Exact Chiral Invariance at Finite Density on Lattice

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**arXiv : 1111.5944*

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Introduction

Exact Chiral Invariance for $\mu \neq 0$

Physical Picture

Summary

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Introduction

♠ A fundamental aspect of the QCD Phase Diagram is the Critical Point in the T-μ_B plane, expected on the basis of symmetries and models.

♠ Would be nice to have a first principles determination of its location in the phase diagram
$\Longrightarrow$ Lattice QCD.

Introduction

♠ A fundamental aspect of the QCD Phase Diagram is the Critical Point in the T-μ_B plane, expected on the basis of symmetries and models.

♠ Would be nice to have a first principles determination of its location in the phase diagram
$\Longrightarrow$ Lattice QCD.

♡ Two light flavours of quarks are crucial for critical point, as is the exact chiral symmetry on the lattice when the quark mass is tuned to zero.

♠ Three (or more) massless flavours have a First Order Chiral transition for $\mu = 0$, while for two massless ones it is Second order.

♡ Temperature dependence of the Chiral Anomaly may be important as well; No CEP if instanton density is small enough below T_{ch} (Pisarski-Wilczek, 1984).

♢ Type of quarks thus become very important. Lattice fermions have a well-known "No-Go" theorem due to Nielsen-Ninomiya.

Popular choices :

- Wilson Fermions – Break *all* chiral symmetries. Well defined flavour and spin, however.

- Kogut-Susskind Fermions – Have some chiral symmetry *but* break flavour and spin symmetry. Flavour singlet axial symmetry is broken.

- Graphene or Creutz-Boriçi Fermions – Have some chiral *and* spin symmetry *but* do break flavour symmetry. Only flavour nonsinglet axial symmetry.

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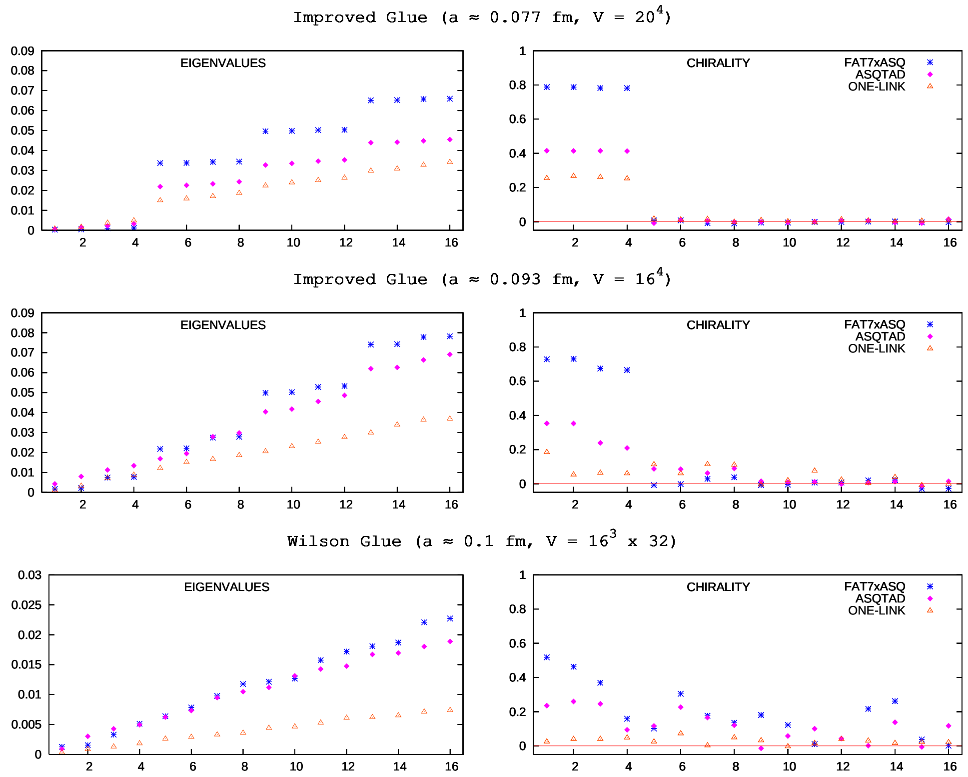
- Overlap/Domain Wall Fermions – Almost like continuum; have *both* correct chiral and flavour symmetry on lattice. Even have an index theorem as well. (Hasenfratz, Laliena & Niedermeyer, PLB 1998; Lüscher PLB 1998.)
- Note that chemical potential, μ_B, has to be introduced without violating the symmetries in order to investigate the entire T-μ_B plane.

♠ The staggered fermions are oft-used, but one needs to have $N_t \geq 13$ for restoration of full flavour symmetry.



