Exact Counting of Black Hole Microstates

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References

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Plan

- Infinite Tower of BPS States
- Connection to Black Hole Entropy
- Going Beyond Bekenstein-Hawking
- Supersymmetric Attractor Formalism
- Black Hole Partition Function
- Stringy Cloak for a Classical Singularity

Infinite Tower of BPS States

Dabholkar & Harvey

Consider Heterotic String on T^4 £ $T^{2.}$ Let the T^2 to be a product of two circles S^1 £ S^1. Take a winding string along a circle with winding number w and momentum n.

It is BPS if it in the right-moving ground state.

It can carry arbitrary left-moving oscillations subject to Virasoro contraint

$$N_L - 1 = nw \equiv N,$$

where N_L is the number operator of 24 left-moving bosons. The spectrum is summarized by the partition function

$$Z(\beta) = 16 \sum d_N e^{-\beta N},$$

where d_N is the degeneracy at level N.

It can be readily evaluated

$$Z(\beta) = \frac{16}{\Delta(e^{-\beta})},$$

in terms of the Jacobi discriminant function

$$\Delta(q) = \eta(q)^{24} = q \prod_{n>1} (1-q^n)^{24}$$

where η is the Dedekind function. The degeneracy is given by the inverse Laplace transform.

$$d_N = \frac{1}{2\pi i}\int_C d\beta e^{\beta N}\frac{1}{\Delta(e^{-\beta})}.$$

To get the leading behavior at large N, we evaluate it at large temperature, $\beta \to 0$. It is convenient to use the modular property

$$\Delta(e^{-\beta}) = (\frac{\beta}{2\pi})^{-12}\Delta(e^{-4\pi^2/\beta}).$$

Then

$$d_N = \frac{1}{2\pi i}\int_C d\beta (\frac{\beta}{2\pi})^{12} e^{\beta N} \frac{1}{\Delta(e^{-4\pi^2/\beta})}.$$

Now use small q asymptotics $\Delta(q) \sim q$

$$d_N = \frac{1}{2\pi i}\int_C d\beta (\frac{\beta}{2\pi})^{12} e^{-4\pi^2/\beta + \beta N}.$$

From this integral representation we see

$$d_N \sim 2\pi(2\pi N)^{-13/2} I_{13}(4\pi\sqrt{N})$$

where $I_{13}(z)$ is a Bessel function. The leading exponential can be found by saddle point approximation. The saddle occurs at

$$\beta = 2\pi/\sqrt{N}$$

The degeneracy has the characteristic exponential growth

$$d_N \sim \exp\left(4\pi\sqrt{nw}\right).$$

This infinite tower of states has played a crucial role in furthering our understanding of stringy duality and black hole physics.

Let's recall a few properties of Bessel functions. It has the integral representation

$$I_\nu(z) = (\frac{z}{2})^\nu \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{dt}{t^{\nu+1}} e^{(t + z^2/4t)},$$

This satisfies the (hyperbolic) Bessel function. Moreover

$$I_{-\nu}(z) = I_\nu(z)$$

The asymptotic expansion is given by

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\{1 - \frac{(\mu - 1^2)}{1!(8z)} + \frac{(\mu - 1^2)(\mu - 3^2)}{2!(8z)^2} - \frac{(\mu - 1^2)(\mu - 3^2)(\mu - 5^2)}{3!(8z)^3} + \ldots\}$$

with $\mu = 4\nu^2$.

Thus the entropy is given by

$$\begin{aligned}\log d_N \quad \sim \quad & 4\pi\sqrt{N} - \frac{27}{2}\log\sqrt{N} \\ & -\log\sqrt{2} - \frac{675}{32\pi\sqrt{N}} \\ & -\frac{675\times 9}{2048\pi^2 N} - \cdots\end{aligned}$$

Heterotic on T^4 $\$$IIA on K_3

Initial evidence came from matching low lying spectrum and effective action but a far more stringent test is obtained by matching the entire tower of BPS states.

The configuration (n,w) is dual to a collection of w D4-branes with n D0-branes sprinkled on them.

Their spectrum must exhibit this exponential growth.

Vafa

The discriminant function made its appearance also in topological YM on K_3 which counts instantons in the 4-brane worldvolume gauge theory. This provided early hints of string-string duality.

Vafa & Witten

Black Hole Entropy

As we increase the string coupling, the state (n, w) corresponds to an extremal black hole with two charges.

Does the entropy of this black hole match the logarithm of the degeneracy at large N?

Classically the black hole has zero area!

But higher derivative α' corrections can correct the solution to give it finite area.

Assuming that the higher curvature terms give a finite area horizon, one can reproduce the entropy of the black holes up to a numerical factor. *Sen*

In particular, the nontrivial dependence of the entropy on the charges n and w can be computed correctly by a simple scaling argument.

