Arithmetic of Calabi-Yau Manifolds

with

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Aims:

- **To explain the fact that the periods of a Calabi–Yau manifold in terms of which we compute many observables of the effective low energy limit of string theory encode important arithmetic information about the manifold.**

- **To speculate about the role of 'quantum corrections' and mirror symmetry.**

Periods of the Quintic

Consider for definiteness, the one parameter family of quintics in $\mathbb{P}_4$

$$\mathcal{M} \;:\; P(x,\psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5\ .$$

$\mathcal{M}$ has $h^{11} = 1$ and $h^{21} = 101$.

In this simple case there is a simple relation between $\mathcal{M}$ and its mirror

$$\mathcal{W} \;=\; \mathcal{M}/\Gamma$$

$$\Gamma \;:\; (x_1,\ x_2,\ x_3,\ x_4,\ x_5) \;\mapsto\; (\zeta^{n_1}x_1,\ \zeta^{n_2}x_2,\ \zeta^{n_3}x_3,\ \zeta^{n_4}x_4,\ \zeta^{n_5}x_5)$$

where $\zeta^5 = 1$ and $\sum_i n_i \equiv 1\ mod\ 5$.

Parametrise the deformations of the complex structure by the periods of the holomorphic $(3,0)$-form Ω

$$\varpi_j(\psi) = \int_{\gamma_j} \Omega\,, \qquad \forall \quad \gamma_j \in H_3(\mathcal{M})$$

$\mathcal{M}$ has $h^{21} = 101$ and $204 = 2{\times}100 + 4$ periods while $\mathcal{W}$ has $h^{21} = 1$ and 4 periods.

These periods are (generalised) hypergeometric functions and satisfy a differential equation of order b_3. In the case of the principal periods

$$\mathcal{L}\,\varpi(\lambda) = 0\ ; \quad \lambda = \frac{1}{(5\psi)^5}$$

where

$$\mathcal{L} = \vartheta^4 - 5\lambda \prod_{i=1}^{4}(5\vartheta + i)\ , \quad \text{with} \quad \vartheta = \lambda \frac{d}{d\lambda}\,.$$

The operator $\mathcal{L}$ is of fourth order and $\lambda = 0$ is a regular singular point with all four indices equal to zero. Thus the solutions near the origin are asymptotic to

$$1,\ \log\lambda,\ \log^2\lambda,\ \log^3\lambda\,.$$

The solution that has no logarithm is the series

$$f_0(\lambda) \;=\; \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5}\,\lambda^m\;.$$

more generally the solutions are of the form

$$\begin{aligned}
\varpi_0(\lambda) &= f_0(\lambda)\\
\varpi_1(\lambda) &= f_0(\lambda)\log\lambda + f_1(\lambda)\\
\varpi_2(\lambda) &= f_0(\lambda)\log^2\lambda + 2f_1(\lambda)\log\lambda + f_2(\lambda)\\
\varpi_3(\lambda) &= f_0(\lambda)\log^3\lambda + 3f_1(\lambda)\log^2\lambda + 3f_2(\lambda)\log\lambda + f_3(\lambda)
\end{aligned}$$

where the $f_j(\lambda)$ are power series. These series will enter into our calculation of the number of rational points of $\mathcal{M}$. Recall that these solutions may be found by the method of Frobenius. That is by seeking solutions of the form

$$\varpi(\lambda,\varepsilon) \;=\; \sum_{m=0}^{\infty} a_m(\varepsilon)\,\lambda^{m+\varepsilon} \quad \text{to the equation} \quad \mathcal{L}\,\varpi(\lambda,\varepsilon) \;=\; \varepsilon^4\lambda^\varepsilon\;.$$

Integral Series

We know what the integers mean for the q-expansion of the yukawa coupling:

$$y_{ttt} = 5\left(\frac{2\pi i}{5}\right)^3 \frac{\psi^2}{\varpi_0(\psi)^2(1-\psi^5)}\left(\frac{d\psi}{dt}\right)^3 = 5 + \sum_{k=0}^{\infty}\frac{n_k k^3 q^k}{1-q^k}\,,$$

where in this expression

$$q = \exp(2\pi i t) \quad \text{and} \quad t = \frac{1}{2\pi i}\frac{\varpi_1(\lambda)}{\varpi_0(\lambda)}\,.$$

