Anomaly at Finite Density & Chiral Fermions on Lattice

*Rajiv V. Gavai & Sayantan Sharma**
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* *arXiv : 0906.5188, submitted to Phys. Rev. D.*

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Introduction

Anomaly for $\mu \neq 0$: Continuum

Two simple ideas for Lattice

Summary

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Introduction

- The Nielsen-Ninomiya Theorem (1981) : For each set of quantum numbers, there are an equal number of *left*-handed and *right*-handed particles in the lattice fermion propagator.

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- No Chiral Anomaly on the lattice for the naıve fermions.

- Canonical – local – fermion formulations (Wilson, Kogut-Susskind, Twisted mass, Creutz-Boriçi..) break the flavour singlet axial U(1).

- Overlap fermions, which are nonlocal, do better. Have an index theorem as well. (Hasenfratz, Laliena & Niedermeyer, PLB 1998; Luscher PLB 1998.)

QCD Phase diagram

♠ A fundamental aspect of the QCD Phase Diagram is the Critical Point in the T-μ_B plane expected on the basis of symmetries and models.

♡ In particular, two light flavours of quark are crucial for it, as is the exact chiral symmetry on the lattice when the quark mass is tuned to zero.

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♠ The oft-used staggered fermions have (some) chiral symmetry but a not so well defined flavour number. Use of Overlap fermions seem desirable.

♡ Note that chemical potential, μ, has to be introduced without violating the symmetries in order to investigate the entire T-μ_B plane.

Introducing Chemical Potential

- Ideally, one should construct the conserved charge, N, as a first step. Adding simply μN leads to a^{-2} divergences in the continuum limit.

- Multiply gauge links in positive/negative time direction by $\exp(a\mu)$ and $\exp(-a\mu)$ respectively. No change in chiral invariance as a result. (Hasenfratz-Karsch 1982; Kogut et al. 1982; Bilic-Gavai 1983).

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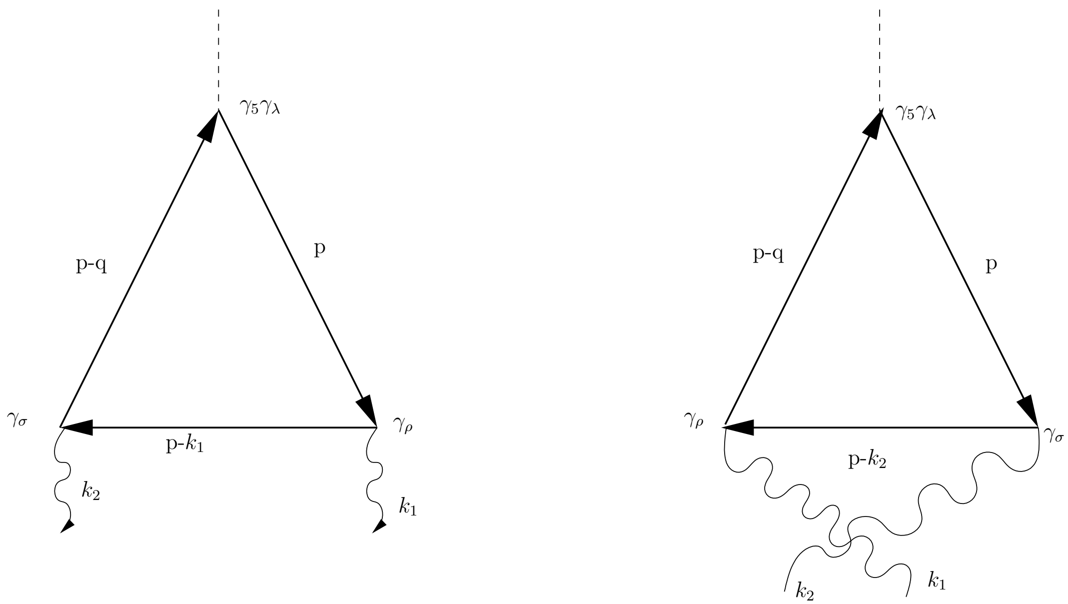
- We (Banerjee, Gavai, Sharma, PRD 2008; PoS Lattice 2008) showed that the resultant overlap fermion action has no chiral invariance for nonzero μ.

Anomaly for $\mu \neq 0$: Continuum results



- Perturbatively we need to compute $\langle \partial_\mu j^5_\mu \rangle$, i.e., the triangle diagrams for $\mu \neq 0$.