- Can we compute the numerical factor?
- Can we justify the assumption of finite horizon?
- The area formula itself can get modified.
- Calculate the sub-leading corrections..

Subsequent developments have dealt with black holes with more charges, that have finite area already classically. The entropy can then be computed from the are formula reliably within supergravity approximation from the Bekenstein-Hawking area formula with precise numerical factors.

Strominger & Vafa

Can we compute corrections??

Going Beyond Bekenstein-Hawking

- We return to the two charge black hole and show that it is indeed possible to incorporate all higher derivative terms (apparently) exactly.
- The inclusion of higher derivative terms gives finite entropy with the precise numerical factor.
- What is more, the entire partition function can reproduced at large N. One can compute the large N corrections exactly to all orders.

Four Ingredients

- Wald's formalism
- Supersymmetric attractor mechanism
- Topological string and exact prepotential
- Microscopic counting.

For the two charge system, all four ingredients are computable. The correction from the higher derivative term is itself the leading order contribution to the entropy.

Wald's Formalism

Consider an action with higher derivative corrections like R^2 etc in addition to the Einstein Hilbert term. Then the Bekenstein-Hawking entropy formula is modified. If there is a Killing horizon, then one can associate an entropy,

$$S = 2\pi \int_{S^2} \epsilon_{ab}\epsilon_{cd}\frac{\delta \mathcal{L}}{\delta R_{abcd}},$$

such that the first law of thermodynamics is satisified.

$$\delta M = T\delta S - \delta W.$$

Thus the higher corrections will modify not only the solution but the area formula itself.

The temperature is defined geometrically in terms of surface gravity

$$T = \frac{\hbar\kappa}{2\pi}.$$

This beautifully general formula assumes its full significance only in a complete quantum theory of gravity such as string theory.

But even with string theory, to really compute the corrections we would have to know all higher order terms, which might be possible in principle but not in practice.

To get better computational control we need something more.

Supersymmetric attractor and topological string

Attractor Formalism

Noticed first for black holes with finite area in N=2 supergravity.

Ferrara, Kallosh, Strominger

The near horizon geometry of a black hole is Bertotti-Robinson (AdS^2 £ S^2) that preserves the full N=2 supersymmetry. *Gibbons*

In supergravity there are many moduli fields

Vector multiplet A_μ, X

Hyper multiplet ϕ_1, ϕ_2

The vector-multiplet moduli couple to the electromagnetic field and vary in the of charged black hole background.

At the horizon they reach an attractor point in the moduli space irrespective of their values at asymptotic infinity.

The vector-multiplet moduli space with $\mathbf{n_v}$ vector multiplets is parameterized by $\mathbf{n_v}$ **+1** complex projective coordinates X^I, $I = 0, 1, \ldots,$ $\mathbf{n_v}$.

There are an infinite number of higher derivative corrections to the Einstein-Hilbert action that are expected to be relevant for the computation of the entropy.

These F-type corrections to the effective action are summarized by the string-loop corrected holomorphic prepotential.

$$F(X^I, W^2) = \sum_{h=0}^{\infty} F_h(X^I) W^{2h}$$

where F_h are computed by the topological string amplitudes at h-loop and give terms in the effective action that go as

$$\int F_h(X) R^2 W^{2h-2}.$$

Here W^2 is the graviphoton field.

Bershadsky, Cecotti, Ooguri, Vafa,

Antoniadis, Gava, Narain, Taylor

The attractor equations are given by

$$p^I = \mathrm{Re}[CX^I] \quad q_I = \mathrm{Re}[CF_I],$$

and $C^2 W^2 = 256$.

The scaling field C is introduced so that (CX^I, CF_I) is non-projective and transforms like (p^I, q_I) as a vector under the $Sp(2n_v + 2; Z)$ symplectic duality group.

The attractor equations are then

determined essentially by symplectic invariance.

The quantum-corrected entropy is given by

$$S_{\mathrm{BH}} = \frac{\pi i}{2}(q_I \bar{C} \bar{X}^I - p^I \bar{C} \bar{F}_I) + \frac{\pi}{2}\mathrm{Im}[C^3 \partial_C F].$$

Cardoso, de Wit, and Mohaupt

(Kappeli)

The first set of attractor equations can be solved

$$CX^I = p^I + \frac{i}{\pi}\phi^I$$

In terms of `potentials' ϕ^I. Then the entropy can be written in a suggestive form.

$$S_{\mathsf{BH}}(q,p) = \mathcal{F}(\phi,p) - \phi^I \frac{\partial}{\partial \phi^I}\mathcal{F}(\phi,p).$$