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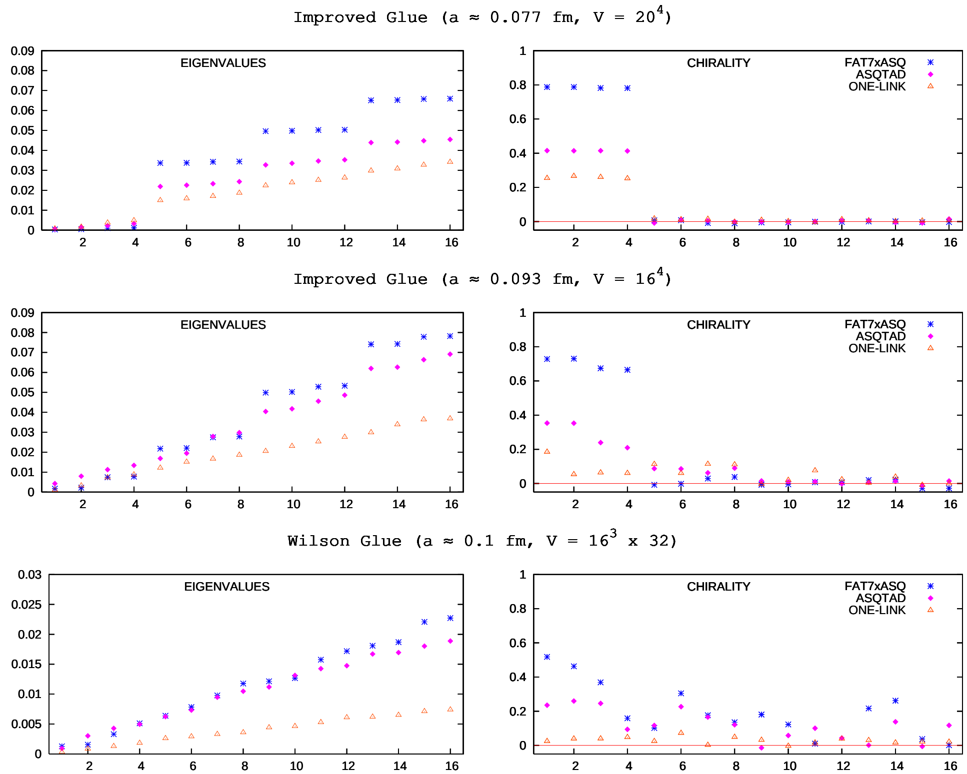
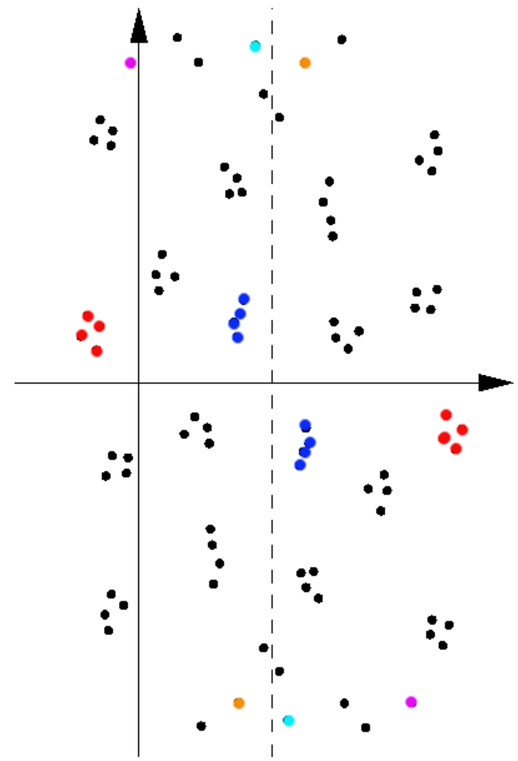


E. Follana et al., PRD 2005



Golterman, Shamir & Svetitsky, PRD 2006.

♡ Rooting appears to have more problems for nonzero μ. Use of Overlap fermions therefore seems desirable.

Introducing Chemical Potential

- Ideally, one should construct the conserved charge, N, as a first step, and add μN. But this leads to a^{-2} divergences in the continuum limit.

- Multiply gauge links in positive/negative time direction by $\exp(a\mu)$ and $\exp(-a\mu)$ respectively. No change in chiral invariance for staggered fermions, as a result. (Hasenfratz-Karsch 1982; Kogut et al. 1982; Bilic-Gavai 1983).

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- Non-locality makes the construction of N difficult for the Overlap case, even non-unique (Mandula, 2007).

- Bloch-Wettig (PRL 2006; PRD 2007) proposal : Use the same prescription as above, i.e., $D_W(0) \to D_W(a\mu)$ in the sign function: $D_{ov} = 1 + \gamma_5 \; sgn(\gamma_5 D_W)$.

