A Simple Idea for Lattice QCD at Finite Density*

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* *Rajiv V. Gavai & Sayantan Sharma, arXiv:1406.0474*

Introduction

♠ Interest in QCD phase diagram led to considerations of Lattice QCD at finite density since early days (Karsch-Hasenfratz '83, Kogut et al. 1983, Bilić-Gavai '84).

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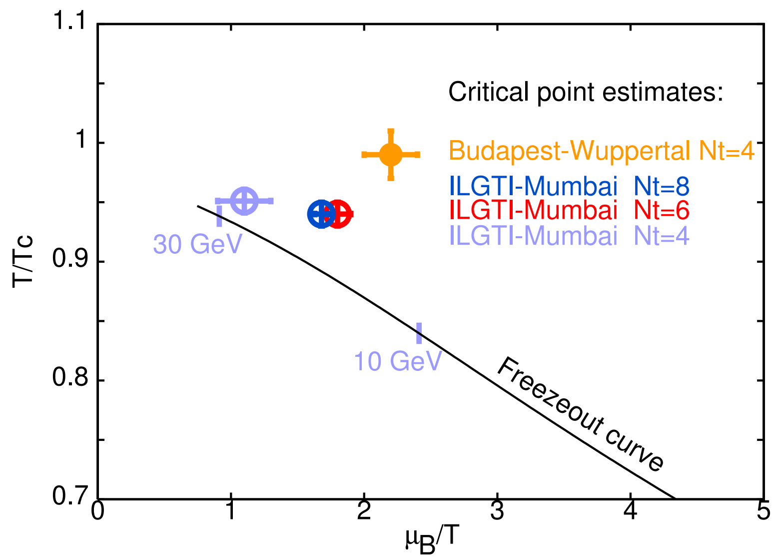
♡ Quark number susceptibility (QNS) soon followed suit, as 'order parameter' (McLerran '87) and as a new thermodynamical quantity (Gottlieb et al. '88, Gavai et al. '89)

♡ Higher order QNS led to the determination of QCD Critical point (Gavai-Gupta, PRD '05, PRD '09 & PoS Lattice 2013).



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♡ Recall that Partition Function $\mathcal{Z}$ is defined by

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♠ A global symmetry & Noether's theorem lead to the corresponding current conservation $\partial_\mu j_\mu(x) = 0 \rightsquigarrow N = \int d^3x j_0(\vec{x}) =$ Constant.

♡ One recasts $\mathcal{Z}$ as an Euclidean path integral over all fields and discretizes it for non-perturbative Lattice evaluations.

♣ A natural transcription of the derivative is forward-backward difference which leads to a N in the point-split form.

◇ Since the the naıvely discretized fermionic action is

$$S^F = \sum_{x,x'} \bar{\psi}(x) \left[\sum_{\mu=1}^{4} D^\mu(x,x') + ma\delta_{x,x'} \right] \psi(x'),$$

where

$$D^\mu(x,x') = \frac{1}{2}\gamma^\mu \left[U^\mu_x \delta_{x,x'-\hat{\mu}} - U^{\mu\dagger}_{x'} \delta_{x,x'+\hat{\mu}} \right],$$

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◇ Using the natural point-split form N amounts to weights $f(a\mu) = 1 + a\mu$ & $g(a\mu) = 1 - a\mu$ to forward and backward time links respectively.

♣ This leads to μ-dependent a^{-2} divergences in energy density and quark number density even in the free theory!

♠ Divergence not new : Presssure or Energy density has it at $T \neq 0$, and has to be subtracted as zero point energy. What is new is the μ-dependence of a divergent term. Problem due to Lattice ?

♡ Hasenfratz-Karsch (PLB 1983) & Kogut et al. (PRD 1983) proposed to modify the weights to $\exp(\pm a\mu)$ to obtain finite results while simultaneously Bilić-Gavai (EPJC 1984) showed $(1 \pm a\mu)/\sqrt{(}1 - a^2\mu^2)$ also lead to finite results for ideal gas.

