FREE FIELD THEORY AS A STRING THEORY?

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+ WORK IN PROGRESS (w/ DUMITRU ASTEFANESEI, SOMDATTA BHATTACHARYA, KAZUYUKI FURUUCHI AND K. 'PATTA' YOGENDRAN)

The Gauge Fields Strings Correspondence

$$\Big\langle \underbrace{\mathcal{O}_1(k_1)}_{\text{Gauge inv. operator}} \mathcal{O}_2(k_2) \ldots\ldots \mathcal{O}_n(k_n) \Big\rangle^{(g)}_{\text{space time}}$$

genus g contribution in $1/N$

$$\stackrel{?}{=} \int_{\mathcal{M}_{g,n}} \Big\langle V_1(k_1,\xi_1)\, V_2(k_2,\xi_2) \ldots\ldots V_n(k_n,\xi_n) \Big\rangle_{\text{world sheet}}$$

vertex op.

HOW DOES GAUGE FIELD THEORY GIVE RISE TO THE MODULI SPACE OF SURFACES OF GENUS g W/ n PUNCTURES?

WHAT CAN WE LEARN ABOUT THE WORLD SHEET THEORY FROM FIELD THEORY?

The Plan

1. The Stringy Reorganisation Of Field Theory Diagrams: Refinement Of 't Hooft's Identification Of Riemann Surfaces In Gauge Theories – Now See The Whole Moduli Space Emerge.

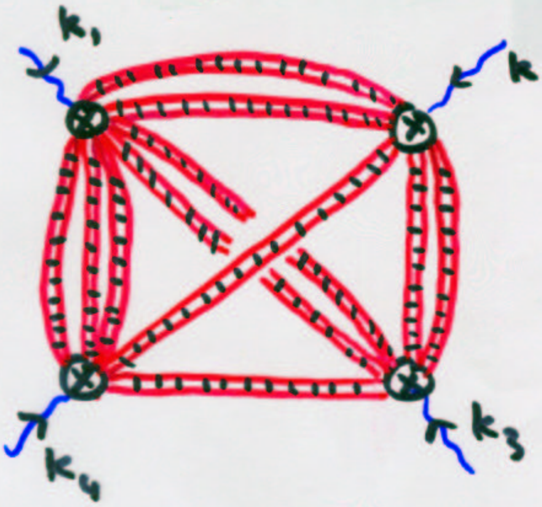
2. Deciphering The World Sheet Theory:

   (i) OPE Of The Worldsheet Theory From The Spacetime OPE.

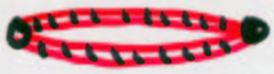
   (ii) Reconstructing Closed String Geometry From World Sheet Correlators (Seeing the AdS geometry; winding modes at finite temperature .....)

The Stringy Reorganisation: An Implementation Of Open-Closed Duality

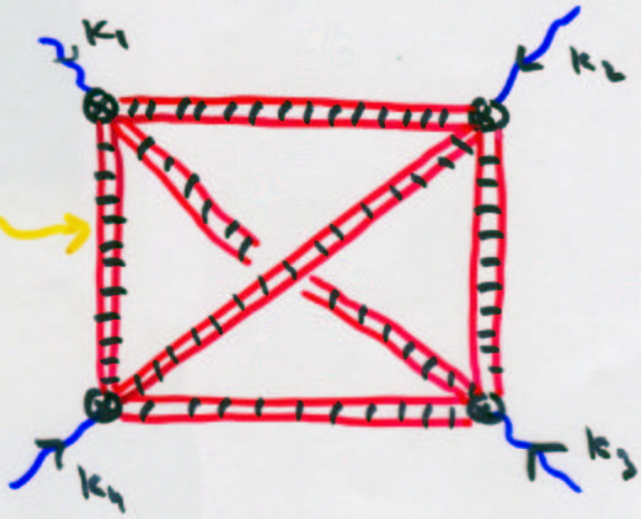
e.g. $G^{\{J_i\}}_{(g)}(k_1 \ldots k_n) = \left\langle \mathrm{Tr}\,\Phi^{J_1}(k_1) \ldots \mathrm{Tr}\,\Phi^{J_n}(k_n) \right\rangle_{\text{genus } g}$