- Denoting by $\Delta^{\lambda\rho\sigma}(k_1, k_2)$ the total amplitude and contracting it with q_λ,

$$
\begin{aligned}
q_\lambda \Delta^{\lambda\rho\sigma} &= -i\ g^2 \mathrm{tr}[T^a T^b] \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\ \left[\gamma^5 \frac{1}{\not p - \not q - i\mu\gamma^4} \gamma^\sigma \frac{1}{\not p - \not k_1 - i\mu\gamma^4} \gamma^\rho \right.\\
&- \gamma^5 \frac{1}{\not p - i\mu\gamma^4} \gamma^\sigma \frac{1}{\not p - \not k_1 - i\mu\gamma^4} \gamma^\rho + \gamma^5 \frac{1}{\not p - \not q - i\mu\gamma^4} \gamma^\rho \frac{1}{\not p - \not k_2 - i\mu\gamma^4} \gamma^\sigma \\
&- \left. \gamma^5 \frac{1}{\not p - i\mu\gamma^4} \gamma^\rho \frac{1}{\not p - \not k_2 - i\mu\gamma^4} \gamma^\sigma \right] .
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- Quadratic Divergent integrals; need cut-off which should be gauge invariant since

$$k_{1\rho} \Delta^{\lambda\rho\sigma}(k_1, k_2) = k_{2\sigma} \Delta^{\lambda\rho\sigma}(k_1, k_2) = 0.$$

- Can be done as for $\mu = 0$ by writing $q_\lambda \Delta^{\lambda\rho\sigma} =$ $(-i)\ \ \mathrm{tr}\ [T^a T^b] g^2 \int \frac{d^4p}{(2\pi)^4} [f(p - k_1, k_2) - f(p, k_2) + f(p - k_2, k_1) - f(p, k_1)]$.

- Due to nonzero μ, the function f has $(p_4^2 + \vec{p}^2) \to ((p_4 - i\mu)^2 + \vec{p}^2)$ in the denominator and terms proportional to μ and μ^2 in the numerator.

- Since the μ^2 terms have Tr $[\gamma^5\gamma^4\gamma^\sigma\gamma^4\gamma^\rho] \sim \epsilon^{4\sigma4\rho}$, they vanish.

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- The final result is $\propto f$ due to the structure above.

- Scaling the integration variable by the cut-off Λ, the μ-dependent terms appear with Λ^{-1}, leading to μ independence as $\Lambda \to \infty$: The same anomaly relation as for $\mu = 0$.

- In agreement with earlier calculations in real time (Qian, Su & Yu ZPC 1994; Gupta-Nayak 1997) or Minkowski space-time (Hsu, Sannino & Schwetz MPLA 2001) .

Anomaly for $\mu \neq 0$: Fujikawa method

- Under the chiral transformation of the fermion fields, given by,

$$\psi' = \exp(i\alpha\gamma_5)\psi \quad \text{and} \quad \bar{\psi}' = \bar{\psi}\exp(i\alpha\gamma_5) \ , \tag{1}$$

the measure changes as

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \mathrm{Det}|\frac{\partial(\bar{\psi}', \psi')}{\partial(\bar{\psi}, \psi)}| = \exp(-2i\alpha \int d^4x \ \mathrm{Tr}\gamma_5) \ . \tag{2}$$

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- Evaluate the trace using the eigenvectors of the operator $\not{D}$ for $\mu = 0$.

- Since $\{\gamma_5, \not{D}\} = 0$, for each finite λ_n , $\phi_n^{\pm} = \phi_n \pm \gamma_5 \phi_n$, eigenvectors of γ_5 with ± 1 eigenvalues, can be used, leading to zero.

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- Define two new vectors, ζ_m and υ_m

$$\zeta_m(\mathbf{x},\tau) = \mathrm{e}^{\mu\tau}\phi_\mathrm{m}(\mathbf{x},\tau) \quad , \quad \upsilon^\dagger_\mathrm{m}(\mathbf{x},\tau) = \phi^\dagger_\mathrm{m}(\mathbf{x},\tau)\mathrm{e}^{-\mu\tau} \ . \tag{3}$$

- Easy to show that ζ_m ($\upsilon^\dagger_m$) is the eigenvector of $\not{D}(\mu)$ ($\not{D}(\mu)^\dagger$) with the same (purely imaginary) eigenvalue λ_m (-λ_m).

- Further, one can show $\sum_m \int \zeta_m(\mathbf{x},\tau) \upsilon^\dagger_m(\mathbf{x},\tau)\ d^4x = \mathbf{I}$ and $\int \upsilon^\dagger_m(\mathbf{x},\tau)\zeta_m(\mathbf{x},\tau)\ d^4x = 1$.