Integers however appear also in the mirror map

$$\begin{aligned}\lambda = {} & q + 154\,q^2 + 179139\,q^3 + 313195944\,q^4 \\ & + 657313805125\,q^5 + 1531113959577750\,q^6 \\ & + 3815672803541261385\,q^7 \\ & + 9970002717955633142112\,q^8 + \ldots\,.\end{aligned}$$

Rational Points

Now ask a very strange (for a physicist) question:

For the quintic $\mathcal{M}$

$$P(x, \psi) = \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5$$

how many solutions of the equation $P(x, \psi) = 0$ are there with integer x_i and how does this number vary with ψ?

Since the x_i are coordinates in a projective space and we are free to multiply the coordinates by a common scale there is no difference between seeking an integral solution and a rational solution, $x_i \in \mathbb{Q}$. This formulation is better because $\mathbb{Q}$ is a field but it is still very hard to answer in general. An easier but still interesting question is how many solutions are there over a finite field.

Field Theory While Standing on One Leg

A field $\mathbb{F}$ is a set on which $+$ and $\times$ are defined and have the usual associative and distributive properties. $\mathbb{F}$ is an abelian group with respect to addition and $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ is an abelian group with respect to multiplication.

Finite fields are uniquely classified by the number of elements which is p^N for some prime p and integer N.

The simplest finite field is $\mathbb{F}_p$ the set of integers $\bmod p$

$\mathbb{F}_7$							
x	0	1	2	3	4	5	6
x^{-1}	*	1	4	5	2	3	6

An old result, going back to Fermat, is $a^p \equiv a$ write this

$$a(a^{p-1} - 1) \equiv 0$$

it follows that

$$a^{p-1} \equiv \begin{cases} 1, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0 \,. \end{cases}$$

There is another elementary fact that is also useful. Consider

$$\sum_{a \in \mathbb{F}_p} a^n = \sum_{a \in \mathbb{F}_p} (ba)^n = b^n \sum_{a \in \mathbb{F}_p} a^n \,.$$

It follows now that

$$\sum_{a \in \mathbb{F}_p} a^n \equiv \begin{cases} 0, & \text{if } p-1 \text{ does not divide } n \\ -1, & \text{if } p-1 \text{ divides } n \,. \end{cases}$$

A Zero'th Order Result

Take now $x \in \mathbb{F}_p^5$ and $5\psi \in \mathbb{F}_p$, $(p \neq 5)$ and let

$$\nu_\lambda = \#\{x \mid P(x,\psi) \equiv 0\}, \quad \lambda = \frac{1}{(5\psi)^5}\,.$$

This number can be computed mod p with relative ease

$$\nu_\lambda \equiv \sum_{x \in \mathbb{F}_p^5} \left(1 - P(x,\psi)^{p-1}\right)$$

Expand the power and use the fact that $\sum x_i^n \equiv \begin{cases} 0, & \text{if } p-1 \text{ does not divide } n \\ -1, & \text{if } p-1 \text{ divides } n\,. \end{cases}$

The result is that

$$\nu_\lambda \equiv {}^{[p/5]}\varpi_0(\lambda) = \sum_{m=0}^{[p/5]} \frac{(5m)!}{(m!)^5}\,\lambda^m\,.$$

p-Adic Numbers

ν_λ is a definite number so we may seek to compute it exactly. We expand

$$\nu_\lambda = \nu_\lambda^{(0)} + \nu_\lambda^{(1)} p + \nu_\lambda^{(2)} p^2 + \nu_\lambda^{(3)} p^3 + \nu_\lambda^{(4)} p^4 + \ldots$$

with $0 \le \nu_\lambda^{(j)} \le p - 1$ and evaluate $\bmod p^2$, $\bmod p^3$, and so on.