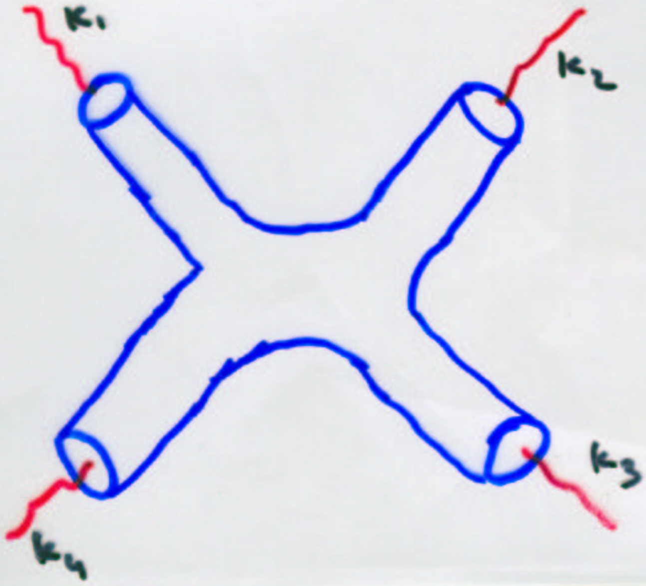


Partially Glue Up Holes Of The Form



"Skeleton Graph"





Dual Graph: Holes Close Up - Vertices Open Up.

Schwinger Parametrisation Of Field Theory Amplitudes

$$G^{\{J_i\}}_{(g)}(k_1 \ldots\ldots k_n) = \sum_{\substack{\text{graphs} \\ \text{(genus } g)}}$$

Internal momenta

$$= \sum_{\substack{\text{graphs} \\ \text{(genus } g)}} \int [d^d p] \int [d\tilde{\tau}] \, e^{-\underbrace{\tilde{P}(k,p,\tilde{\tau})}_{\text{Gaussian in } (k,p)}}$$

$$\left[\text{Using } \frac{1}{p^2} = \int_0^\infty d\tilde{\tau} \, e^{-\tilde{\tau} p^2} \right]$$

[See e.g. Itzykson + Zuber]

$$= \sum_{\substack{\text{graphs} \\ \text{(genus } g)}} \int_0^\infty \frac{\prod_i d\tilde{\tau}_i}{\Delta(\tilde{\tau})^{d/2}} \exp\left[-\overbrace{P(\tilde{\tau},k)}^{\text{Quadratic in } k}\right]$$

One Schwinger Parameter $\tilde{\tau}$ For Each Edge

$\left[\text{Total } \# = \frac{1}{2}\sum_i J_i\right]$

Thus, For E.G. A 6 Pt. Fn. Might Be Represented As An Integral Over 20,000 Parameters!

Explicit Expressions Can Be Written For $\Delta(\tilde{\tau})$ And $P(\tilde{\tau}, k)$ In Terms Of Graph Structure.

[Nakanishi, Symanzik, Lam & Le Brun ...]