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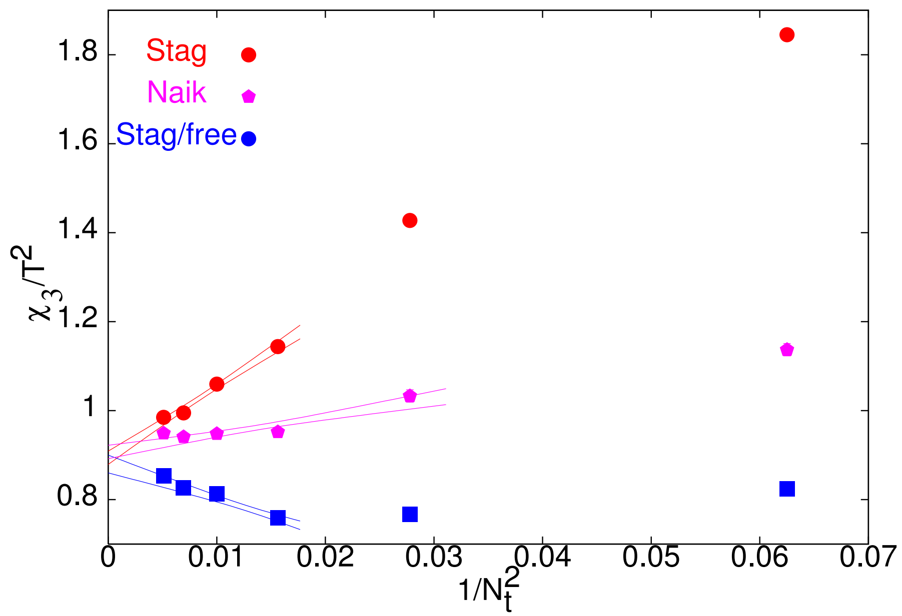
♡ However, the analytical proof was *only* for free quarks & thus pert. theory. Numerical computations had to be performed to show that it worked for the non-perturbative interacting case as well (Gavai-Gupta PRD 67, 034501 (2003)) :



♣ These computations, and all the others above, employed staggered fermions, which break flavour & spin symmetry for nonzero lattice spacing a. Chiral Models or symmetry-based model considerations suggest that $N_f = 2$ can have a critical point but not $N_f \geq 3$.

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- Need exact chiral *and* flavour symmetry for fermions for reliable QCD phase diagram investigations.

- Neuberger Overlap fermions, defined by $aD_{ov} = 1 + \gamma_5 \, \mathrm{sign}(\gamma_5 D_{Wilson})$, satisfy the Ginsparg-Wilson relation, $\{\gamma_5, D\} = aD\gamma_5 D$ and have exact chiral symmetry on lattice.

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- Overlap/Domain Wall Fermions – Almost like continuum; have *both* correct chiral and flavour symmetry on lattice. Even have an index theorem as well.
  (Hasenfratz, Laliena & Niedermeyer, PLB 1998; Lüscher PLB 1998.)

- Their *non-locality* makes it difficult to define conserved charge on the lattice, however. (Kikukawa & Yamada, NPB 1999; Mandula PRD 2009.)

♢ Bloch-Wettig (PRL 2006; PRD 2007) proposal : Use the same prescription for timelike links as for the local fermions and employ $D_{Wilson}(a\mu))$ in the overlap matrix.

♣ The resultant overlap fermion action also has no a^{-2} divergences (Gattringer-Liptak, PRD 2007; Banerjee, Gavai, Sharma, PRD 2008) in the free case.

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It is :

$$\begin{aligned} S^F &= \sum_n [\bar{\psi}_{n,L}(aD_{ov} + a\mu\gamma^4)\psi_{n,L} + \bar{\psi}_{n,R}(aD_{ov} + a\mu\gamma^4)\psi_{n,R}] \\ &= \sum_n \bar{\psi}_n[aD_{ov} + a\mu\gamma^4(1 - aD_{ov}/2)]\psi_n \ . \end{aligned}$$

- Easy to check that under the chiral transformations, $\delta\psi = i\alpha\gamma_5(1 - aD_{ov})\psi$ and $\delta\bar{\psi} = i\alpha\bar{\psi}\gamma_5$, it is invariant for all values of $a\mu$ and a.

- It reproduces the continuum action in the limit $a \to 0$ under $a\mu \to a\mu/M$ scaling, M being the finite irrelevant parameter in overlap action.

- Order parameter exists for all $a\mu$ and T. It is
$\langle\bar{\psi}\psi\rangle = \lim_{am\to 0}\lim_{V\to\infty}\langle \mathrm{Tr}\ \frac{(1-aD_{ov}/2)}{[aD_{ov}+(am+a\mu\gamma^4)(1-aD_{ov}/2)]}\rangle$.

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- It, however, has a^{-2} divergences which cannot be removed by exponentiation of the μ-term (Narayanan-Sharma, JHEP 2011).

- The Overlap fermion dilemma : Either exact chiral invariance on lattice or divergences in $a \to 0$ limit.

Tackling the Divergences

- Opt for exact chiral invariance & learn to tackle the divergences.

- Note that contrary to common belief, μ-dependent divergences are **NOT** a lattice artifact. Indeed lattice regulator simply makes it easy to spot them. Using a momentum cut-off Λ in the continuum theory, one can show the presence of $\mu\Lambda^2$ terms in number density easily (Gavai-Sharma, 1406.0474).