In terms of a `free energy' function

$$\mathcal{F}(\phi, p) = -\pi \mathrm{Im}\left[F\left(p^I + \frac{i}{\pi}\phi^I, 256\right)\right].$$

The potentials are determined by the second set of attractor equations

$$q_I = -\frac{\partial}{\partial \phi^I}\mathcal{F}(\phi, p).$$

Ooguri, Strominger, Vafa

Our System

We have a special case when the Calabi-Yau is

$\mathbf{K_3}$ **£ $\mathbf{T^2}$**. There are **23** 2-cycles of which we take w_1 to be the 2-torus itself and w_a, a =2, …, 23 to be the **22** 2-cycles of K_3.

$$
\begin{aligned}
F_0 &= -\frac{1}{2}C_{ab}X^a X^b(\frac{X^1}{X^0}) \\
F_1 &= \frac{i}{128\pi}\log\Delta(q) \\
F_h &= 0, \quad h > 1.
\end{aligned}
$$

The perturbative state **(n, w)** on the heterotic side is dual to **w** 4-branes wrapping $\mathbf{K_3}$ with **n** 0-branes sprinkled on it.

The 4-cycle is dual to the 2-form $\mathbf{w_1}$ and hence we have a nonzero magnetic charge an $\mathbf{p}^1 = \mathbf{w}$ r magnetic charges are zero.

The 0-brane couples electrically to the graviphoton field and hence $\mathbf{q_0 = n}$ and all other electric charges are zero.

In the large volume limit we can approximate the second term by and solve the attractor equations for the charges

$\mathbf{q_A = (q^0, 0, 0, \ldots, 0)}$ and $\mathbf{p^A = (0, p_1, 0, \ldots, 0)}$.

The solution is

with ϕ^1 $$\phi^0 = -2\pi\sqrt{\frac{p^1}{q_0}}, \qquad \phi^a = 0,$$

Leading order entropy

The leading order entropy now comes not from the Einstein-Hilbert term but from the corrections and is given by

$$S = 4\pi\sqrt{p^1 q_0} = 4\pi\sqrt{wn},$$

In precise agreement with the logarithm of degeneracy at large N

Black Hole Partition Function

Given the free energy it is natural to define a partition function

$$Z_{\mathrm{BH}}(\phi^I, p^I) = e^{\mathcal{F}(\phi^I, p^I)}$$
$$\equiv \sum_{q_I} \Omega(q_I, p^I) e^{-\phi^I q_I},$$

$\Omega(\mathbf{q_I}, \mathbf{p^I})$ are the black hole degeneracies.

Can we test this??

Black hole degeneracy

The exact black hole degeneracy is proportional to

$$\int \prod_{a=2}^{22} d\phi^a d\phi^1 d\phi^0 \exp\left(\mathcal{F}(\phi^A, p^A) + \phi^0 q_0\right).$$

Use the free energy. Do the 22 Gaussian $\{\phi^a\}$ integrals. The ϕ^0 integral is trivial at large N, the integrand is independent of ϕ^0. We are left with a single integral over ϕ^1 .

We get the same integral representation that we encountered in our microscopic counting for d_N. Hence black hole degeneracy $\Omega(p^1, q_0)$ then has is given by