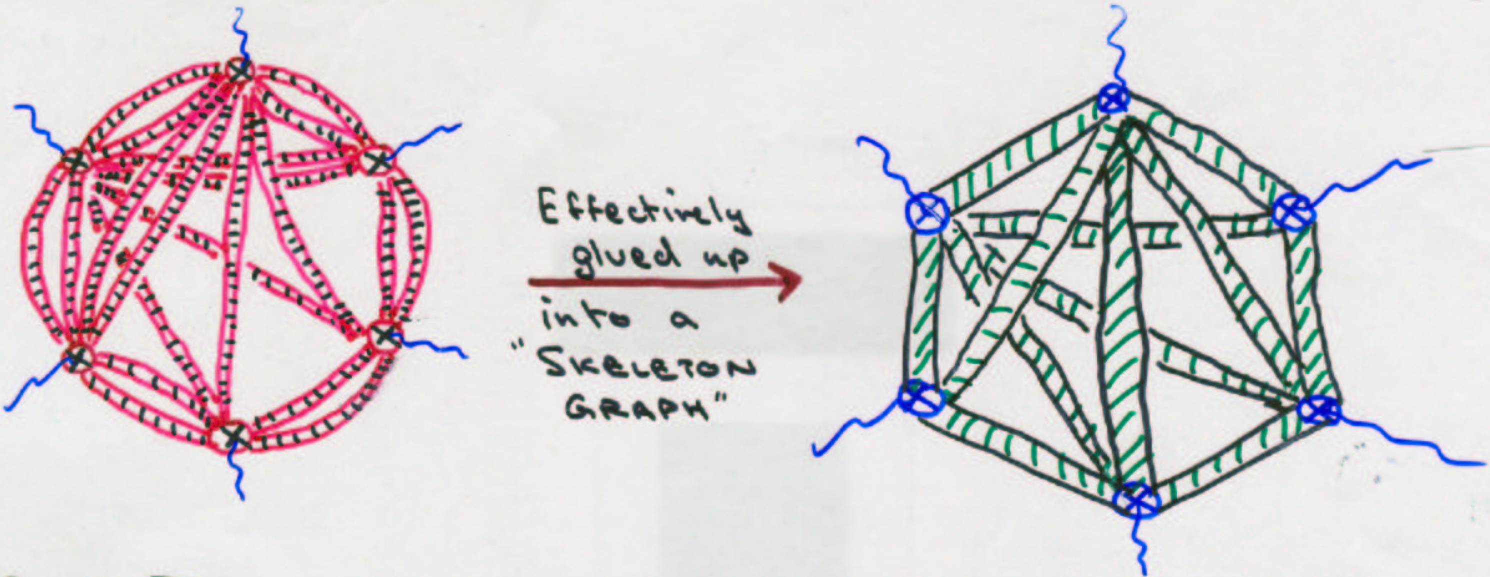
$$\Delta(\tilde{\tau}) = \sum_{T_1} \left(\prod^{\ell} \tilde{\tau} \right)$$

[T_1 = 1-tree: obtained by cutting graph at ℓ edges to get a connected tree. $\prod^{\ell}$ Over all parameters of cut edges]

$$P(\tilde{\tau}, k) = \frac{1}{\Delta(\tilde{\tau})} \sum \left(\prod^{\ell+1} \tilde{\tau} \right) \left(\Sigma k \right)^2$$

[T_2 = 2-tree: obtained by cutting graph at $\ell+1$ edges to get two disconnected trees. Σk = momentum flowing into either tree.]

SIMPLIFICATION:



CAN REPLACE PARALLEL EDGES (homotopically defm'ble)

WITH SCHWINGER PARAMETERS $\tilde{\tau}_{r\mu_r}$ $(\mu_r = 1 \dots m_r)$

BY A SINGLE EFFECTIVE SCHWINGER PARAMETER

$$\frac{1}{\tau_r} = \sum_{\mu_r=1}^{m_r} \frac{1}{\tilde{\tau}_{r\mu_r}}$$

MULTIPLICITY OF PARALLEL EDGES

POSSIBLE BECAUSE

$$P(\tilde{\tau}, k) = P_{\text{Skel}}(\tau, k)$$

$$\Delta(\tilde{\tau}) = \left(\frac{\prod \tilde{\tau}}{\prod \tau}\right) \Delta_{\text{Skel}}(\tilde{\tau})$$

[Note: RHS defined in terms of skeleton graph structure]

CORRESPONDENCE WITH ELECTRICAL NETWORKS

FEYNMAN GRAPH $\longleftrightarrow$ ELEC. NETWORK

EXTERNAL MOMENTA (k_i) $\longleftrightarrow$ EXT. CURRENTS

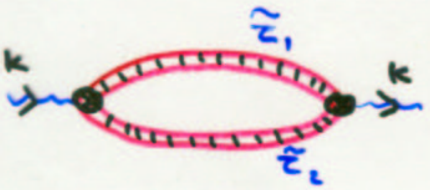
INTERNAL MOMENTA (p_i) $\longleftrightarrow$ INT. CURRENTS

SCHWINGER MODULI (τ) $\longleftrightarrow$ RESISTANCES

$\tilde{P}(k, p, \tau)$ $\longleftrightarrow$ POWER DISSIPATED IN ORIGINAL CIRCUIT

$P(\tau, k)$ $\longleftrightarrow$ POWER DISSIPATED IN EQUIV. CIRCUIT (after eliminating int. currents)