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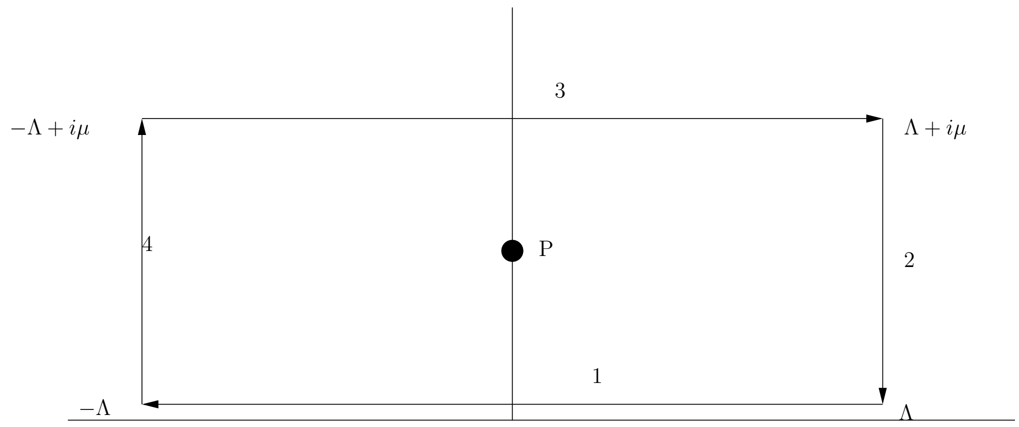
- The expression for the number density is

$$n = \frac{2iT}{V} \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{(\omega_n - i\mu)}{p^2 + (\omega_n - i\mu)^2} \equiv \frac{2iT}{V} \int \frac{d^3p}{(2\pi)^3} \sum_{\omega_n} F(\omega_n, \mu, \vec{p}), \qquad (1)$$

where $p^2 = p_1^2 + p_2^2 + p_3^2$. Our gamma matrices are all Hermitian.

- Vaccum contribution is removed by subtracting $n(T = 0, \mu = 0)$.

- In the usual contour method, but with a cut-off Λ, one has in the $T = 0$ but $\mu \neq 0$ case the following in the complex p_0-plane:



(2)

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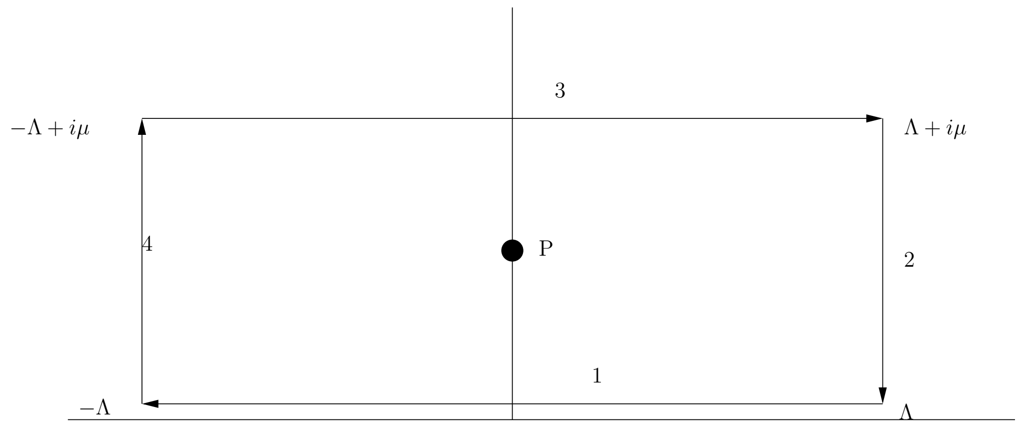

- The $\mu\Lambda^2$ terms arise from the arms 2 & 4 in figure above. (Gavai-Sharma, arXiv 1406.0474) :

$$
\begin{aligned}
\text{Sum of } 2+4 &= \int \frac{d^3p}{(2\pi)^3} \left(\int_2 + \int_4 \right) \frac{d\omega}{\pi} \frac{\omega}{p^2 + \omega^2} \\
&= -\frac{1}{2\pi} \int \frac{d^3p}{2\pi^3} \ln \left[\frac{p^2 + (\Lambda + i\mu)^2}{p^2 + (\Lambda - i\mu)^2} \right] .
\end{aligned} \tag{2}
$$

- One usually *assumes* this term to cancel for $\mu \neq 0$ by setting Λ infinite. However, since $\Lambda \gg \mu$, expanding in μ/Λ, one finds the leading Λ^3 terms indeed cancel but there is a nonzero coefficient for the $\mu\Lambda^2$ term.

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- Ignoring the contribution from the arms 2 & 4 amounts to a subtraction of the 'free theory divergence' in continuum !

- Indeed, the integrals 2 & 4 are symmetric under $\omega \to -\omega$ but the limits are not. Even without a cut-off, it is clear that the resultant subtraction is divergent. $\because$ μ-dependent integrals $\Longrightarrow$ μ-dependent divergence.