- Using these eigenvector spaces of $\not{D}(\mu)$, trace of γ_5 can again be shown to be zero for all non-zero λ_m, leading to Tr $\gamma_5 = n_+ - n_-$ for $\mu \neq 0$ as well.

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- Both perturbatively, and nonperturbatively, we have shown that the anomaly does not change at finite density, as may have been expected naıvely.

- If chiral transformation on lattice is chosen to depend on μ, so that Bloch-Wettig proposal has chiral invariance for $\mu \neq 0$, then the resulting index theorem has μ*-dependent* zero modes which determine the anomaly, unlike in the continuum.

- It is undesirable for other reasons we pointed out in Lattice 2008.

A "Gauge-like" Symmetry

- A non-unitary transformation of the fermion fields of the QCD action in the presence of μ, given by $\psi^{'}(\mathbf{x},\tau) = \mathrm{e}^{\mu\tau}\psi(\mathbf{x},\tau)$, $\bar{\psi}^{'}(\mathbf{x},\tau) = \bar{\psi}(\mathbf{x},\tau)\mathrm{e}^{-\mu\tau}$, makes the action μ-independent: $S_F(\mu) \to S'_F(0)$.

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- Easy to see that it works for any local fermion action, including for the lattice action, with $\mu\tau$ generalized $f(\mu a_4) * n_4$.

- Generalization for non-local cases, Overlap fermions ? not possible ?

Two simple ideas for Lattice

- Only fermions confined to the domain wall are physical, so introduce a chemical potential only to count them:

$$D_{ov}(\hat{\mu})_{xy} = (D_{ov})_{xy} - \frac{a\hat{\mu}}{2a_4\ M}\left[(1-\gamma_4)U_4(y)\delta_{x,y-\hat{4}} + (\gamma_4+1)U_4^{\dagger}(x)\delta_{x,y+\hat{4}}\right] . \tag{4}$$

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- As the Bloch-Wettig proposal, this too breaks chiral invariance but D_{ov} is defined by the usual sign-function. But clearly Simpler !

- Expect a^{-2}-divergences as $a \to 0$. Follow the same prescription used for the Pressure computation (which diverges at zero temperature as Λ^4). Use Large N_τ and the same lattice spacing a for subtraction.

- Consider two Observables : $\Delta\epsilon(\mu, T) = \epsilon(\mu, T) - \epsilon(0, T)$ and Susceptibility, $\sim \partial^2 \ln \mathcal{Z} / \partial \mu^2$.