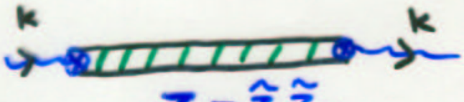
E.G.



$$= \int d^d p \int_0^\infty d\tilde{\tau}_1 \, d\tilde{\tau}_2 \, e^{-\boxed{[\tilde{\tau}_1 p^2 + \tilde{\tau}_2 (k-p)^2]}}$$

$\tilde{P}(k, p, \tilde{\tau})$

$\parallel$



$$\underbrace{\tau = \frac{\tilde{\tau}_1 \tilde{\tau}_2}{\tilde{\tau}_1 + \tilde{\tau}_2}} \qquad = \int_0^\infty \frac{d\tilde{\tau}_1 \, d\tilde{\tau}_2}{(\tilde{\tau}_1 + \tilde{\tau}_2)^{d/2}} \, e^{-\boxed{\frac{\tilde{\tau}_1 \tilde{\tau}_2}{\tilde{\tau}_1 + \tilde{\tau}_2} k^2}}$$

$P(\tilde{\tau}, k)$

TWO PARALLEL RESISTORS $\tilde{\tau}_1$, $\tilde{\tau}_2$ REPLACED BY SINGLE EFFECTIVE RESISTOR.

GENERALISES TO MULTIPLE PARALLEL EDGES

HENCE, FOR EACH GRAPH:

$$\int_0^\infty \frac{\prod_i d\tilde{\tau}_i}{\Delta(\tilde{\tau})^{d/2}}\, e^{-P(\tilde{\tau},k)} = C^{\{m_r\}} \int_0^\infty \prod_r \left(\frac{d\tau_r}{\tau_r^{(m_r-1)(\frac{d}{2}-1)}}\right) \frac{e^{-P_{skel}(\tau,k)}}{\Delta_{skel}(\tau)^{d/2}}$$

From Jacobian: $(\tilde{\tau} \to \tau)$

[Info. about J_i enters only here $\sum m_{r(i)} = J_i$]

Summing over graphs w/ diff. m_r

$$\Rightarrow G^{\{J_i\}}_{(g)}(k_1 \ldots k_n) = \sum_{\substack{\text{skeleton}\\ \text{graphs}\\ \text{(genus } g)}} \int_0^\infty \frac{\prod_r d\tau_r f^{\{J_i\}}(\tau)}{\Delta_{skel}(\tau)^{d/2}}\, e^{-P_{skel}(\tau,k)}$$