- Why was the lattice result then a 'discovery' in 80's ? Close look at Kapusta's textbook reveals use of T itself as cut-off, and hence $T > 0$. $T = 0$ obtained only as a limit of the final result.

- Computations within an interacting theory can not avoid $T = 0$. e.g. $T = 1/N_t a$ is zero for $N_t \to \infty$ for any a.

- In fact, in $\mathcal{Z} = \mathrm{Tr}\ \exp[-\beta(H - \mu N)]$, normal ordering ensures 'Dirac sea' does not contribute. The μ-dependent divergence is its manifestation as Euclidean path integrals do not normal-order!

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- One may as well follow the canoninical μN form, and a prescription of subtracting the free theory divergence by hand on Lattice. *Advantage:* If it works, one has several computational benefits in computing the higher order susceptibilities needed in critical point search.

- Indeed, for any fermion it leads to
  $M(a\mu) = M(a\mu = 0) + a\mu \sum_{x,y} N(x,y)$, and therefore,
  $M' = \sum_{x,y} N(x,y)$, and $M'' = M''' = M''''... = 0$,
  in contrast to the exp($\pm a\mu$)-prescription where *all* derivatives are nonzero:
  $M', M'''... \neq 0$ and $M'', M'''', M''''''... \neq 0$.

- Consequently, one has fewer terms in the Taylor coefficients, especially as the order increases. E.g., in the 4th order susceptibility, $\mathcal{O}_4 = -6 \ \mathrm{Tr}\ (M^{-1}M')^4$ in the linear case, compared to
$\mathcal{O}_4 = -6 \ \mathrm{Tr}\ (M^{-1}M')^4 + 12 \ \mathrm{Tr}\ (M^{-1}M')^2 M^{-1}M'' - 3 \ \mathrm{Tr}\ (M^{-1}M'')^2 - 3 \ \mathrm{Tr}\ M^{-1}M'M^{-1}M''' + \mathrm{Tr}\ M^{-1}M''''$.

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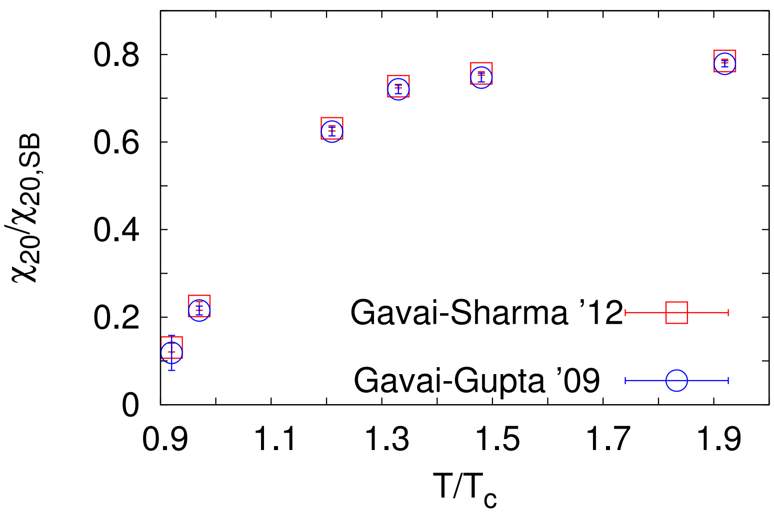
- $\mathcal{O}_8$ has one term in contrast to 18 in the usual case. $\Longrightarrow$ Less Number of M^{-1} computations needed & perhaps lesser cancellations too.

- The resultant computer time savings can be up to a factor of two, with still better error control (due to less cancellations). Moreover, higher orders crucially needed to establish the reliability can perhaps be more easily obtained.

- Makes Lattice QCD at finite density thus simpler.

Testing the idea

♠ Using our proposed μN term (Gavai-Sharma PRD 2010) to evaluate (Gavai-Sharma, arXiv 1111.5428, PRD 2012) the baryon susceptibility at $\mu = 0$,



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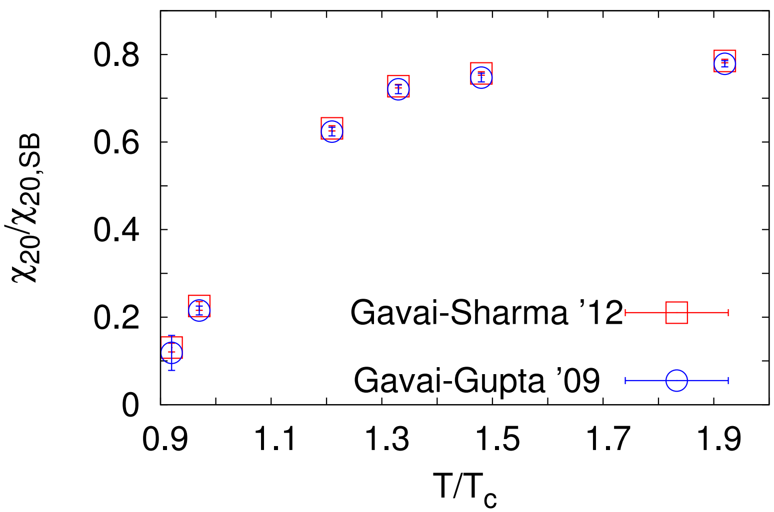