This leads naturally into p-adic analysis. Given an $r \in \mathbb{Q}$ we write

$$r = \frac{m}{n} = \frac{m_0}{n_0} p^\alpha$$

where m_0, n_0 and p have no common factor. The p-adic norm of r is defined to be

$$||r||_p = p^{-\alpha}, \quad ||0||_p = 0$$

and is a norm, that is it has the properties:

$$\begin{aligned} ||r||_p &\ge 0, \\ ||r_1 r_2||_p &= ||r_1||_p \, ||r_2||_p \\ ||r_1 + r_2||_p &\le ||r_1||_p + ||r_2||_p \end{aligned}$$

Counting the Number of Points Exactly

Denote by ν_λ the number of solutions to the equation $P(x,\psi)=0$ over $\mathbb{F}_p$.

$$\nu_\lambda = {}^p f_0(\Lambda) + \left(\frac{p}{1-p}\right) {}^p f_1'(\Lambda) + \frac{1}{2!}\left(\frac{p}{1-p}\right)^2 {}^p f_2''(\Lambda) + \frac{1}{3!}\left(\frac{p}{1-p}\right)^3 {}^p f_3'''(\Lambda) + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4 {}^p f_4''''(\Lambda) + \mathcal{O}(p^5)\ .$$

This expression holds for $5 \not| \, p-1$. In the expression

$$\Lambda = \mathrm{Teich}(\lambda) = \lim_{n\to\infty} \lambda^{p^n} \quad \text{and} \quad {}^p f_0(\Lambda) = \sum_{m=0}^{p-1} \frac{(5m)!}{(m!)^5}\Lambda^m\ .$$

Now, as we have said, the number of rational points is determined by the periods and there are $b^3 = 2h^{21} + 2$ of these. The Hodge number h^{21} counts the number of parameters on which the complex structure depends and, in simple cases, this corresponds to the number of ways of deforming the defining polynomial

$$P(x,c) = \sum_{\vec{v}} c_{\vec{v}}\, x^{\vec{v}} \quad ; \quad x^{\vec{v}} = x_1^{v_1}\, x_2^{v_2}\, x_3^{v_3}\, x_4^{v_4}\, x_5^{v_5} \; .$$

The directions in which $P(x,c)$ can be deformed correspond to the monomials $x^{\vec{v}}$ considered subject to the ideal $(\partial P/\partial x_i)$. A special role is played by fundamental monomial

$$Q = x_1 x_2 x_3 x_4 x_5$$

which is related by mirror symmetry to the Kähler form of the mirror.

Return now to our special one parameter family of polynomials

$$P(x,\psi) = \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 \,.$$

$\mathcal{M}$ has $2h^{21}(\mathcal{M}) + 2 = 204 = 2 \times 100 + 4$ periods while $\mathcal{W}$ has $2h^{21}(\mathcal{W}) + 2 = 4$.

$$\begin{array}{ccccccc} 1 & \longrightarrow & Q & \longrightarrow & Q^2 & \longrightarrow & Q^3 \\ & & x^v & \longrightarrow & Q\, x^v & & \end{array}$$

This leads to 1 fourth order differential operator $\mathcal{L}_{\vec{1}}$ and 100 second order operators $\mathcal{L}_{\vec{v}}$.

There are tenth order monomials that are not included in the above scheme and which require special attention. The generators of the ideal are

$$x_1^4 \simeq \psi\, x_2 x_3 x_4 x_5 \quad \text{\& cyclic.}$$

Thus

$$x^{(4,3,2,1,0)} \simeq \psi\, x^{(0,4,3,2,1)} \simeq \cdots \simeq \psi^5\, x^{(4,3,2,1,0)} \,.$$

We can also perform the sum in our expression for the number of points to give

$$\nu_\lambda = \sum_{m=0}^{p-1} \beta_m \Lambda^m$$

with coefficients

$$\beta_m = \lim_{n\to\infty} \frac{a_{m(1+p+p^2+\ldots+p^{n+1})}}{a_{m(1+p+p^2+\ldots+p^n)}} = (-1)^m G_{5m} G_{-m}^5$$

When we include the contributions of the other periods for the case $5|p-1$ we find

$$p\nu_\lambda^* = (p-1)^5 + \sum_{\vec{v}} \sum_{m=0}^{p-2} (-1)^m \Lambda^m G_{5m} \prod_{j=1}^{5} G_{-(m+kv_j)}$$

where $k = (p-1)/5$. The contribution of $\vec{v} = (0,0,0,0,0)$ gives our previous expression. The quintic $\vec{v}$'s correspond to the other 200 periods and give the extra terms that arise when $5|p-1$. These terms have a natural interpretation as the exceptional divisors of the mirror manifold. The monomial of degree 10 contributes only for the conifold when $\psi^5 = 1$.