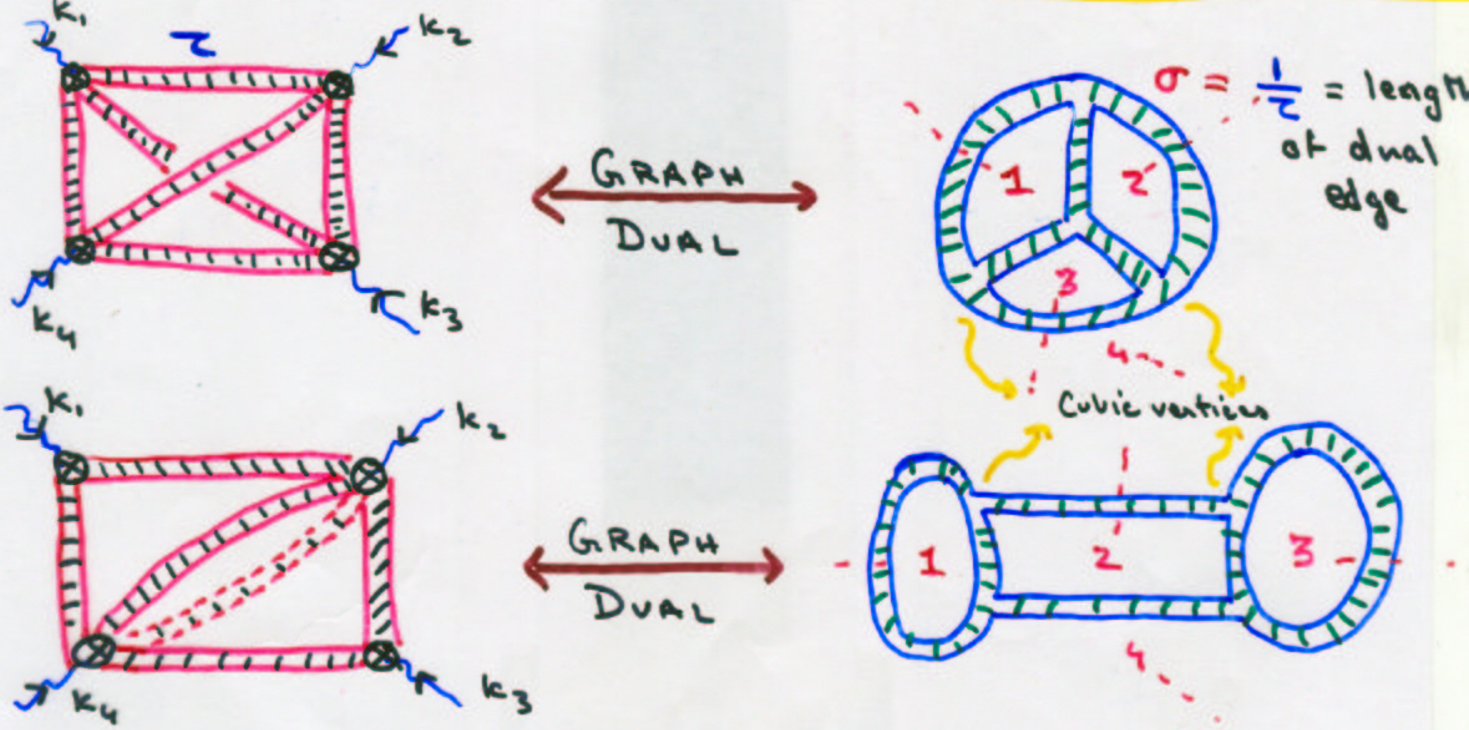
THUS SUM OVER MODULI SPACE OF SKELETON GRAPHS (lengths + connectivity).

GENERICALLY (for J_i > min. # $\propto n$) SKEL. GRAPHS HAVE TRIANGULAR FACES. (all poss. connections consistent w/ genus)

[For e.g. planar skeleton graphs consist of triangulations of sphere w/ n vertices. $\Rightarrow$ $3(n-2)$ edges]

Moduli Space Of Skeleton Graphs (genus g w/ n marked vertices) $\equiv$ (via Graph Duality) Moduli Space Of Riemann Surfaces (genus g w/ n marked points + radius to each point)

$$= \boxed{\mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong \hat{\mathcal{M}}_{g,n}}$$



Dual Graphs Give Cubic SFT Cell Decomp. Of Moduli Space w/ $\sigma = \frac{1}{\tau}$ = length of strips.

[Cells match smoothly onto each other at boundaries $\sigma \to 0$.]

{Giddings, Martinec, Witten
⋮
Zwiebach

+ Parallel Math. Developments {Strebel, Penner, Kontsevich... }

THEREFORE

$$G^{\{J_i\}}_{(g)}(k_1 \dots k_n) = \sum_{\substack{\text{Skel.}\\ \text{graphs}\\ \text{(genus } g)}} \int_0^\infty \frac{\prod_r d\tau_r \, f^{\{J_i\}}(\tau)}{\Delta_{\text{Skel}}(\tau)^{d/2}} e^{-P_{\text{Skel}}(\tau,k)}$$

$$= \int_{\mathcal{M}_{g,n} \times \mathbb{R}_+^n} [d\sigma] \, \rho^{\{J_i\}}(\sigma) \, e^{-\sum_{i,j}^n k_i \cdot k_j \, g_{ij}(\sigma)}$$

$$\sigma = \frac{1}{\tau}$$

FUNCTIONS WHICH CAN BE EXPLICITLY WRITTEN FOR ANY n-PT. FUNCTION

- THUS THIS PROCEDURE GIVES AN EXPLICIT FORM FOR INTEGRAND ON MODULI SPACE = CORRELATOR OF WORLDSHEET CFT.
- <u>BUT</u> THE COORDINATES σ GIVE AN AWKWARD PARAMETRISATION OF MODULI SPACE – CHANGE OF VARIABLES TO USUAL COORDINATISATION NOT EASY TO WORK OUT.