$$\Omega(p^1, q_0) \sim 2\pi(2\pi N)^{-13/2} I_{13}(4\pi\sqrt{N})$$

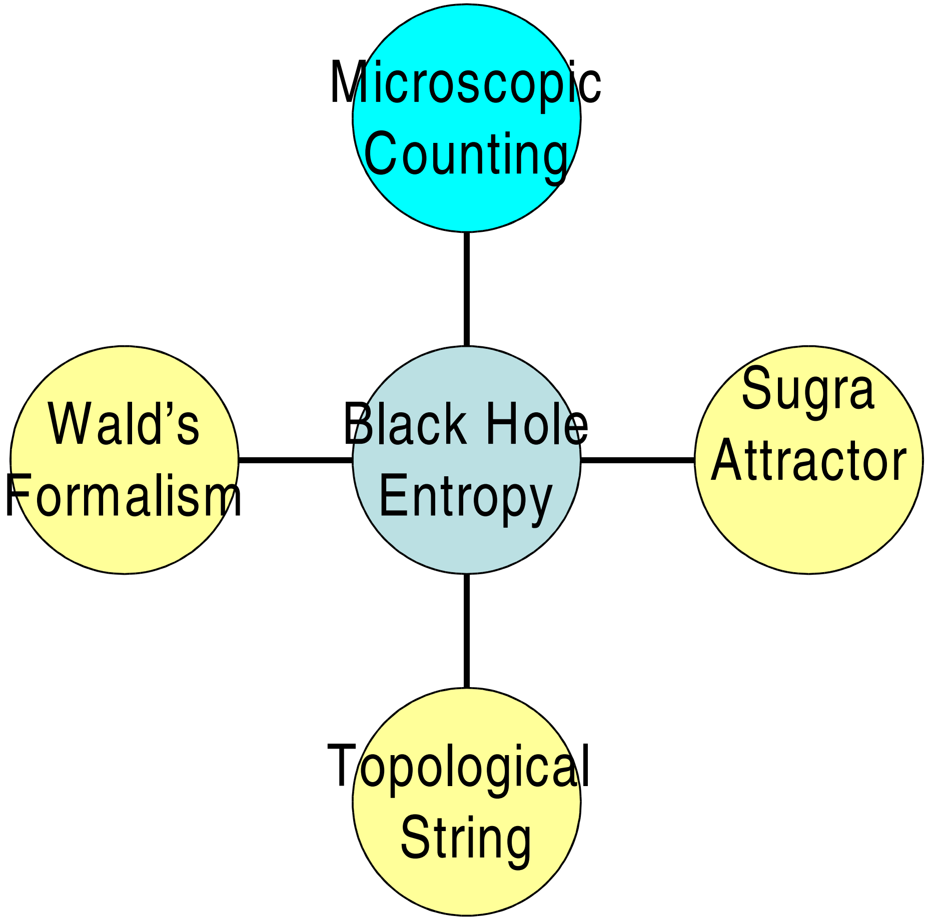
$$\sim d_N$$

We thus see that the Boltzmann entropy of the OSV ensemble matches with the microscopic entropy to all orders

$$\log d_N \sim 4\pi\sqrt{N} - \frac{27}{2}\log\sqrt{N} - \log\sqrt{2} - \frac{675}{32\pi\sqrt{N}} - \frac{675\times 9}{2048\pi^2 N} - \cdots$$

There are exponentially small corrections that are nonperturbative in the string coupling. On the microscopic side they can be computed systematically using the Rademacher expansion.

A. D., Denef, Moore, & Pioline, in progress


Microscopic Counting
Wald's Formalism
Black Hole Entropy
Sugra Attractor
Topological String

Stringy Cloak for a Null Singularity

- Classically, the spacetime geometry of these black holes is singular. There is a null singularity that is not separated from the asymptotic observer by a regular horizon.
- Once we include the corrections, the singularity is `cloaked’ and we obtain a spacetime with a regular horizon.

The near horizon geometry is **$AdS_2 \times S^2$.** The radius of the 2-sphere is large in Einstein frame and goes as $\sqrt{N}$. The string coupling is determined by the attractor equations and vanishes

as $1/\sqrt{N}$ on the horizon. Thus in the string frame, the horizon is still string scale consistent with the heuristic idea of a `stretched horizon'. Is there an exact conformal field theory?

- The entropy is no longer given by the Bekenstein-Hawking relation but rather by

$$S = A/2$$

- Here A is the quantum-corrected area. Wald's correction is of the same order of magnitude as Bekenstein-Hawking and together we have A/4 + A/4 = A/2.

- This works for a large class of black holes in CY3 compactifications that have classically vanishing area.
- On the heterotic side for N=2 supersymmetry, this can be seen by a `universality’ argument of Sen. The microscopic entropy depends only on the central charge of the left-moving CFT, which is always 24. On the macroscopic side, the α' corrections of the N=2 sigma model are inherited from the N=4 theory.

Conclusions

- Black Holes that have zero classical area have precisely computable entropy using Wald's formalism and sugra attractor which is in agreement with the microscopic counting.
- The black hole partition function agrees with the microscopic partition function in an asymptotic expansion to all orders in inverse charge but there are nonperturbative corrections.

A curious fact--The Jacobi discriminant function makes its appearance in a rather different context in the effective action of the heterotic string.

$$\mathrm{Re}\int \log\left(\Delta(q)\right)\mathrm{tr}R\wedge R$$

The argument is $q = \exp(2\pi i\tau)$ where τ is the spacetime axion-dilaton field. The leading term gives the coupling for Green-Schwarz mechanism and the q expansion sums the contribution due to 5-brane instantons.

The asymptotic expansion is given by

$$
\begin{aligned}
I_\nu(z) \quad \sim \quad & \frac{e^z}{\sqrt{2\pi z}}\{1 - \frac{(\mu - 1^2)}{1!(8z)} \\
& + \frac{(\mu - 1^2)(\mu - 3^2)}{2!(8z)^2} \\
& - \frac{(\mu - 1^2)(\mu - 3^2)(\mu - 5^2)}{3!(8z)^3} \\
& + \ldots\}
\end{aligned}
$$

with $\mu = 4\nu^2$.