Rational Points over $\mathbb{F}_p$: Dwork's Character

Let

$$\Theta : \mathbb{F}_p \longrightarrow \mathbb{C}_p^*$$

be a non-trivial additive ($\Theta(x + y) = \Theta(x)\Theta(y)$) character of order p ($\Theta(x)^p = 1$). (This is a p-adic version of a character of a commutative group $G \longrightarrow \mathbb{C}$.) Thus

$$\sum_{y \in \mathbb{F}_p} \Theta(yP(x,\psi)) = p\,\delta(P(x,\psi))$$

$$p\,\nu_\lambda = \sum_{x \in \mathbb{F}_p^5} \sum_{y \in \mathbb{F}_p} \Theta(yP(x,\psi))$$

Dwork constructed such character in terms of Gauss sums

$$G_n = \sum_{x \in \mathbb{F}_p^*} \Theta(x)\,\mathrm{Teich}^n(x)$$

and in terms of these one can expand the character in the form

$$\Theta(x) = \frac{1}{p-1} \sum_{m=0}^{p-2} G_{-m}\,\mathrm{Teich}^m(x)\,.$$

Incorporating these considerations

$$\nu_\lambda = p^4 + \sum_{\vec{v}} \gamma_{\vec{v}} \sum_{m=0}^{p-2} \beta_{\vec{v},m}\, \mathrm{Teich}^m(\Lambda)\ ,$$

where the $\beta_{\vec{v},m}$ are given in terms of the Gauss sums or, equivalently, in terms of p-adic Γ functions.

- For $5 \not| (p-1)$ we only have a contribution from $\vec{v} = (0,0,0,0,0)$
- The coefficients $\beta_{\vec{v},m}$ are closely related to the coefficients in the series expansions of the periods around the regular singular point $\lambda = 0$.

Explicitly to order p:

$$\nu_\lambda = {}^{[p-1]}f_0(\lambda^p) + p\ {}^{[p-1]}f_1'(\lambda^p)$$

$$- \delta_p\, p \sum_{\vec{v}} \frac{\gamma_{\vec{v}}}{\prod_{i=1}^5 (v_i k)!}\ {}^{[p-1]}{}_2F_1(a_{\vec{v}}, b_{\vec{v}}; c_{\vec{v}}; \lambda^p) + \ldots$$

The tenth order polynomial $\vec{v} = (4, 3, 2, 1, 0)$, corresponds to a "period" that is zero everywhere, except when $\psi^5 = 1$. For these values of ψ the variety is not smooth anymore: it has 125 isolated singularities that are double points ("conifold" singularities). The calcualtion for the number of rational points makes sense even for these singular cases. A little simplification reveals the contribution to ν_λ of $\vec{v} = (4, 3, 2, 1, 0)$ as

$$24p^2(p-1)\delta(Teich(\psi)^5 - 1)\ .$$

The Zeta-Function

Consider now $N_r(\lambda) = \frac{\nu_\lambda - 1}{p-1}$ which are the numbers of projective solutions of $P = 0$ over $\mathbb{F}_{p^r}$ and form

$$\zeta(T,\lambda) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r(\lambda)\, T^r}{r}\right).$$

If $\mathcal{M}$ is a point then $N_r = 1$ for all r and

$$\sum_{r=1}^{\infty} \frac{N_r T^r}{r} = \sum_{r=1}^{\infty} \frac{T^r}{r} = -\log(1-T) \implies \zeta_{\text{pt}}(T) = \frac{1}{1-T}$$

Thus for a point

$$\prod_p \zeta_{\text{pt}}(p^{-s}) = \prod_p \frac{1}{1-p^{-s}} = \zeta_R(s)\,.$$

The Weil Conjectures

- **Rationality** (Dwork): $\zeta(T)$ is a rational function of T

- **Functional equation** (Groethendieck):

$$\zeta\left(\frac{1}{p^d T}\right) = \pm p^{d\chi/2}\, T^{\chi}\, \zeta(T)$$

where χ is the Euler characteristic and d is the real dimension of $\mathcal{M}$.