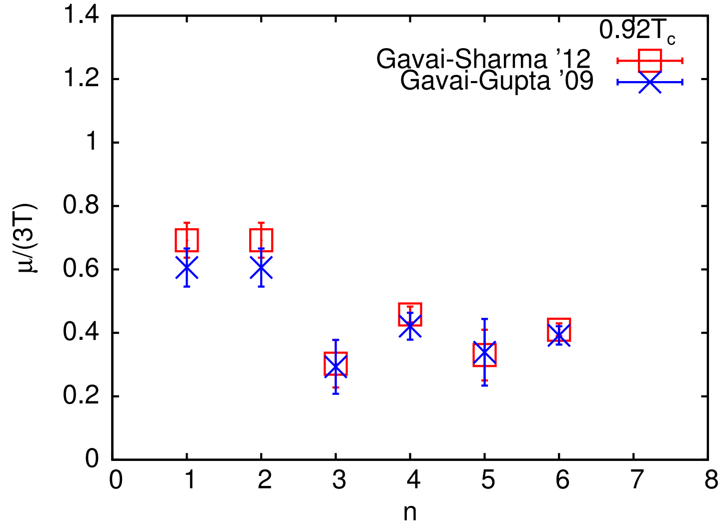

♡ ALL NLS Coefficients do have the same sign for the new method.

♠ The estimates for radius of convergence are comparable as well.

But, Divergence . . . ?

- In order to test whether the divergence is truely absent, one needs to take the continuum limit $a \to 0$ or equivalently $N_t \to \infty$ at fixed $T^{-1} = aN_t$.

- We tested it for quenched QCD. For $m/T_c = 0.1$, we employed $N_t = 4$, 6, 8, 10 and 12 lattices and 50-100 independent configurations and computed different susceptibilities at $T/T_c = 1.25$, & 2.

