Abelian Higgs Mechanism

i.e. giving mass to a $U(1)$ gauge boson via a complex scalar.

A complex scalar has the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi - \frac{m^2}{2}\phi^* \phi$$

which is invariant under ~~$\phi \to e^{i\alpha(x)}\phi$~~

$$\phi \to e^{i\alpha}\phi$$

However if α is now a function of x^μ, we get

$$\partial_\mu \phi \to \partial_\mu\left(e^{i\alpha(x)}\phi\right)$$

$$= e^{i\alpha(x)}\partial_\mu \phi + i\left(\partial_\mu \alpha(x)\right)e^{i\alpha(x)}\partial_\mu \phi$$

If we have a vector field A_μ that transforms as $A_\mu \to A_\mu + \partial_\mu \alpha(x)$ then

we can define $D_\mu \phi \equiv \partial_\mu \phi - i A_\mu \phi$

then $D_\mu \phi \to e^{i\alpha(x)} D_\mu \phi$

i.e. it transforms in the same way as the field $\phi(x)$. D_μ is called a "covariant derivative."

The Lagrangian ~~given~~ ~~by~~ for the vector field (from Electromagnetism) is

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu}$$ where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$F_{\mu\nu}$ is invariant under $A_\mu \to A_\mu + \partial_\mu \alpha$

Therefore the total Lagrangian we can write is

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}(D_\mu \phi)^* (D^\mu \phi) - \frac{m^2}{2} \phi^* \phi$$

There are more terms we can write for the scalar field, in particular $\sim (\phi^* \phi)^2$

$$V(\phi) = \frac{m^2}{2} \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi)^* (D^\mu \phi) - V(\phi)$$

Since ϕ is invariant under $\phi \to e^{i\alpha(x)}\phi$

we can choose to absorb the imaginary part of ϕ in $\alpha(x)$ and work with real component only; i.e if we re-write

$$\phi = \varphi\, e^{i\theta(x)} \qquad \text{where}$$

$$\varphi \cos\theta(x) \equiv \mathrm{Re}\,\phi$$

$$\varphi \sin\theta(x) \equiv \mathrm{Im}\,\phi$$

(this is similar to writing $z = x + iy = r e^{i\theta}$)

then the choice $\alpha(x) = -\theta(x)$ allows us to work in a "gauge" where we only have real φ.

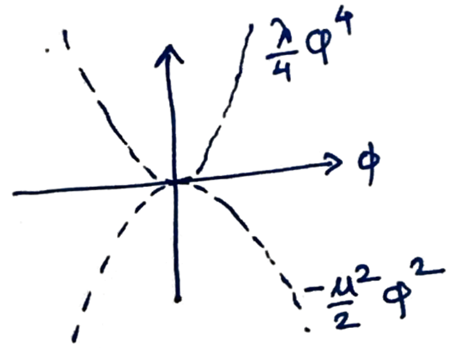
The potential in terms of φ is

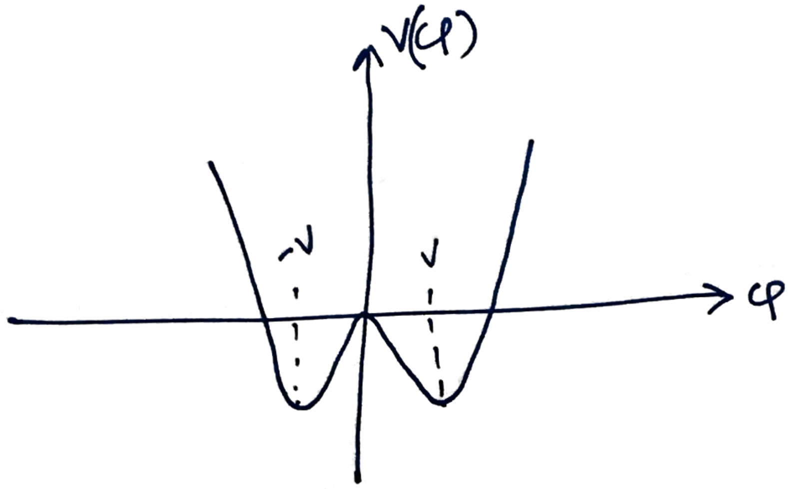
$$V(\varphi) = +\frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4$$

λ)

If $m^2 < 0$ $\quad (m^2 = -\mu^2$ where $\mu^2 > 0)$

$$V(\varphi) = -\frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4$$





The minimum is no longer at zero.

We need to expand the field around this new vacuum.

