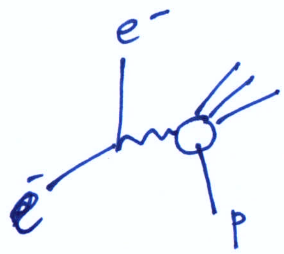
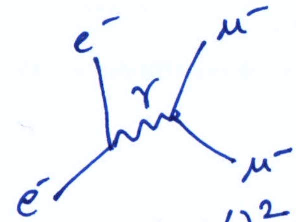
Last time:



We want to find an expression for this process

$ep \to eX$

Step 1: Look at $e^-\mu \to e^-\mu$



can be written as

$$\overline{|M|^2} = \frac{Q_e^2 Q_m^2}{4t} L^e_{\mu\nu} L^{m\,\mu\nu} \qquad \left(\begin{aligned} t &= (k-k')^2 \\ &\equiv q^2 \end{aligned} \right)$$

where $L^e_{\mu\nu}(k,k') = 4\left(k_\mu k'_\nu + k'_\mu k_\nu - (k\cdot k')\, g_{\mu\nu}\right)$

similarly $L^m_{\mu\nu}(p,p') = 4\left(p_\mu p'_\nu + p'_\mu p_\nu - (p\cdot p')\, g_{\mu\nu}\right)$

This gives $L^e_{\mu\nu} L^{m\,\mu\nu}$

$=$

<u>Step 2</u>: Model hadronic tensor $H_{\mu\nu}$ on Leptonic tensor $L_{\mu\nu}$.

Since $L_{\mu\nu}$ is symmetric in $\mu \leftrightarrow \nu$, we write $H_{\mu\nu}$ as all possible symmetric terms. Available momenta are p & q only.

$$H_{\mu\nu} = -g_{\mu\nu} H_1 + \dots$$

Current conservation demands $q_\mu L^{\mu\nu} = 0$

and $q_\mu H^{\mu\nu} = 0$

This can be used to simplify $H_{\mu\nu}$ to

$$H_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M^2}\right) H_1 + \frac{H_2}{M^2}\left(p^\mu - \frac{p.q}{q^2} q^\mu\right)\left(p^\nu - \frac{p.q}{q^2} q^\nu\right)$$

This gives $L_{\mu\nu} H^{\mu\nu} = \left(\cos^2\frac{\theta}{2} H_2 - \cancel{\frac{q^2}{2M^2}} 2\sin^2\frac{\theta}{2} H_1\right)$

comparing with $e\mu \to e\mu$, we can make a first guess for forms of H_1 & H_2.

In particular, if the proton was a point-like particle, we would have

$$(p+q)^2 = M^2 \Rightarrow \quad 2p\cdot q + q^2 = 0$$

$$\Rightarrow 2M\left(\frac{p\cdot q}{M}\right) + q^2 = 0$$

Choosing $\nu = \frac{p\cdot q}{M}$ & $Q^2 = -q^2$ as our variables, we have

$$H_2(Q^2,\nu) = \delta\left(\nu - \frac{Q^2}{2M}\right)$$

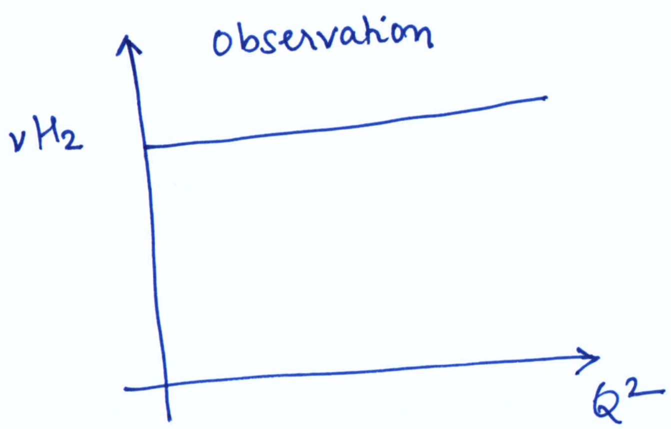
$$H_1(Q^2,\nu) = \frac{Q^2}{2M^2}\,\delta\left(\nu - \frac{Q^2}{2M}\right)$$