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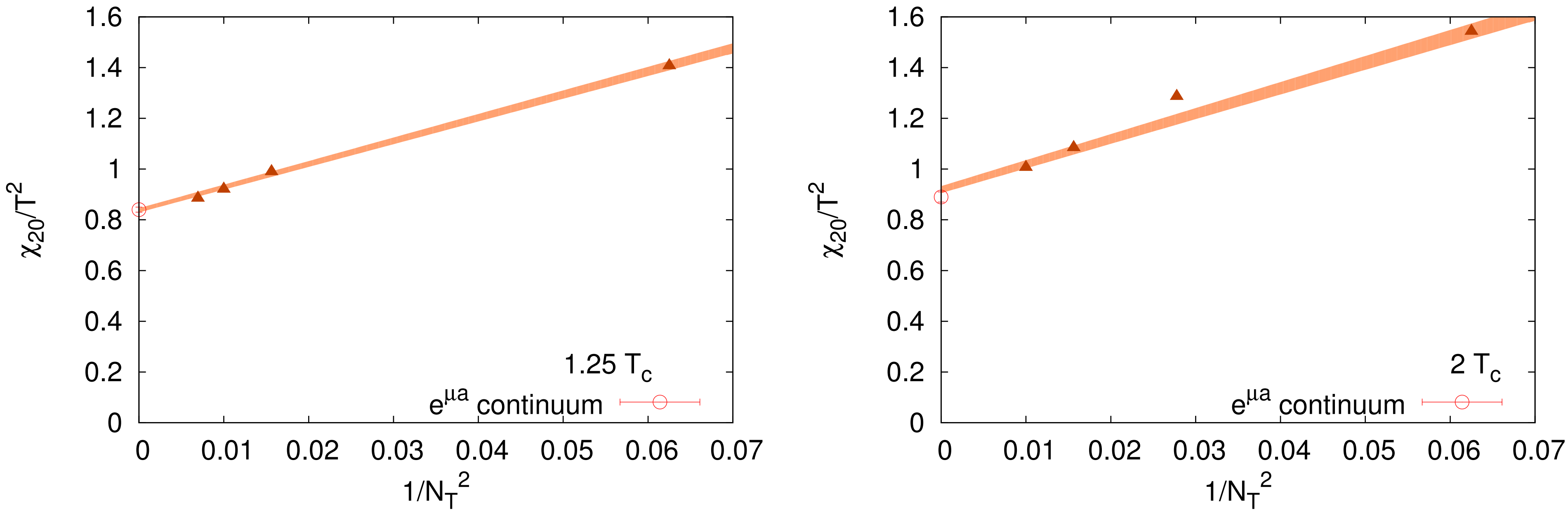
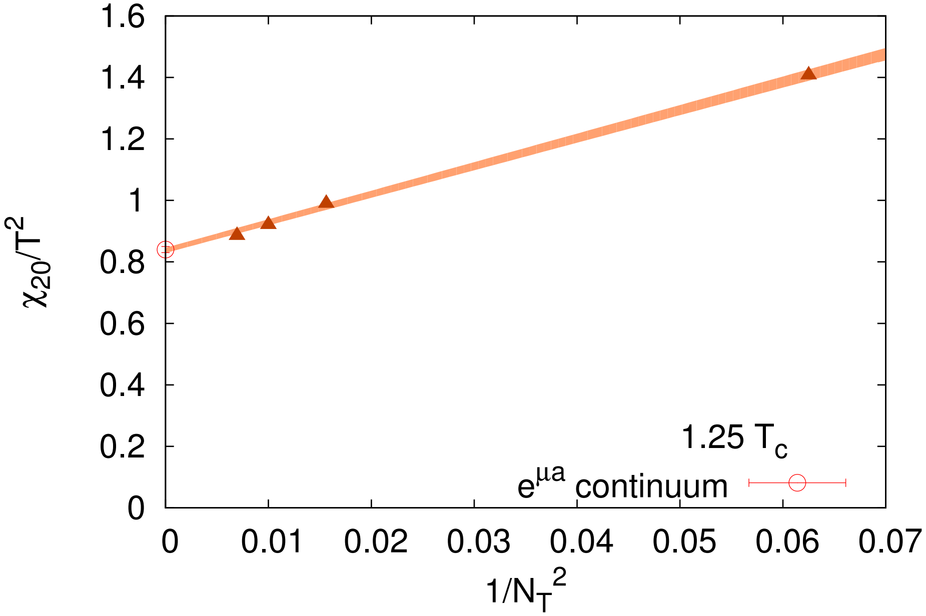
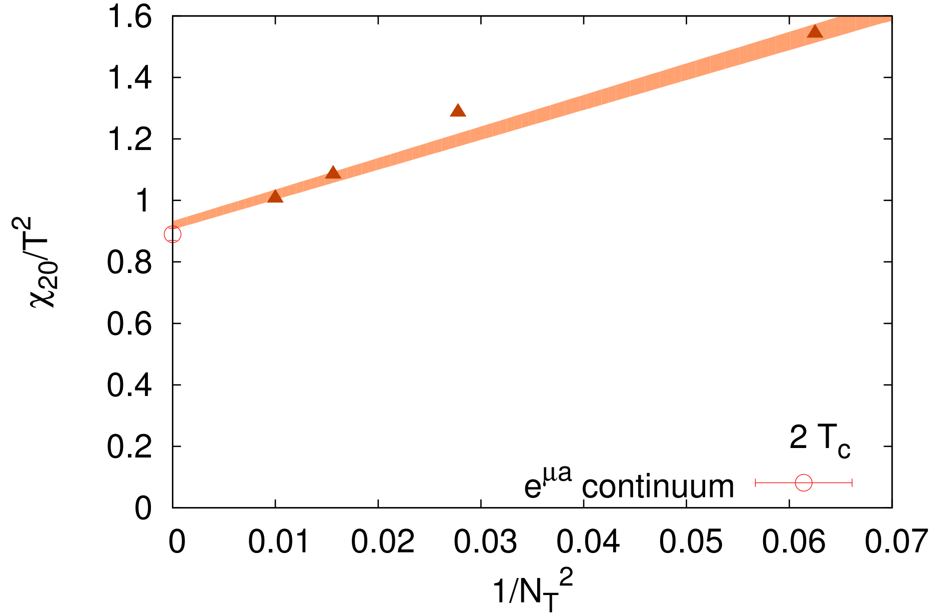
- $1/a^2$-term for free fermions on the corresponding $N^3 \times \infty$ lattice was subtracted from the computed values of the susceptibility.

- Expect χ_{20}/T^2 to behave as $\chi_{20}/T^2 = c_1(T) + c_2(T)N_T^2 + c_3(T)N_T^{-2} + \mathcal{O}(N_T^{-4})$.



- Absence of any quadratically divergent term is evident in the positive slope of the data. Logarithmic divergence cannot be ruled out with our limited N_t data.