The minimum is given by

$$\frac{\partial V}{\partial \phi} = 0 \Rightarrow -\mu^2\varphi + \lambda\varphi^3 = 0$$

$$\Rightarrow \varphi(-\mu^2 + \lambda\varphi^2) = 0$$

$$\Rightarrow \varphi = 0 \quad \text{or} \quad \varphi^2 = \frac{\mu^2}{\lambda}$$

$$\frac{\partial^2 V}{\partial \phi^2} > 0 \text{ at } \varphi = 0$$

$\Rightarrow$ Minimum at $\pm v$ where $v \equiv \sqrt{\frac{\mu^2}{\lambda}}$

$$D_\mu \phi = \partial_\mu \phi - i A_\mu \phi .$$

$$(D_\mu \phi)^* (D^\mu \phi) = |\partial_\mu \phi|^2 + i A_\mu \left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right) + A_\mu A^\mu |\phi|^2$$

Therefore when ϕ is at its minimum at v, (v is called the vacuum expectation value),

$$\langle \phi \rangle = v$$

then we have a term $v^2 A_\mu A^\mu$ that looks like a <u>mass term</u>.

This is the Higgs mechanism.

$\varphi \equiv (v + \varphi_r)$ (Expanding the radial component around the minimum)

gives the potential

$$V(\varphi) = -\frac{\mu^2}{2} (v + \varphi_r)^2 + \frac{\lambda}{4} (v + \varphi_r)^4$$

Terms proportional to φ_r^2 are

$$-\frac{\mu^2}{2} \varphi_r^2 + \frac{6\lambda}{4} v^2 \varphi_r^2 = 2\lambda v^2 \varphi_r^2$$

$\Rightarrow$ Mass of ~~xxxx~~ radial mode $\Rightarrow m_{\varphi_r}^2 = 4\lambda v^2$

(from $\frac{1}{2} m_{\varphi_r}^2 \varphi_r^2$)

Higgs Mechanism for SU(2)

~~Notice~~ We start with Φ which is a doublet of SU(2)

i.e. $\Phi \rightarrow e^{ig\alpha^a(x)t^a}\Phi$

$t^a \equiv \frac{\sigma^a}{2}$ where σ^a are the Pauli matrices.

The corresponding covariant derivative is

$$D_\mu \equiv \partial_\mu - ig W_\mu^a t^a$$

(where W_μ^a are three gauge fields similar to A_μ in the abelian case)

We write the Lagrangian in the same way:

$$\mathcal{L} = \frac{1}{2}(D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi) - \frac{1}{4}W_{\mu\nu}W^{\mu\nu}$$

where $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$

$$W_\mu = W_\mu^a t^a$$

SU(2) has 3 generators i.e. 3 degrees of freedom

$$\Phi = \begin{pmatrix} \phi_2 + i\phi_3 \\ \phi_0 + i\phi_1 \end{pmatrix} \rightarrow \text{4 degrees of freedom.}$$

Similar to the Abelian case ~~where~~ we absorbed the imaginary part completely in the gauge transformation, here too we can go to a gauge where only one degree of freedom remains.

$$\Phi \equiv \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}$$

$$\cancel{(D_\mu \Phi)^\dagger (D^\mu \Phi) = \mathcal{L}}$$

$$D_\mu \Phi = \begin{pmatrix} 0 \\ \partial_\mu \phi_0 \end{pmatrix} - \frac{ig}{2} \begin{pmatrix} W_{3\mu} & W_{1\mu} - iW_{2\mu} \\ W_{1\mu} + iW_{2\mu} & -W_{3\mu} \end{pmatrix} \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}$$

$$(D^\mu_{,} \phi)^\dagger = \begin{pmatrix} 0 & \partial_\mu \phi_0 \end{pmatrix} + \frac{ig}{2} \overset{(0\ \phi_0)}{\begin{pmatrix} W_{3\mu} & W_{1\mu} - iW_{2\mu} \\ W_{1\mu} + iW_{2\mu} & -W_{3\mu} \end{pmatrix}} \cancel{\begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}}$$

$$V(\Phi) = -\frac{\mu^2}{2}\Phi^\dagger\Phi + \frac{\lambda}{4}(\Phi^\dagger\Phi)^2$$

$$= -\frac{\mu^2}{2}(0\ \phi_0)\begin{pmatrix}0\\ \phi_0\end{pmatrix} + \frac{\lambda}{4}\left((0\ \phi_0)\begin{pmatrix}0\\ \phi_0\end{pmatrix}\right)^2$$

$$= -\frac{\mu^2}{2}\phi_0^2 + \frac{\lambda}{4}\phi_0^4$$

→ The same minimisation as before applies.