- Complicates The Ease Of Extraction Of Information About World Sheet Correlators.
- However, Can Still Hope To Check Properties Expected Of A World Sheet Correlator.

e.g. Consistency w/ A Worldsheet OPE (for the worldsheet vertex ops. $V_i(k_i, \xi_i)$)

$$V_I(k_1, \xi_1)\, V_J(k_2, \xi_2) = \sum_K \underbrace{\tilde{C}^K_{IJ}}_{\text{Worldsheet structure constants}}(k_1, k_2, \xi_1 - \xi_2)\, V_K(k, \xi)$$

ξ_1, ξ_2 are the natural cplx. coords for insertions

In fact, for a CFT

$$\tilde{C}^K_{IJ}(k_1, k_2, \xi_1 - \xi_2) \sim \frac{\tilde{C}^K_{IJ}(k_1, k_2)}{(\xi_1 - \xi_2)^{\Delta_K - \Delta_J - \Delta_I}}$$

But We Do Not Know $\xi(\sigma)$ - change of variables - Explicitly.

Still Can Try To See This World Sheet OPE From The Spacetime OPE....

From Spacetime OPE To Worldsheet OPE

$$\langle \mathcal{O}_I(k_1)\,\mathcal{O}_J(k_2)\,\mathcal{O}_L(k_3)\,\mathcal{O}_M(k_4)\rangle_{g=0}$$

$$= \sum_K \underbrace{C_{IJ}^{K}(k_1 k_2)}_{\text{Sp-time Structure Constants}} \langle \mathcal{O}_K(k)\,\mathcal{O}_L(k_3)\,\mathcal{O}_M(k_4)\rangle_{g=0}$$

$$\int_{\mathcal{M}_{0,4}\times\mathbb{R}_+^4} \langle V_I(k_1,\xi_1)\,V_J(k_2,\xi_2)\,V_L(k_3,\xi_3)\,V_M(k_4,\xi_4)\rangle_{WS}$$

$$= \sum_K C_{IJ}^{K}(k_1 k_2) \int_{\mathcal{M}_{0,3}\times\mathbb{R}_+^3} \langle V_K(k,\xi)\,V_L(k_3,\xi_3)\,V_M(k_4,\xi_4)\rangle_{WS}$$

- This follows generally from the gauge-string correspond.

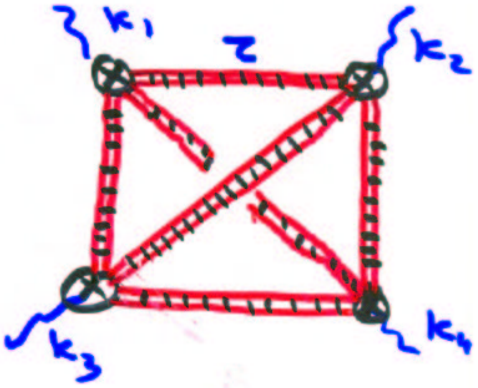
- But Looks As If It Is Coming From A World Sheet OPE.

Can We Check, In Our Representation Of The Worldsheet Correlators, That This Is So?

WRITE 4-pt. FUNCTION

(e.g. $\langle \mathrm{Tr}\,\Phi^3(k_1)\,\mathrm{Tr}\,\Phi^3(k_2)\,\mathrm{Tr}\,\Phi^3(k_3)\,\mathrm{Tr}\,\Phi^4(k_4)\rangle_{g=0}$)

IN SCHWINGER REPN.



THE SPACETIME OPE ARISES FROM

$\tilde{k}^2 = |k_1 - k_2|^2 \longrightarrow \infty$

$$\int \frac{\prod_{a=1}^{6} d\tau_a}{\Delta_4(\tau)^{d/2}}\, e^{-P_4(\tau_a, k_i)}$$

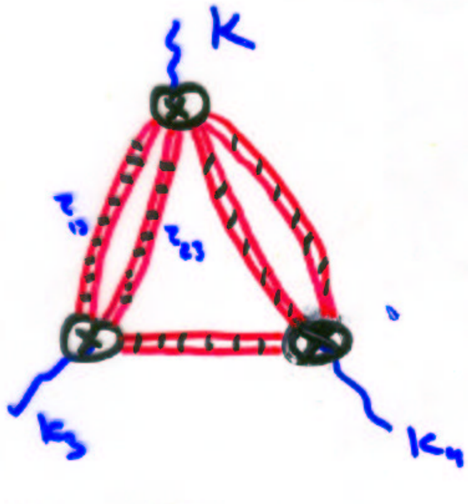
As FOR 3-PT. FN.