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- We (Banerjee, Gavai, Sharma, PRD 2008; PoS Lattice 2008; Gattringer-Liptak, PRD 2007) showed that although the resultant overlap fermion action has no a^{-2} divergences, unfortunately it has no chiral invariance for nonzero μ either.

♠ Exact chiral invariance for a lattice fermion operator D is assured if it satisfies the Ginsparg-Wilson relation : $\{\gamma_5, D\} = aD\gamma_5 D$.

♠ In particular, the chiral transformations (Lüscher, PLB 1999) $\delta\psi = i\alpha\gamma_5(1 - \frac{a}{2}D)\psi$ and $\delta\bar{\psi} = i\alpha\bar{\psi}(1 - \frac{a}{2}D)\gamma_5$, leave the action $S = \sum \bar{\psi}D\psi$ invariant.

♣ The chiral invariance is lost for nonzero μ, since

$$\delta S = i\alpha \sum_{x,y} \bar{\psi}_x \Big[\gamma_5 D(a\mu) + D(a\mu)\gamma_5 - \frac{a}{2}D(0)\gamma_5 D(a\mu) - \frac{a}{2}D(a\mu)\gamma_5 D(0)\Big]_{xy} \psi_y,$$

under Lüscher's chiral transformations.

♣ However, the sign function definition of Bloch-Wettig merely ensures

$$\gamma_5 D(a\mu) + D(a\mu)\gamma_5 - a\ D(a\mu)\gamma_5 D(a\mu) = 0,$$

which is not sufficient to make $\delta S = 0$.

What if ...

♠ the chiral transformations were $\delta\psi = i\alpha\gamma_5(1 - \frac{a}{2}D(a\mu))\psi$ and $\delta\bar{\psi} = i\alpha\bar{\psi}(1 - \frac{a}{2}D(a\mu))\gamma_5$?

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- Leads to symmetry groups *different* at each μ. Recall we wish to investigate $\langle\bar{\psi}\psi\rangle(a\mu)$ to explore if chiral symmetry is restored.

- The symmetry group remains *same* at each T with $\mu = 0$ $\Longrightarrow \langle\bar{\psi}\psi\rangle(am = 0, T)$ is an order parameter for the chiral transition.

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- Not allowed since $\gamma_5 D(a\mu)$ is not Hermitian, unless $a\mu = 0$.

- Symmetry transformations should not depend on "external" parameter μ. Chemical potential is introduced for charges N_i with $[H, N_i] = 0$. At least the symmetry should not change as μ does.

- We propose a way to introduce μ for overlap fermions without breaking lattice exact chiral symmetry.

- We also show :

  1) why it is physically more motivated to do so, and

  2) it may have the correct anomaly for small enough a.

Exact Chiral Invariance for $\mu \neq 0$

- Choosing $\psi_L = (1-\gamma_5)\psi/2$ & $\psi_R = (1+\gamma_5)\psi/2$, and $\bar\psi_L = \bar\psi(1+\gamma_5)/2$ & $\bar\psi_R = \bar\psi(1-\gamma_5)/2$, chiral invariance in continuum can be made manifest :

  $S_{QCD} = \int d^3x \ d\tau [\bar\psi_L(\not{D} + \mu\gamma^4)\psi_L + \bar\psi_R(\not{D} + \mu\gamma^4)\psi_R - F^{\mu\nu}F_{\mu\nu}/4]$.

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- Our proposal (Gavai-Sharma arXiv :1111.5944) is to follow the same idea in writing down the lattice action at nonzero μ to preserve exact chiral invariance.

- Chiral projectors for overlap fermions can be defined as $\psi_L = [1 - \gamma_5(1 - aD_{ov})]\psi/2$ & $\psi_R = [1 + \gamma_5(1 - aD_{ov})]\psi/2$, leaving the antiquark field decomposition as in the continuum.
  $\Longrightarrow$ Ready to define our new action.

- Overlap action for nonzero μ is

$$
\begin{aligned}
S^F &= \sum_n [\bar\psi_{n,L}(aD_{ov} + a\mu\gamma^4)\psi_{n,L} + \bar\psi_{n,R}(aD_{ov} + a\mu\gamma^4)\psi_{n,R}] \\
&= \sum_n \bar\psi_n [aD_{ov} + a\mu\gamma^4(1 - aD_{ov}/2)]\psi_n \ .
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$$= \sum_n \bar\psi_n[aD_{ov} + a\mu\gamma^4(1 - aD_{ov}/2)]\psi_n \ .$$

- Easy to check that under the chiral transformations, $\delta\psi = i\alpha\gamma_5(1 - aD_{ov})\psi$ and $\delta\bar\psi = i\alpha\bar\psi\gamma_5$, it is invariant or all values of $a\mu$ and a.

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- It reproduces the continuum action in the limit $a \to 0$ under $a\mu \to a\mu/M$ scaling, M being the irrelevant parameter in overlap action.