- **Riemann Hypothesis** (Deligne):

$$\zeta(T) = \frac{P_1(T)P_3(T)\ldots P_{2d-1}(T)}{P_0(T)P_2(T)\ldots P_{2d}(T)}$$

with $P_i(T)$ a polynomial with coefficients in $\mathbb{Z}$ of degree b_i. Furthermore

$$P_i(T) = \prod_{j=1}^{b_i}(1-\alpha_{ij}\,T)\,,\ |\alpha_{ij}| = p^{i/2} \text{ and } P_0(T) = 1-T\,,\ P_{2d}(T) = 1-p^d T\,.$$

The ζ-Function

We now work over $\mathbb{F}_{p^r}$ and let $N_r(\psi)$ denote the number of projective solutions to $P(x, \psi) = 0$. The ζ-function is defined by the expression

$$\zeta(T, \psi) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r(\psi) T^r}{r}\right)$$

We are led to decompose N_r into a sum of contributions $N_r = N_{r,0} + \sum_v N_{r,v}$.

$$\zeta_{\mathcal{M}}(T, \psi) = \frac{R_0(T, \psi) \prod_v R_v(T, \psi)}{(1-T)(1-pT)(1-p^2T)(1-p^3T)}$$

$$\zeta_{\mathcal{W}}(T, \psi) = \frac{R_0(T, \psi)}{(1-T)(1-pT)^{101}(1-p^2T)^{101}(1-p^3T)} .$$

In all cases, apart from the conifold, R_0 is a quartic

$$R_0 = 1 + a_0\, T + b_0\, pT^2 + a_0\, p^3T^3 + p^6T^4 .$$

The Euler Curves

Classical analysis gives an expression for the hypergeometric functions in terms of Euler's integral which is of the form

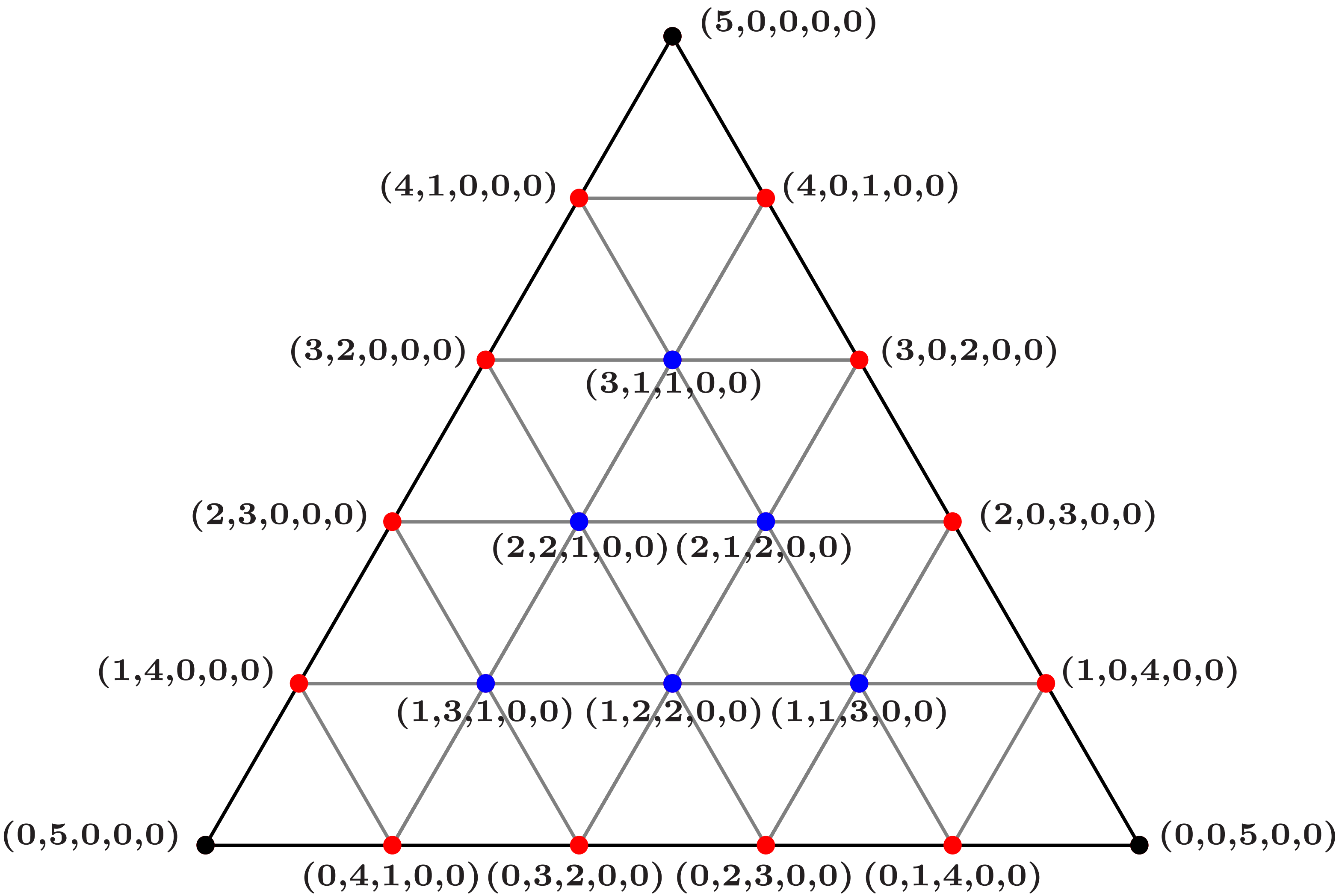
$$\int dx\, x^{-\alpha/5}(1-x)^{-\beta/5}(1-x/\psi^5)^{-(1-\beta/5)}\,.$$