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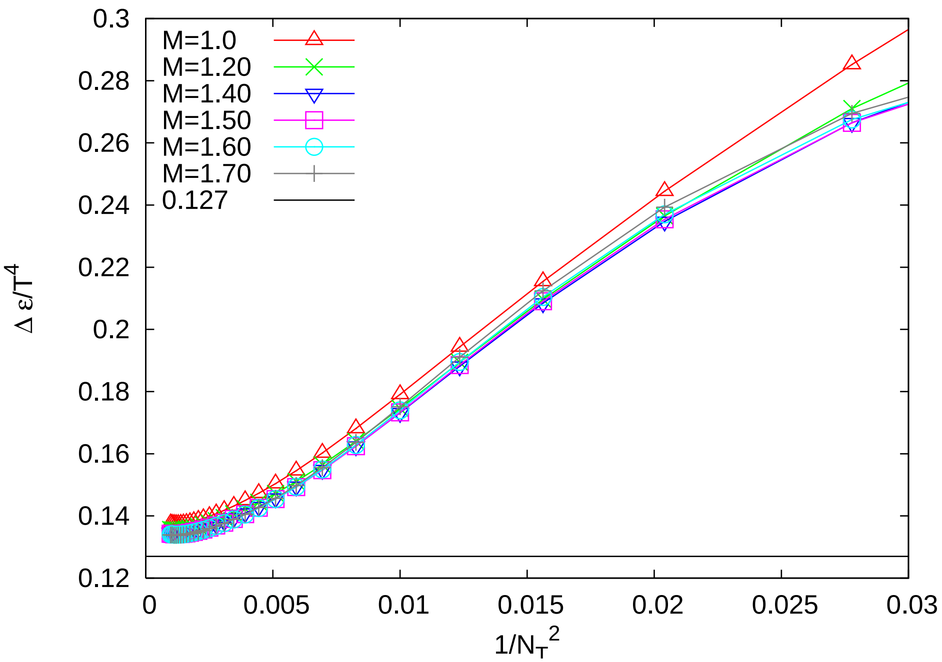
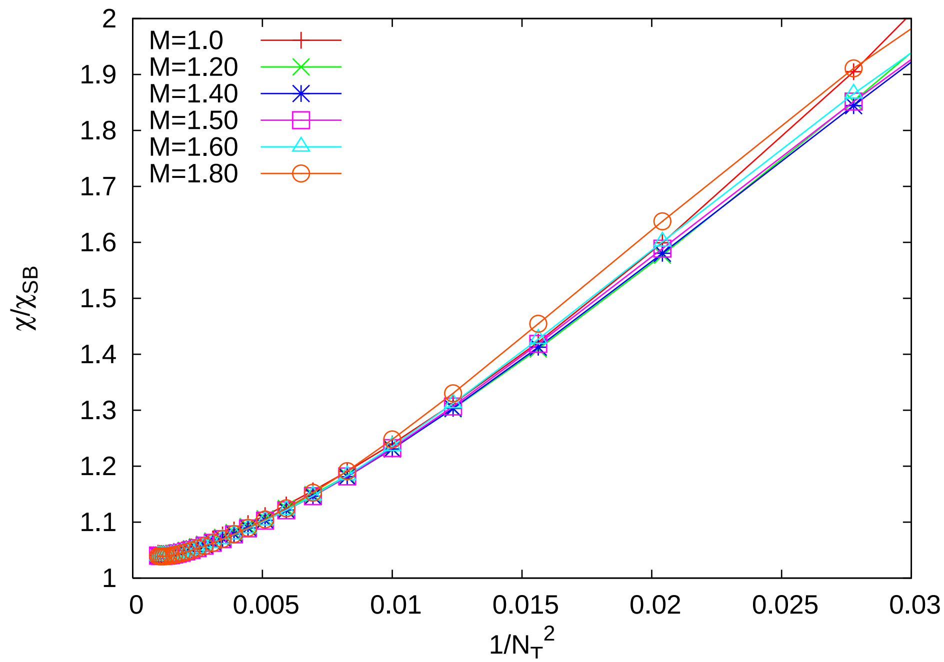
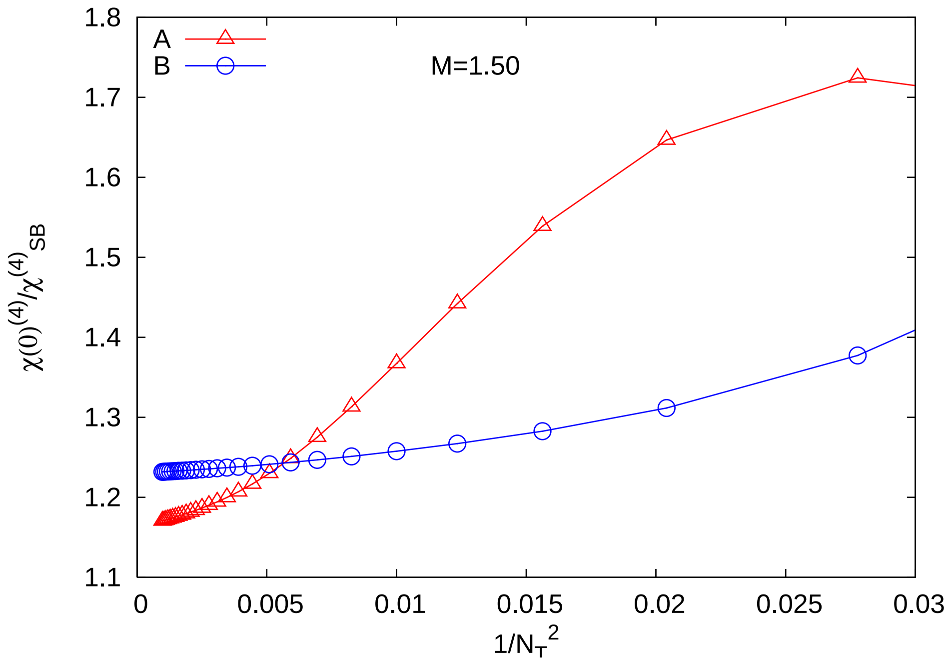
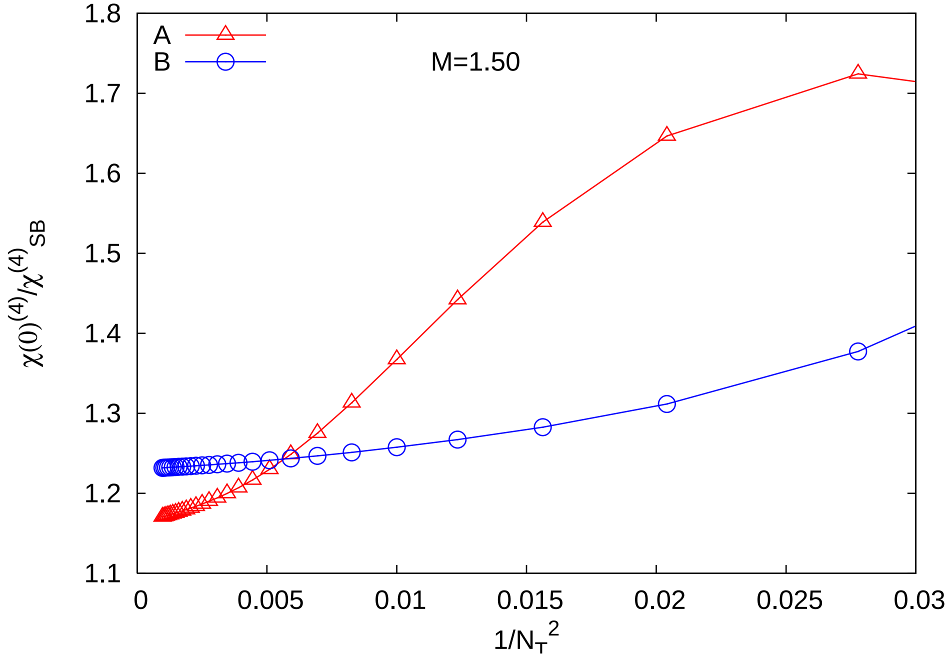