- Order parameter exists for all μ and T. It is $\langle\bar\psi\psi\rangle = \lim_{am\to 0} \lim_{V\to\infty} \langle \mathrm{Tr}\ \frac{(1-aD_{ov}/2)}{[aD_{ov}+(am+a\mu\gamma^4)(1-aD_{ov}/2)]} \rangle$.

Physical Picture via Domain Wall Fermions

- Domain wall fermions are defined using an extra—fifth—dimension. Gauge fields reside in the 4-d world only.

- Fermion fields at the boundary of the 5th dimension define the physical fermion fields of 4-d :

  $\Psi = P_{-}\psi(x,1) + P_{+}\psi(x,N_5)$, $\bar{\Psi} = \bar{\psi}(x,1)P_{+} + \bar{\psi}(x,N_5)P_{-}$,

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  with $P_\pm$ as the continuum chiral projectors.

- 4-d Fermion determinant is obtained by dividing out the heavier 5-d bulk modes from that of 5-d. It then is $\det D^{(5)}(ma) / \det D^{(5)}(ma = 1)$.

- We (RVG-Sharma, 1111.5944) therefore propose to count only the 5-d zero modes as the number density N, and couple μ only to the boundary modes: For nonzero μ, we add $a\mu \sum_{(x,s)} [\bar{\Psi}(x, s) \ \gamma_4 \Psi(x, s)]$.

- We then follow Edwards-Heller (PRD 63 (2001) 094505) method to check what it leads to in the large N_5-limit.

- Using $H_W = \gamma_5 D_W$, one introduces η_i (and $\bar{\eta}_i$) fields residing at each site i along the 5th direction. These are related to the 5-d fermion fields in an asymmetric way:

  $\bar{\eta}_i = \bar{\psi}_i \gamma_5 (a_5 H_W P_- - 1)$ and $\eta_i = P_+ \psi_{i-1} + P_- \psi_i$

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- Integrating the η-fields successively one by one, one obtains $\det D^{(5)}$ in terms of a transfer matrix $T = (1 + a_5 H_W P_+)^{-1} (1 - a_5 H_W P_-)$.

- Since we couple μ only to the boundary fermions, T clearly remains unaffected, as does the bulk mode determinant. This is in contrast to the Bloch-Wettig idea. For their proposal, T is μ-dependent as is the bulk mode determinant.

- All that changes for our case then is that in the determinant ratio above $\det D^{(5)}(ma)$ gets replaced by $\det D^{(5)}(ma, \mu a) =$ $\det[D^{(5)}(ma) + a\mu\{(a_5 H_W P_- - 1)^{-1}\gamma_4 P_- - T^{-N_5}(a_5 H_W P_+ + 1)^{-1}\gamma_4 P_+\}]$.

- After some algebra, one can show that the $N_5 \to \infty$, $a_5 \to 0$ limit leads to the $same$ action we proposed above.

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- After some algebra, one can show that the $N_5 \to \infty$, $a_5 \to 0$ limit leads to the $same$ action we proposed above.

- Please note that the form of the Dirac matrix we obtain is identical to that of Narayanan-Sharma, JHEP 1110, 151.

- They needed sources to define chiral symmetry as arising from their chiral rotations. These chiral symmetry transformations were local. Chiral rotation on quark fields could, however, not be defined there.

- We employ the standard (nonlocal) chiral transformations for quark fields, leading as a direct consequence to an order parameter on the lattice for all μ.

Chiral Anomaly

- Our $D(a\mu)$, defined above, is not γ_5-hermitian. In general, it is not clear whether it may be diagonalizable.

- Since an M-scaling is essential for correct continuum limit, and since $a\mu/M$ can be made small for small enough a, one can look at the chiral anomaly in that approximation at $\mathcal{O}((a\mu/M)^2)$.

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- For $a\mu = 0$, D is diagonalizable. Its eigenvalues come in pairs (λ, λ^*). Using these properties, we showed that chiral anomaly arises, and is governed by the number of zero modes, as usual.

- First order perturbation theory in $a\mu/M$ was used to show that both these statements remain unchanged for nonzero μ on a fine enough lattice.

Summary

- Using the analogy with continuum QCD, we demanded manifest chiral invariance in terms of L and R-fields on the lattice to obtain $aD_{ov} + a\mu\gamma^4(1 - aD_{ov}/2)$ as the exact chiral invariant form.

- The chiral invariance on lattice is exact for all μ and T, leading to an order parameter to test SSB in QCD the phase diagram.

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- The chiral invariance on lattice is exact for all μ and T, leading to an order parameter to test SSB in QCD the phase diagram.

- Chiral anomaly remains unaffected in continuum by nonzero μ (Gavai-Sharma PRD 2010). It is indeed so for our proposal for small enough lattice spacing a.