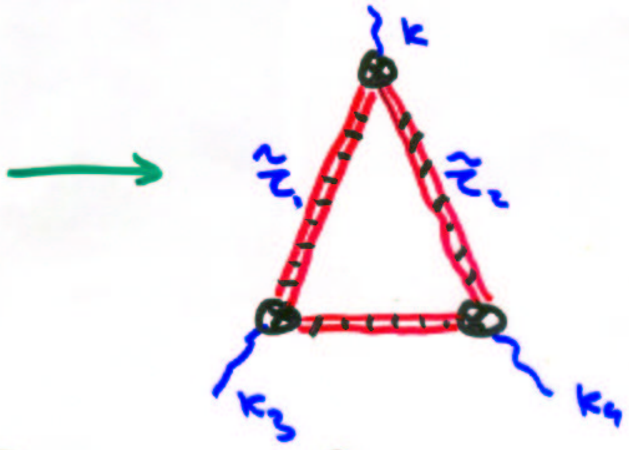
$$= \int \frac{\prod_{r=1}^{5} d\tau_r\, d\tau}{[\Delta_3(\tau_r) + \tau\tilde{\Delta}]^{d/2}}\, e^{-[P_3(\tau_r, k_3, k_4, k) + \tilde{P}(\tau, \tilde{k}, \tau_r, k_4)]}$$

3-PT. FN. EXP.

(Separating out τ and $\tilde{k}$ dep.)

$$= \int \frac{\prod_{r=1}^{3} d\tilde{\tau}_r}{\Delta_3(\tilde{\tau})^{d/2}}\, e^{-P_3(\tilde{\tau}, k_3, k_4, k)} \times \int_0^1 d\alpha_i \int d\tau \left[\frac{1}{(1+\tau f(\tilde{\tau}_i))^{d/2}} \times e^{-\frac{\tau\tilde{k}^2}{4(1+\tau f)}\left[1 + \frac{\tilde{k}\cdot}{\tilde{k}^2} + \frac{g(\tilde{\tau}\tilde{k})}{\tilde{k}^2}\right]}\right]$$





$\tilde{\tau}$ ARE EFFECTIVE SCHWINGER PARAMETERS FOR SKEL. GRAPH. $\alpha_i = \tau^{i}/\tilde{\tau}_r$

CHANGE VARIABLES: $\xi_1 = \dfrac{\tau}{1+\tau f(\tilde{\tau},\alpha)}$

$$\int \frac{\prod_{r=1}^{3} d\tilde{\tau}_r}{\Delta_3(\tilde{\tau})^{d/2}} e^{-P_3(\tilde{\tau},k_3,k_4,k)} \times$$

$$\int_0^1 \prod_{i=1}^{3} d\alpha_i \int d\xi_1 \left(1-\xi_1 f(\tilde{\tau},\alpha)\right)^{\frac{d}{2}-2} \times$$

$$e^{-\frac{\xi_1 \tilde{k}^2}{4}}\left[1+\frac{\tilde{k}\cdot k'}{\tilde{k}^2}+\frac{g(\tilde{\tau},\alpha,k)}{\tilde{k}^2}\right]$$

TERMS IN THE SPACETIME OPE COME FROM DOING THE ξ_1 INTEGRAL AND EXPANDING ABOUT $\tilde{k}^2=\infty$.

e.g. for $d=4$

$$\int d\xi_1\, e^{-\frac{\xi_1 \tilde{k}^2}{4}}\left[1+\frac{\tilde{k}\cdot k'(\tilde{\tau},\alpha)}{\tilde{k}^2}+\frac{g(\tilde{\tau},\alpha,k)}{\tilde{k}^2}\right]$$

$$\sim \frac{1}{\tilde{k}^2}\left[1+\frac{\tilde{k}\cdot k'(\tilde{\tau},\alpha')}{\tilde{k}^2}+\frac{g(\tilde{\tau},\alpha,k)}{\tilde{k}^2}\right]^{-1}$$

$$\sim \frac{1}{\tilde{k}^2} + \left(\text{terms w/ higher powers of } \frac{1}{\tilde{k}^2}\right)$$

