frac{\sigma^3}{2} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$

eigenvalue $T_3 = \pm 1/2$ for up/down-type fermio

$\pm g W^3_\mu$ can be written as $T_3 W^3_\mu$

Total coupling:

$$-\frac{i}{2}\left[(2T_3) g W^3_\mu + Y g' B_\mu\right]$$

$$= \cancel{\frac{i}{2}}\left[\cancel{2}T_3 g\, (s_\theta \cancel{A_\mu} + c_\theta \cancel{Z_\mu}) + \cancel{\ }\right]$$

$$= -i\left[T_3 g\, (s_\theta A_\mu + c_\theta Z_\mu) + \frac{Y}{2} g' (c_\theta A_\mu - s_\theta Z_\mu)\right]$$

$$= -ig\left[\left(T_3 s_\theta + \frac{Y}{2}\frac{g'}{g} c_\theta\right) A_\mu + \left(T_3 c_\theta - \frac{Y}{2}\frac{g'}{g} s_\theta\right) Z_\mu\right]$$

$$\frac{g'}{g} = \tan\theta \qquad \frac{g'}{g}\cos\theta = \sin\theta.$$

Term proportional to A_μ

$$-ig\left[T_3+\frac{Y}{2}\right]A_\mu \sin\theta \equiv -i\underbrace{e}_{=g\sin\theta}\left[T_3+\frac{Y}{2}\right]A_\mu$$

Term proportional to Z_μ.

$$-ig\left[T_3\, c_\theta-\frac{Y}{2}\tan\theta\sin\theta\right]Z_\mu$$

$$=-ig\sin\theta\left[T_3\frac{\cos\theta}{\sin\theta}-\frac{Y}{2}\tan\theta\right]Z_\mu$$

$$=-ie\left[T_3\cot\theta-\frac{Y}{2}\tan\theta\right]Z_\mu.$$

For neutrino $T_3=+1/2 \Rightarrow Y=-1$

$\Rightarrow T_3+Y/2=0 \Rightarrow$ no A_μ coupling.

For electron (with same Y) $T_3=-1/2$ $Y=-1$

$T_3+Y/2=-1$ (electric charge)

$$\Rightarrow \boxed{Q=T_3+Y/2}$$

Similar for quarks.

We can work out all charges.

	$SU(2)$	$U(1)_Y$
$L = \begin{pmatrix} \nu \\ e \end{pmatrix}$	2	-1
$Q = \begin{pmatrix} u \\ d \end{pmatrix}$	2	$-1/3$
e_R	1	$+2$
u_R	1	$4/3$
d_R	1	$-2/3$

For right-handed fields: no $W_\mu^\pm$ couplings

$$D_\mu^R = \partial_\mu - i g' \frac{Y_R}{2} B_\mu$$

For electrons $-\frac{ig'}{2} Y_R \, \bar{e}_R \left(c_\theta A_\mu - s_\theta Z_\mu \right) e_R$

coupling to $A_\mu = -\frac{ig'}{2} Y_R c_\theta = -ig \frac{g'}{\sqrt{g^2+g'^2}} \frac{Y_R}{2}$

$$= -ig \sin\theta \frac{Y_R}{2}$$

$$= -ie \frac{Y_R}{2}$$

$$\Rightarrow Y_R = +2$$

Higgs terms (a.k.a fermion mass terms)

$$\mathcal{L} = y_e \bar{L} \Phi e_R + h.c.$$

$$= y_e \left(\bar{\nu}_L \ \bar{e}_L \right) \begin{pmatrix} 0 \\ v+\varphi \end{pmatrix} e_R + h.c.$$

$$= y_e (v+\varphi) \bar{e}_L e_R + h.c.$$

$$= \underbrace{y_e v}_{m_e} \left(1+\frac{\varphi}{v}\right) \bar{e}_L e_R + h.c$$

For quark terms.

$$\mathcal{L}_d = y_d \bar{Q} \Phi d_R + \cancel{y_u \Phi} + h.c$$

$$\rightarrow \cancel{y_d}\, m_d \left(1+\frac{\varphi}{v}\right) \bar{d}_L d_R + c.c.$$

What about u-quarks?