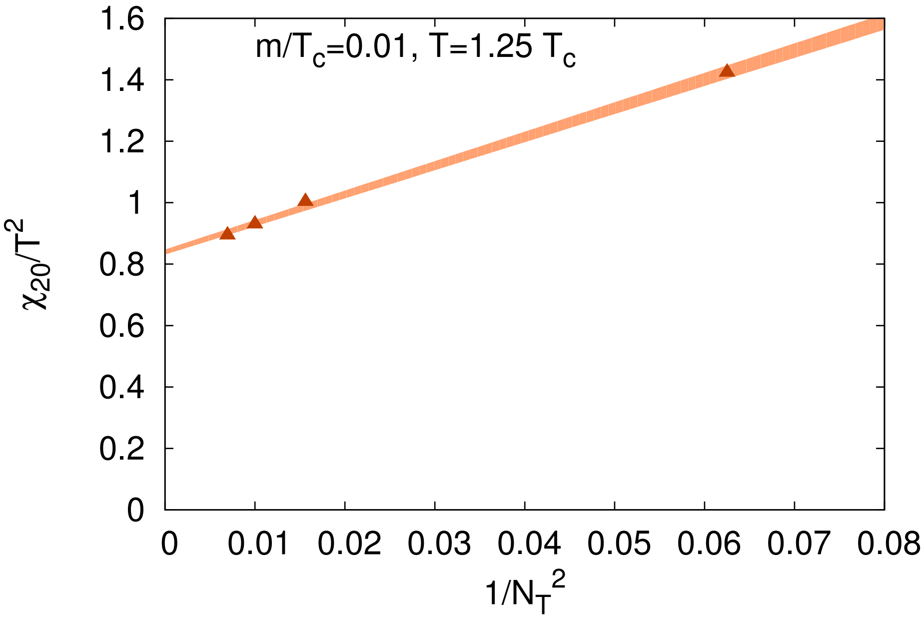




- Absence of any quadratically divergent term is evident in the positive slope of the data. Logarithmic divergence cannot be ruled out with our limited N_t data.

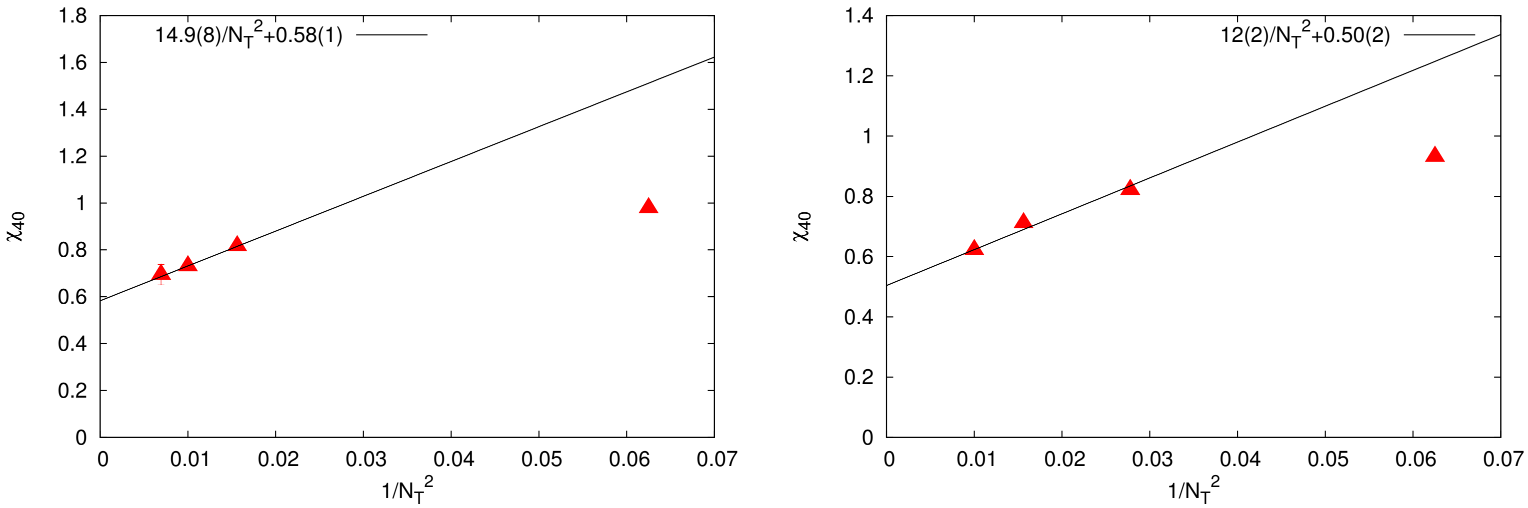
- Furthermore, our extrapolated continuum result coincides with the earlier result obtained with the $\exp(\pm a\mu)$ action (Swagato Mukherjee PRD 2006).

- We lowered the mass by a factor of 10 to $m/T_c = 0.01$ & repeated the exercise at a lower temperature on $T/T_c = 1.25$.



- Again no divergent term is evidently present in the slope of the data.