If we think of Euler's integral as $\int \frac{dx}{y}$ then we are led to curves

$$\mathcal{E}_{\alpha\beta}(\psi):\quad y^5 = x^\alpha(1-x)^\beta(1-x/\psi^5)^{5-\beta}\,.$$

v	α	β
$(4,1,0,0,0)$	2	3
$(3,2,0,0,0)$	1	4
$(3,1,1,0,0)$	2	4
$(2,2,1,0,0)$	4	3

$$\mathcal{E}_{\alpha\beta} = \begin{cases} \mathcal{A} & \alpha+\beta=5 \\ \mathcal{B} & \alpha+\beta\neq 5 \text{ and } \alpha\neq\beta\,. \end{cases}$$


(5,0,0,0,0)
(4,1,0,0,0)
(4,0,1,0,0)
(3,2,0,0,0)
(3,0,2,0,0)
(3,1,1,0,0)
(2,3,0,0,0)
(2,0,3,0,0)
(2,2,1,0,0)
(2,1,2,0,0)
(1,4,0,0,0)
(1,0,4,0,0)
(1,3,1,0,0)
(1,2,2,0,0)
(1,1,3,0,0)
(0,5,0,0,0)
(0,0,5,0,0)
(0,4,1,0,0)
(0,3,2,0,0)
(0,2,3,0,0)
(0,1,4,0,0)

For the curve $\mathcal{A}$ there is a corresponding ζ-function

$$\zeta_{\mathcal{A}}(T) \;=\; \frac{R_{\mathcal{A}}(T)^2}{(1-T)(1-pT)}\,.$$

Now the existence of nontrivial fifth roots of unity is important for the mirror construction. Such roots of unity exist in $\mathbb{F}_{p^r}$ precisely when $5|p^r-1$. For given p let $\rho = 1, 2 \text{ or } 4$ be the smallest r for which $5|p^r-1$.

The $R_{\vec{v}}$ pair up in the following way:

$$\begin{aligned} R_{(4,1,0,0,0)}(T)\, R_{(3,2,0,0,0)}(T) &= R_{\mathcal{A}}(p^\rho T^\rho)^{1/\rho} \\ R_{(3,1,1,0,0)}(T)\, R_{(2,2,1,0,0)}(T) &= R_{\mathcal{B}}(p^\rho T^\rho)^{1/\rho}\,. \end{aligned}$$

So the ζ-function for $\mathcal{M}$ takes the form

$$\boxed{\zeta_{\mathcal{M}}(T,\psi) \;=\; \frac{R_0(T,\psi)\, R_{\mathcal{A}}(p^\rho T^\rho,\psi)^{\frac{30}{\rho}}\, R_{\mathcal{B}}(p^\rho T^\rho,\psi)^{\frac{20}{\rho}}}{(1-T)(1-pT)(1-p^2T)(1-p^3T)}\,.}$$