We need $\begin{pmatrix} v+\varphi \\ 0 \end{pmatrix}$ kind of field.

CP conjugate:

$$i\sigma^2 \Phi^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v+\varphi \end{pmatrix} = \begin{pmatrix} v+\varphi \\ 0 \end{pmatrix}$$

$$\mathcal{L}_u = y_u \, \bar{Q} \, \Phi^c \, u_R + c.c.$$

$$= m_u \left(1 + \frac{\varphi}{v}\right) \bar{u}_L u_R + c.c.$$

How to add generations ?

— Easy answer

$$\mathcal{L}_d = \sum_i y_{d\,i} \; \bar{Q}_i \, \Phi \, d_{R_i} + c.c$$

However, no reason (i.e. no gauge-invariance or other symmetry) that requires $\bar{Q}$ & d_R to be from the same generation

$\Rightarrow$ Most general answer

$$\mathcal{L}_d = \sum_{ij} (Y_d)_{ij} \; \bar{Q}_i \, \Phi \, d_{Rj} + c.c$$

Similarly

$$\mathcal{L}_u = \sum_{ij} (Y_u)_{ij} \; \bar{Q}_i \, \Phi^c \, u_{Rj} + c.c$$

In component notation, this is.

$$\mathcal{L}_d = \sum (y_d)_{ij} v \, \bar{d}_{Li} \left(1 + \frac{\varphi}{v}\right) d_{Rj} + c.c$$

$$= \bar{d}_{Li} \, (M_d)_{ij} \, d_{Rj} \left(1 + \frac{\varphi}{v}\right) + c.c$$

Mass matrix.

Similarly,

$$\mathcal{L}_u = \bar{u}_{Li} \, (M_u)_{ij} \, u_R \left(1 + \frac{\varphi}{v}\right) + c.c.$$

To diagonalise $(M_d)_{ij} \longrightarrow (\tilde{M}_d)_{ij}$

$$\tilde{d}_{Li} = (V_L^d)_{ij} \, d_{Lj}$$

$$\tilde{d}_{Rj} = (V_R^d)_{ij} \, d_{Rj}$$

$$\bar{d}_{Li} (M_d)_{ij} d_{Rj} = \tilde{d}_{Li} \underbrace{\left[(V_L^d)^\dagger M \, V_R^d \right]_{ij}}_{\displaystyle \| \atop \displaystyle (\tilde{M})_{ij}} \tilde{d}_{Rj}$$

$$(\tilde{M})_{ij} = \begin{bmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{bmatrix}$$

$\tilde{d}_L$, $\tilde{d}_R$ are mass eigenstates.

& d_L, d_R are gauge eigenstates

Similarly, we should have.

$$\tilde{u}_{L_i} = (V^u_L)_{ij}\, u_{L_j} \;\Rightarrow\; u_L = V^{u\dagger}_L\, \tilde{u}_L$$

$$\tilde{u}_R = (V^d_L)_{ij}\, u_{R_j} \;\Rightarrow\; u_R = (V^u_R)^\dagger\, \tilde{u}_R$$

The only place where u, d states mix are in W-interactions.

$$\mathcal{L}_W = \frac{ig}{2}\, \bar{Q}_L \gamma^\mu Q_L W_\mu$$

$$= \frac{ig}{2}\left[\bar{u}_L \gamma^\mu d_L W_\mu^+ + \bar{d}_L \gamma^\mu u_L W_\mu^-\right]$$

In terms of mass eigenstates

$$\bar{u}_L \gamma^\mu d_L = \cancel{(\tilde{u}_L)_i (V^{*u}_L)_{ij}\, \gamma^\mu\, V^d}$$

$$= \tilde{u}_{L_i} (V^u_L)^*_{ij}\, \gamma^\mu\, (V^d_L)^\dagger_{jk}\, \tilde{d}_{L_j}$$

$$= \tilde{u}_{L_i}\, \underbrace{(V^u_L V^{d\dagger}_L)_{ik}}_{V_{CKM}}\, \gamma^\mu\, \tilde{d}_{L_k}$$

$$V_{CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

– why only 1 phase?