using $\delta(ax) = \frac{1}{a}\delta(x)$

$$\left[\begin{aligned} &\int dx\, \delta(ax) \\ &= \frac{1}{a}\int d(ax)\, \delta(ax) \\ &= \frac{1}{a}\int dx\, \delta(x) \end{aligned}\right]$$

$$\nu H_2(Q^2) = \delta\left(\cancel{1} - \frac{Q^2}{2M\nu}\right)$$

$$2MH_1(Q^2) = \frac{Q^2}{2M\nu}\,\delta\left(\gamma - \frac{Q^2}{2M\nu}\right) = \frac{Q^2}{2M\nu}(\nu H_2)$$



What this tells us is that the assumption of point-like particles inside the proton is basically correct.

$$H_2(Q^2,\nu) \to \nu H_2(Q^2/2M\nu) \to F_2(x) \quad \text{where } x = \frac{Q^2}{2M\nu} \propto Q^2.$$

But F_2 does not depend on Q^2 either!

Parton model

Assume there are multiple "partons" inside the proton. If the proton carries momentum p^μ, the momentum carried by the parton is xp^μ

where $0 < x < 1$.

This means, energy of parton is xE, mass is xM_p.

$(\text{parton momentum} + \text{photon momentum})^2 = x^2p^2 + 2p\cdot q + q^2$

If $x^2p^2 \ll q^2 \Rightarrow \boxed{x = -q^2/2p\cdot q}$

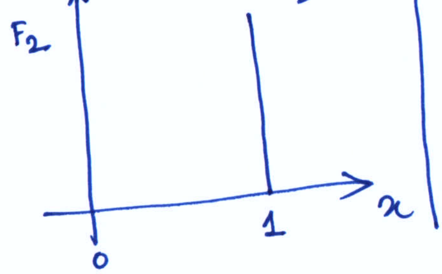
From DIS experiments, we know νH_2 is constant in Q^2

$$\nu H_2 \rightarrow F_2(x, Q^2) \quad \text{where } x = \text{momentum fraction.}$$

$$M H_1 \rightarrow \frac{1}{2x} F_2(x, Q^2)$$

We want to find a deeper modeling of F_2 in terms of constituent partons.

If there was only one parton inside the proton, we expect this.



If there are three, we expect this

Assume we can factorise F_2 into contributions from each parton

$$F_2(x) = \sum Q_i^2 \, x \, f_i(x)$$

parton distribution function.

Charge of parton w.r.t electric charge of electron.

$$\Rightarrow \quad \frac{F_2(x)}{x} = \sum Q_i^2 f_i(x)$$

Idea 1: We know have (2 u-quarks + 1 d-quark) works to get the right charge/spin/hadron pattern.

proton

$$\frac{F_2^p(x)}{x} = \cancel{\Sigma}\left[\left(\frac{2}{3}\right)^2 u^p(x) + \left(\frac{1}{3}\right)^2 d^p(x)\right] = \frac{4u^p(x) + d^p(x)}{9}$$

$$\frac{F_2^{\cancel{p}n}(x)}{x} = \cancel{\Sigma}\left(\frac{1}{3}\right)^2 d^n(x) + \left(\frac{2}{3}\right)^2 u^n(x)$$

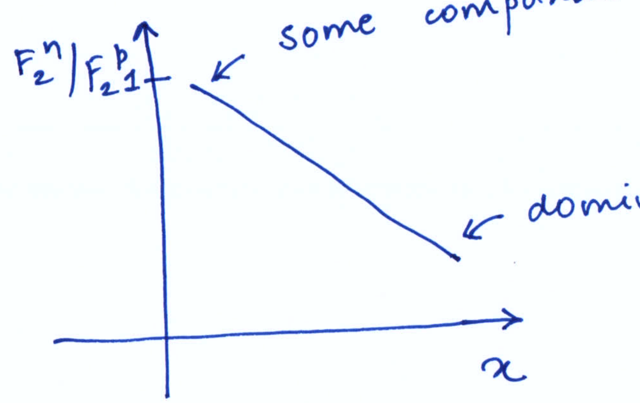
Since neutron is made up of udd, we expect $u^p = d^n$

$$\Rightarrow \quad \frac{F_2^n}{x} = \left(\frac{1}{3}\right)^2 u^p(x) + \left(\frac{2}{3}\right)^2 d^p(x) = \frac{u^p(x) + 4d^p(x)}{9}$$

Look at F_2^n / F_2^p

Expectation $\rightarrow$ constant factor for all x.