$$\phi_0 = (v+\varphi)$$

At $\phi_0 = v$,

$$D_\mu\Phi\Big|_v = -\frac{ig}{2}\begin{pmatrix}W_{3\mu} & W_{1\mu}-iW_{2\mu}\\ W_{1\mu}+iW_{2\mu} & -W_{3\mu}\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}$$

$$= -\frac{igv}{2}\begin{pmatrix}W_{1\mu}-iW_{2\mu}\\ -W_{3\mu}\end{pmatrix}$$

$$(D_\mu\Phi)^\dagger\Big|_v = +\frac{igv}{2}\begin{pmatrix}W_{1\mu}+iW_{2\mu}\\ -W_{3\mu}\end{pmatrix}$$

$$(D^\mu\Phi)^\dagger(D_\mu\Phi)\Big|_v = \frac{g^2v^2}{4}\left(W_{1\mu}^2 + W_{2\mu}^2 + W_{3\mu}^2\right)$$

All three gauge bosons have mass $= \frac{1}{2}gv$

It is customary to define

$$W_\mu^{\pm} = W_\mu^1 \mp i W_\mu^2$$

The mass terms then look like

$$\frac{1}{2} M_W^2 W_\mu^+ W^{\mu -} + \frac{1}{2} M_W^2 W_3^2$$

We now apply this to our world where the symmetry is $\boxed{SU(2) \times U(1)}$

This means that the Higgs ~~xxx~~ ~~xxx~~ transforms under both these groups.

The covariant derivative is.

$$D_\mu = \partial_\mu - ig \underbrace{W_\mu^a t^a}_{SU(2)} - \frac{ig'}{2} \underbrace{B_\mu}_{U(1)}$$

(we have normalised the U(1) coupling for convinience.)

$$\Phi = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}$$

(again, we have absorbed three d.o.f in the gauge transformation)

The treatment of $V(\Phi)$ remains the same.

$$D_\mu \Phi\Big|_v = \left\{ -\frac{ig}{2}\begin{pmatrix} +W_{3\mu} & W^+_\mu \\ W^-_\mu & -W_{3\mu}\end{pmatrix} - \frac{ig'}{2}\begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu\end{pmatrix}\right\}\begin{pmatrix}0\\ v\end{pmatrix}$$

$$= -\frac{i}{2}\begin{pmatrix} +gW_{3\mu}+g'B_\mu & gW^+_\mu \\ gW^-_\mu & -gW_{3\mu}+g'B_\mu\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}$$

$$= -\frac{iv}{2}\begin{pmatrix} gW^+_\mu \\ -gW_{3\mu}+g'B_\mu\end{pmatrix}$$

$$(D_\mu \Phi^+)\Big|_v = \frac{iv}{2}\left(gW^-_\mu \quad -gW_{3\mu}+g'B_\mu \right)$$

~~Define~~ $\cancel{B_\mu = \cos\theta_W A_\mu + \sin\theta_W Z_\mu}$

$\cancel{W_{3\mu} = -\sin\theta Z_\mu + \cos\theta A_\mu}$

$$(D^\mu\Phi)^+(D_\mu\Phi)\Big|_v = \frac{v^2}{4}\Big\{ g^2 W^+_\mu W^{\mu -} - (gW_{3\mu}+g'B_\mu)(gW_{3\mu}+g'B_\mu)\Big\}$$

this combination $(gW_{3\mu}+g'B_\mu)$ is the only one to get mass.

define

$$Z_\mu = \frac{g W_{3\mu} + g' B_\mu}{\sqrt{g^2+(g')^2}}$$

$$(D^\mu \Phi)^+ (D_\mu \phi) = \frac{v^2}{4} \left\{ g^2 \, W_\mu^+ W^{\mu -} + \sqrt{(g^2+g'^2)} \, Z_\mu Z^\mu \right\}$$

$$M_W^2 = \frac{g v^2}{2}$$

$$M_Z^2 = \frac{\sqrt{(g^2+g'^2)} \, v^2}{2}$$

$$\frac{M_W^2}{M_Z^2} = \frac{g}{\sqrt{g^2+g'^2}} \equiv \cos\theta_W$$

where θ_W is called the "Weinberg Angle".

$$\rho = \frac{M_W^2}{M_Z^2 \cos\theta_W}$$ is an important

~~SU(2) x U(1) x str~~

observable.

The doublet representation predicts $\rho = 1$.

$$Z_\mu = \cos\theta_W \; W_{3\mu} + \sin\theta_W \; B_\mu$$

define the orthogonal component

$$A_\mu = -\sin\theta_W \; W_{3\mu} + \cos\theta_W \; B_\mu.$$

or equivalently,

$$B_\mu = \sin\theta_W \; Z_\mu + \cos\theta_W \; A_\mu$$

$$W_{3\mu} = \cos\theta_W \; Z_\mu - \sin\theta_W \; A_\mu.$$

Note that the $A_\mu A^\mu$ combination does not occur in $(D_\mu \Phi)^\dagger (D^\mu \Phi)$ and therefore remains massless. We identify this with a <u>photon</u> (corresponding to electromagnetism.)