- 4th Order Susceptibility : A) $\hat{\mu}/s$ and B) $\hat{\mu}/M$



- 4th Order Susceptibility : A) $\hat{\mu}/s$ and B) $\hat{\mu}/M$

- Divergences can be eliminated; M-dependence milder.

- Slow convergence to the expected continuum value; Can be improved by using higher-link derivatives, or even variations of the coefficients of the $\hat{\mu}$-term.

Extending to Local Fermions

- Propose to introduce μ in general by
  $S_F = \sum_{x,y} \bar\Psi(x) M(\mu; x, y) \Psi(y) = \sum_{x,y} \bar\Psi(x) D(x, y) \Psi(y) + \mu a \sum_{x,y} N(x, y)$.
  Here D can be the the staggered, overlap, the Wilson-Dirac or any other suitable fermion operator and $N(x, y)$ is the corresponding point-split and gauge invariant number density.

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- This leads to $M' = \sum_{x,y} N(x, y)$, and $M'' = M''' = M''''... = 0$, in contrast to the popular exp($\pm a\mu$)-prescription where *all* derivatives are nonzero:
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- Lot fewer terms in the Taylor coefficients, especially as the order increases. E.g., in the 4th (8th) order susceptibility, the $\mathcal{O}_4$ ($\mathcal{O}_8$) has one (one) term in contrast to 5 (18) in the usual case.

- Number of M^{-1} computations needed are lesser.

Summary

- We showed, both perturbatively and non-perturbatively, that the introduction of nonzero μ leaves the anomaly unaffected. The zero modes of the Dirac operator for $\mu = 0$ govern it; nonzero μ simply scales the eigenvectors.

- A "gauge-like" symmetry in the continuum can be traced as the reason for anomaly preservation. Local lattice actions can easily adopt it. But how to ensure that for the overlap like non-local fermions remains unclear.

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- Overlap fermions at finite density could be studied by simply adding the μ-term linearly. The chiral symmetry breaking is similar but the inverse propagator simpler.

- Extending to staggered fermions, it may be less costly to implement this idea and may permit extensions to higher orders.

What if . . .

♠ the chiral transformations were $\delta\psi = \alpha\gamma_5(1 - \frac{a}{2}D(a\mu))\psi$ and $\delta\bar{\psi} = \alpha\bar{\psi}(1 - \frac{a}{2}D(a\mu))\gamma_5$?

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- Symmetry transformations should not depend on "external" parameter μ. Chemical potential is introduced for charges N_i with $[H, N_i] = 0$. At least the symmetry should not change as μ does.
- Moreover, symmetry groups *different* at each μ. Recall we wish to investigate $\langle\bar{\psi}\psi\rangle(a\mu)$ to explore if chiral symmetry is restored.
- The symmetry group remains *same* at each T with $\mu = 0$ $\Longrightarrow \langle\bar{\psi}\psi\rangle(am = 0, T)$ is an order parameter for the chiral transition.