Observation



<u>Idea 2:</u>

Postulate a "sea" of quarks. Because total charge is fixed, $u^s = \bar{u}^s = d^s = \bar{d}^s = s^s = \bar{s}^s \equiv S(x)$

$\uparrow$ sea quark density.

$$\underset{\substack{\uparrow \\ \text{total} \\ u\text{-content}}}{u^p} = \underset{\substack{\uparrow \\ \text{valence}}}{u_v^p} + \underbrace{(u_s^p + \bar{u}_s^p)}_{\text{sea}}$$

, similarly for d

$$s^p = s_s^p + \bar{s}_s^p$$

$$\frac{F_2^p}{x} = \left(\frac{2}{3}\right)^2 u^p + \left(\frac{1}{3}\right)^2 d^p + \left(\frac{1}{3}\right)^2 s^p$$

$$= \frac{[4u_v^p + d_v^p]}{9} + \frac{3}{4} S(x)$$

same sea component.

$$\frac{F_2^n}{x} = \frac{[4d_v^p + u_v^p]}{9} + \frac{3}{4} S(x)$$

Now we check again

Requirement of parton model is sum of momentum carried by all partons = momentum of proton

i.e $\sum_i \int x p f_i(x) = p$

$\Rightarrow \quad \sum \int x f_i(x) = 1.$

However, observations give $\sum \int x f_i(x) \approx 0.5$

$\Rightarrow$ Half of momentum missing

$\Rightarrow$ neutral particle (i.e. does not interact with photon) inside the proton.

$\rightarrow$ "Gluons" of QCD.

Remember also, we had

$$R \, \delta = \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum |e_q|^2}{e_\mu} = 3 \times \left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right)$$

$\uparrow$

required to match observation.

Quantum Chromo-dynamics.

The quark field is a triplet of SU(3)

i.e. $\psi_q = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}$ that transforms as

$$\psi_q \to \exp\{ i g_s t^a \alpha^a(x) \} \psi_q$$

under SU(3) t^a are the eight generators of SU(3) algebra.

The covariant derivative is given by

$$D_\mu = \left(\partial_\mu - i g_s t^a G_\mu^a \right)$$

where G_μ^a are the eight gluon fields.
(similar to 3 W's of SU(2))

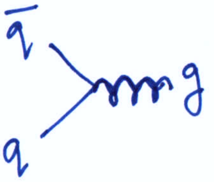
The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \bar{\psi}_q (i \gamma^\mu D_\mu) \psi$$

$$G_{\mu\nu}^{a} = D_\mu G_\nu^{a} - D_\nu G_\mu^{a}$$

$$= \partial_\mu G_\nu^{a} - \partial_\nu G_\mu^{a} - i g_s \, G_\mu^{b} G_\nu^{c} f^{abc}$$

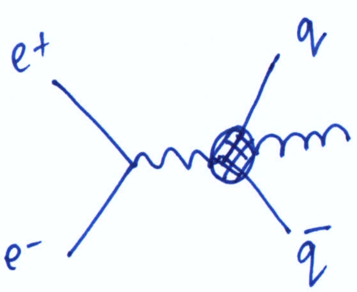
$$\bar{\psi}_q \left(i\gamma^\mu \left(\partial_\mu - i g_s t^a G_\mu^a \right) \right) \psi_q$$

gives a vertex  $\equiv \; i g_s t^a \gamma^\mu$

$(t^a)_{ij}$

ij : colour indices $\equiv \bar{r}g, \; \bar{g}b$, etc...