EACH OF THESE TERMS IS A <u>3 PT-FN.</u> (IN SCHWINGER REPN.) $\left\langle \mathcal{O}_\kappa(k)\,\mathrm{Tr}\,\Phi^3(k_3)\,\mathrm{Tr}\,\Phi^3(k_4)\right\rangle_{g=0}$

ALTERNATIVELY, EXPAND OUT THE INTEGRAND IN ξ_1 (2nd and 3rd terms in the exponent) AND INTEGRATE TERM BY TERM

$$\int d\xi_1 \, e^{-\frac{\xi_1 \tilde{k}^2}{4}} \left[1 + \sum_{n>0} \xi_1^n \, (\quad) \right]$$

$$= \frac{1}{\tilde{k}^2} + (\text{terms w/ higher powers of } \tfrac{1}{\tilde{k}^2})$$

THEREFORE, INDIVIDUAL TERMS IN THE SPACETIME OPE COME FROM TERMS IN THE ξ_1 EXPANSION.

CAN IDENTIFY THE ξ_1 EXPANSION w/ WORLDSHEET OPE EXPANSION:-

[For large $\tilde{k}^2$, integrand gets its contribution from $\boxed{\xi_1 \sim \frac{1}{\tilde{k}^2} \to 0}$. $\xi_1 \to 0$ is the same as $\tau \to 0$.

TRANSLATING THIS INTO MODULI SPACE, EXACTLY THE REGION WHERE TWO PUNCTURES COME TOGETHER - HENCE WORLDSHEET OPE]

- Since The Power Series Expansion Is In ξ_1 (And Not In τ), Suggests ξ_1 As The More Usual Coordinate On Moduli Space $\propto |\xi_1 - \xi_2|^2$.
- Identifying Power Series In ξ_1 With Expansion In $\frac{1}{k^2}$ Suggests A Simple Relation Between C_{IJ}^k + $\tilde{C}_{IJ}^k$

$$C_{IJ}^k \sim \tilde{C}_{IJ}^k \times \frac{1}{(\Delta_K - \Delta_I - \Delta_J)!}$$

- Plausible That Associativity Of Sp. Time OPE coeffs $\Rightarrow$ Associativity Of W.S. OPE
- Existence Of A Consistent W.S. OPE Would Be Evidence For The Existence Of A Consistent World Sheet CFT.

Probing The Bulk Geometry

The Three Point Fn. In Yang-Mills Already Allows You To "See" The AdS.

$$\left\langle \mathrm{Tr}\, \Phi^{I_1}(k_1)\, \mathrm{Tr}\, \Phi^{I_2}(k_2)\, \mathrm{Tr}\, \Phi^{I_3}(k_3) \right\rangle_{g=0}$$

$$= \int_0^\infty \prod_{r=1}^{3} \frac{d\sigma_r\, \sigma_r^{(\ )}}{\hat{\Delta}(\sigma)^{d/2}}\, e^{-\frac{1}{\hat{\Delta}(\sigma)}\left[\sigma_1 k_1^2 + \sigma_2 k_2^2 + \sigma_3 k_3^2\right]}$$

Can See The AdS By Change Of Variables

$$\frac{1}{p_i} = \frac{\sigma_i}{\hat{\Delta}(\sigma)} \qquad \left[\hat{\Delta}(\sigma) = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right]$$

$$= \int \frac{dx_0}{x_0^{d+1}} \int_0^\infty \prod_{i=1}^{3} \Big(\underbrace{dp_i\, p_i^{\Delta_i - \frac{d}{2} - 1} (x_0)^{\Delta_i} e^{-x_0^2 p_i}\, e^{-\frac{k_i^2}{p_i}}}_{K_{\Delta_i}(k_i; x_0)} \Big)$$

$K_{\Delta_i}(k_i; x_0) \rightarrow$ Momentum Space Bulk To Bdy. Propagator In AdS_{d+1}

p_i Are Schwinger Parameters For The External Legs.

Can We See Non-Trivial Geometries?

Yang-Mills on $\mathbb{R}^3 \times S^1$. $\longleftrightarrow$ Non Extremal Black Hole Background