- Higher order susceptibility show similar finite result in continuum limit:


14.9(8)/N_T^2+0.58(1)
χ40
1/N_T^2
12(2)/N_T^2+0.50(2)
χ40
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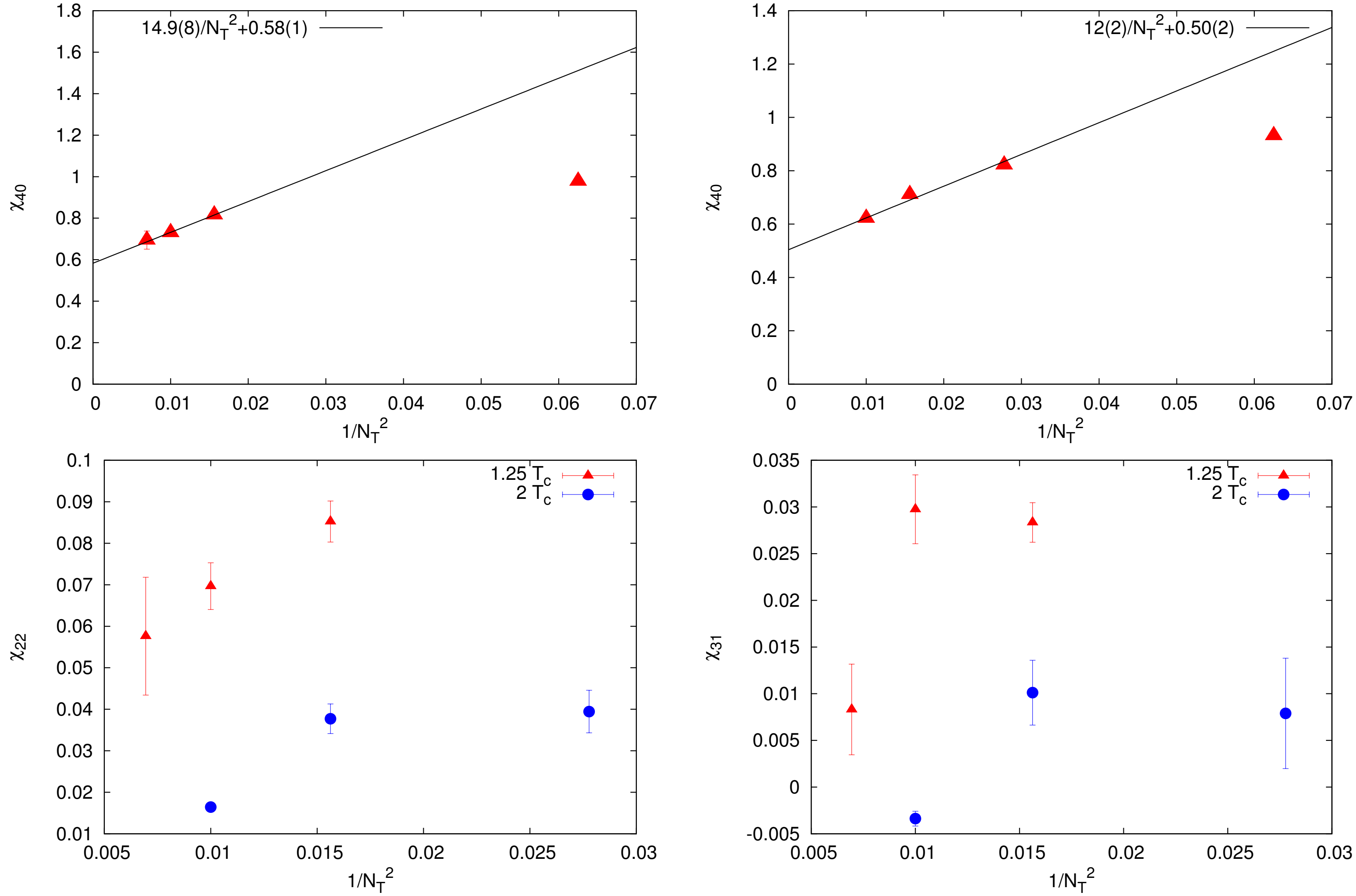
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2 T_c
χ22
1/N_T^2
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2 T_c
χ31
1/N_T^2

Summary

- Actions linear in μ can be employed safely, and may have computational advantages.

- Divergence in the quark number susceptibility can be subtracted off by the corresponding free theory result. Continuum extrapolation yields the same result for both the linear and the exponential form, as it must.

Summary

- Actions linear in μ can be employed safely, and may have computational advantages.
- Divergence in the quark number susceptibility can be subtracted off by the corresponding free theory result. Continuum extrapolation yields the same result for both the linear and the exponential form, as it must.
- Interactions do not induce any additional divergence at finite T or μ once the zero temperature divergence is removed. This has been well known perturbatively but seems to hold non-perturbatively as well.
- Conserved charge N should not get renormalized.