Why QCD? Why not just have 3 colours of quarks and be done with it?

 $\equiv$ 3-jet event

Observation of 3-jet events!

Why not just U(1) (like photon?)

— More complicated answer

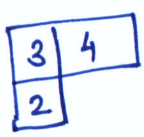
(1) Normalisation of cross section matches SU(3) prediction

(2) SU(3) automatically "confines" quarks within proton if we make the assumption that only SU(3) <u>singlets</u> are allowed to exist in nature.

Review of Young's Tableaux. = combinatoric trick!

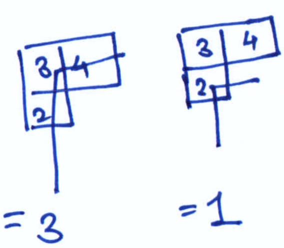
$\square \equiv$ triplet

Dimension is calculated by the following rules



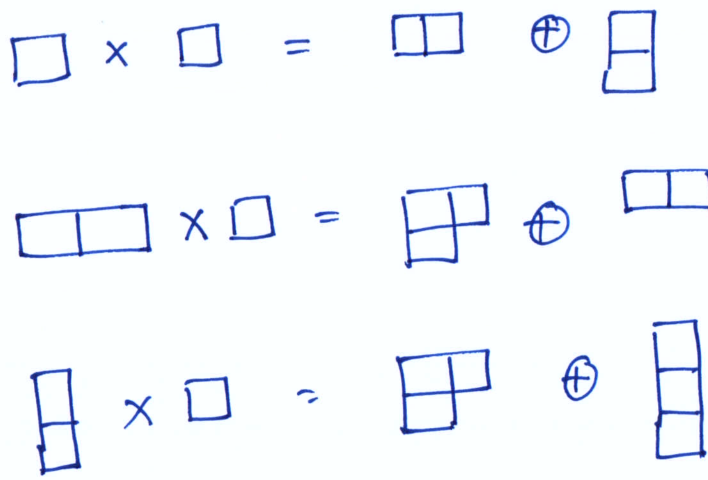
① Number increasing left → right and decreasing top → bottom

② construct paths. and count all possible path lengths (# of boxes crossed)



$$\text{dimension of representation} = \frac{\prod \text{all numbers}}{\prod \text{all path lengths}}$$

Two representations can be combined by combining their tableaux.

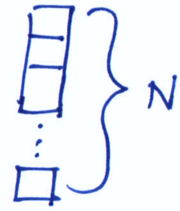


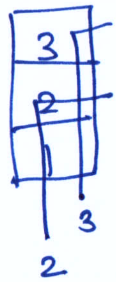
Rules:

1. Add boxes from the second term one by one to the first
2. You can add a box to the right or ~~at~~/ the below
3. Row lengths should remain same or decrease as you go down

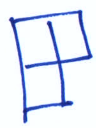
4. column lengths should remain same or decrease left → right

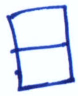
What are ~~colour~~ singlets?

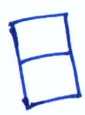
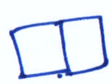
For SU(N), a singlet looks like 

because e.g.  $= \dfrac{3\times2\times1}{3\times2\times1} = 1.$

How do we make  from 

 $\otimes$  =  $\oplus$  ✓

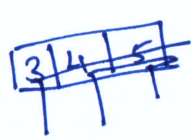
How do you make ?

 $\times$  =  $\oplus$ 

bar to denote it is a <u>different</u> representation from  which is also 3-dim.

$\Rightarrow \quad 3 \times 3 = \bar{3} + \cancel{3}\,6$

$\bar{3} \times 3 = 1 + 8$

i.e. $3 \times 3 \times 3 = 1 + 8 + 8 +$  $= \dfrac{\cancel{3}\times\cancel{4}^{\,2}\times 5}{\cancel{3}\times\cancel{2}} = 10$

$= 1 \oplus 8 \oplus 8 \oplus 10 = 27$!! ✓

$\Rightarrow$ All possible states have either

quark $\times$ anti-quark (i.e. $3 \times \bar{3}$) like $\pi^\pm, \pi^0$

<u>OR</u>

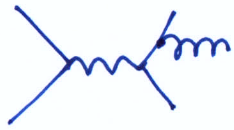
3 quarks like p, n.

$$\psi \to \exp\{ i g_s t^a \alpha^a(x)\} \psi$$

$$\psi^* \to \exp\{-i g_s (t^a)^* \alpha^a(x)\} \psi^*$$

$$= \exp\{+ i g_s t^a \alpha^a(x)\} \psi^*$$

Let us calculate. $e^+ e^- \to q\bar{q}g$.