For 2 generations.

suppose there is a phase $(\bar{d}\ \bar{s}) \begin{pmatrix} c_\theta & s_\theta e^{i\delta} \\ -s_\theta e^{-i\delta} & c_\theta \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$

define $s' = e^{i\delta} s$ $\qquad \bar{s}' s' = \bar{s} s$

$$\text{terms} = \begin{cases} c_\theta \bar{d} d + s_\theta \bar{d} (e^{i\delta} s) \\ -s_\theta \cancel{\bar{d}s} (e^{-i\delta}) \bar{s} d + c_\theta \bar{s} s \end{cases} = \begin{matrix} c_\theta \bar{d} d + c_\theta \bar{s}' s' \\ s_\theta \bar{d} s' \mp s_\theta \bar{s}' d \end{matrix}$$

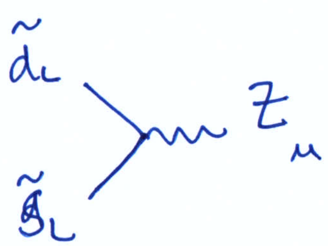
$$= (\bar{d}\ \bar{s}') \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

→ absorbed.

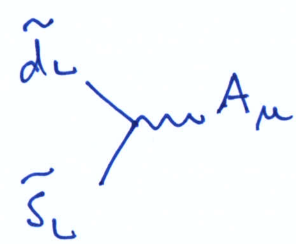
Out of 3 phases, 2 can be absorbed ⇒ 1 left.

What about a vertex like this?

$\boxed{\text{No flavour-changing Neutral Current (FCNC) at tree level}}$



or



we had:

$$g \sin\theta\ \bar{d}_L \left(T_3 \frac{\cos\theta}{\sin\theta} + \frac{Y}{2} \frac{\sin\theta}{\cos\theta} \right) \gamma^\mu d_L\ Z_\mu.$$

$$+\ g \sin\theta\ \bar{d}_L \left(+ T_3 \cancel{\sin\theta} + \frac{Y}{2} \right) \gamma^\mu d_L\ A_\mu$$

⇒ every time, it will be $(V_L^d)^\dagger (V_L^d) = \mathbb{1}$

Similar for other fermions.

Normally, the lagrangian is written in mass eigenstats

$\Rightarrow$ $\tilde{u}$ renamed to u

$\tilde{d}$ renamed to d etc.

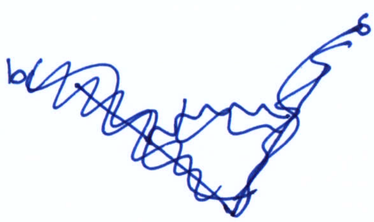
<u>GIM</u> <u>mechanism</u>

Glashow

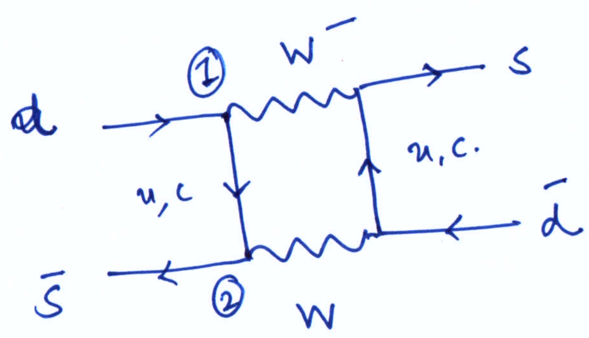
Iliopoulos

Maiani

What about FCNC at 1-loop?



consider only two generations first.



at ① u- term contribution = V_{ud} = ~~cos~~ c_{12}

c-term contribution = V_{cd} = $-s_{12}$

at ② u-term contribution = V_{us} = s_{12}

c-term contribution = V_{cs} = c_{12}

$$\text{Total} = -c_{12} s_{12} + s_{12} c_{12} = 0 \; !$$

$\Rightarrow$ As long as we have full ~~generation~~ $SU(2)$ structure (i.e. $\begin{pmatrix} u_i \\ d_i \end{pmatrix}$) in each generation, there are no FCNC events at 1-loop in SM.

This was used to predict the existance of a charm quark.