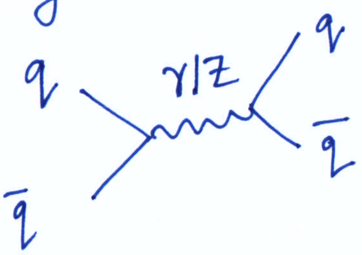


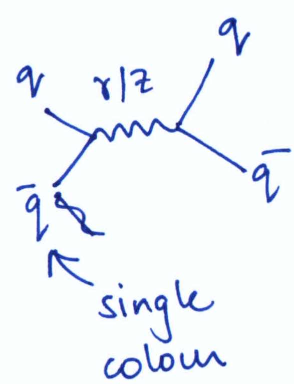
$+$

Remember  $= i g_s t^a \gamma^\mu$

$$\overline{|M|^2} = \frac{(\sum e_q^2)e^2}{4s^2} L^e_{\mu\nu} H^{\mu\nu}$$

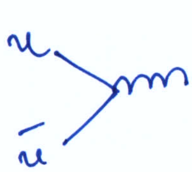
Things to note: ① EW interactions are "blind" to colour. i.e. if we have a purely EW process

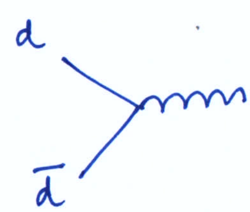
$\left| \sum_{\text{colour}} \right.$ 

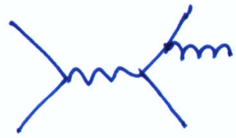
$\left. \right|^2 = 3 \times \left| \right.$ 

$\left. \right|^2$

② QCD is blind to "flavour". i.e. u-quarks are no diffrent than d-quarks when it comes to QCD.

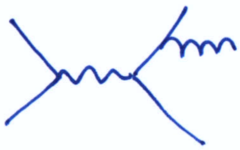
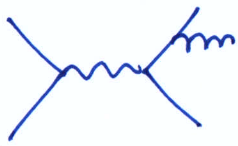
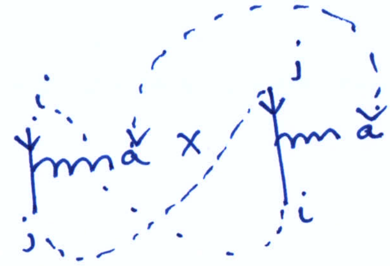


$= i g_s (t^a) \gamma^\mu =$


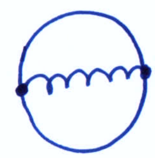


Look at colour indices alone

$\rightarrow \quad \bar{u}_i \, (t^a)_{ij} \, v_j \quad =$ 

 $\times$ ()* $=$ 

$\bar{u}_i \, (t^a)_{ij} \, v_j \quad \bar{v}_{j'} \, (t^b)_{j'i'} \, u_{i'}$

$=$ 

$= \sum_a (t^a)_{ij} (t^a)_{ji}$

$= Tr(t^a t^a)$

when we sum over ~~all d.o.f~~ spins.

$$\sum_s \tilde{u}_{j,s}(p) \, \bar{u}_{i,s}(p) = \not{p} \, \delta_{ij}$$

This looks similar to $\sum (\sigma^i)(\sigma^i)^*$

$$= \sum |S_x|^2 + S_y^2 + S_z^2$$

↳ Casimir operator.

$$C_F \equiv Tr(t^a t^a) = \frac{N^2-1}{2N} = 4/3$$

↳ SU(N)