The Conifold

For the conifold $\psi^5 = 1$ the ζ-function seems to be especially simple

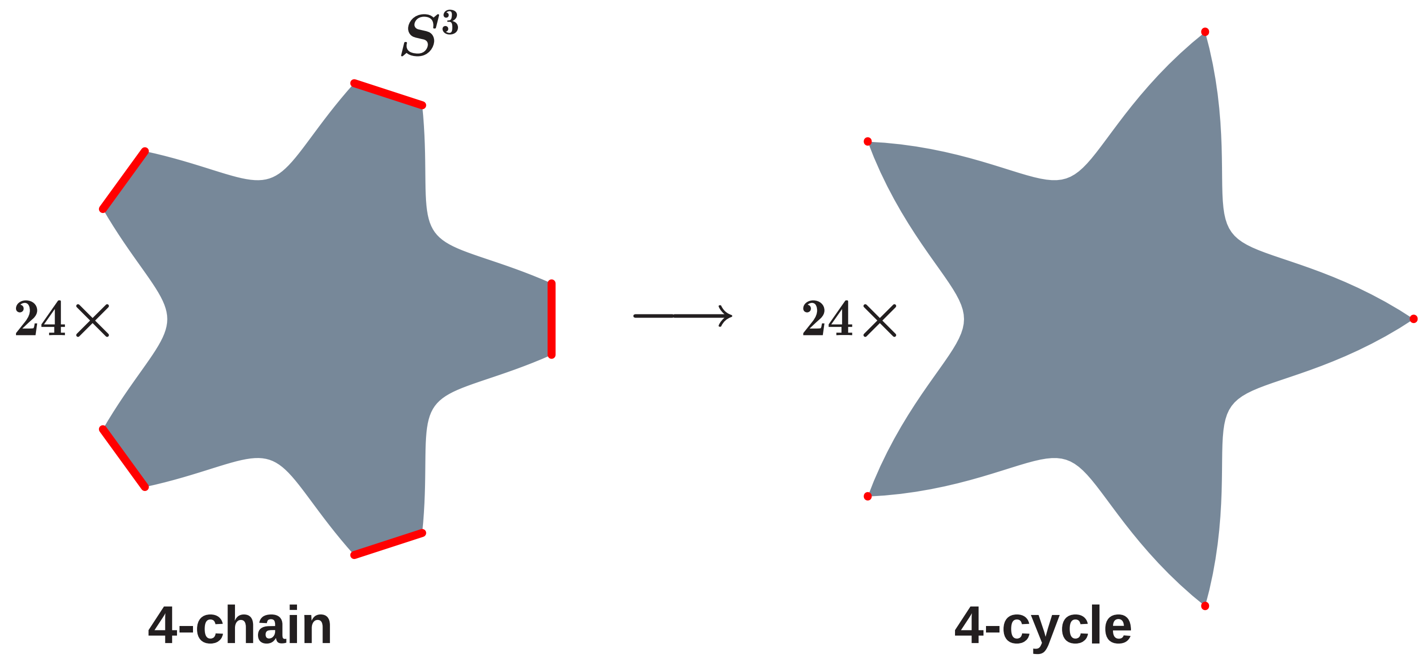
$$\zeta(T,1) \;=\; \frac{(1-\epsilon\, pT)\,(1-a_p\,T+p^3T^2)\,(1-pT)^{100}}{(1-T)(1-pT)(1-p^2T)(1-p^3T)\,(1-p^2T)^{24}} \quad ; \quad \rho = 1$$

where $\epsilon = \left(\frac{5}{p}\right) = \pm 1$ and a_p is the p-th coefficient in the q-expansion of the eigen-form, g, found by Schoen; it is the unique cusp form of weight 4 for the group $\Gamma_0(25)$.

$$g = \eta(q^5)^4\left[\eta(q)^4 + 5\eta(q)^3\eta(q^{25}) + 20\eta(q)^2\eta(q^{25})^2 + 25\eta(q)\eta(q^{25})^3 + 25\eta(q^{25})^4\right]$$

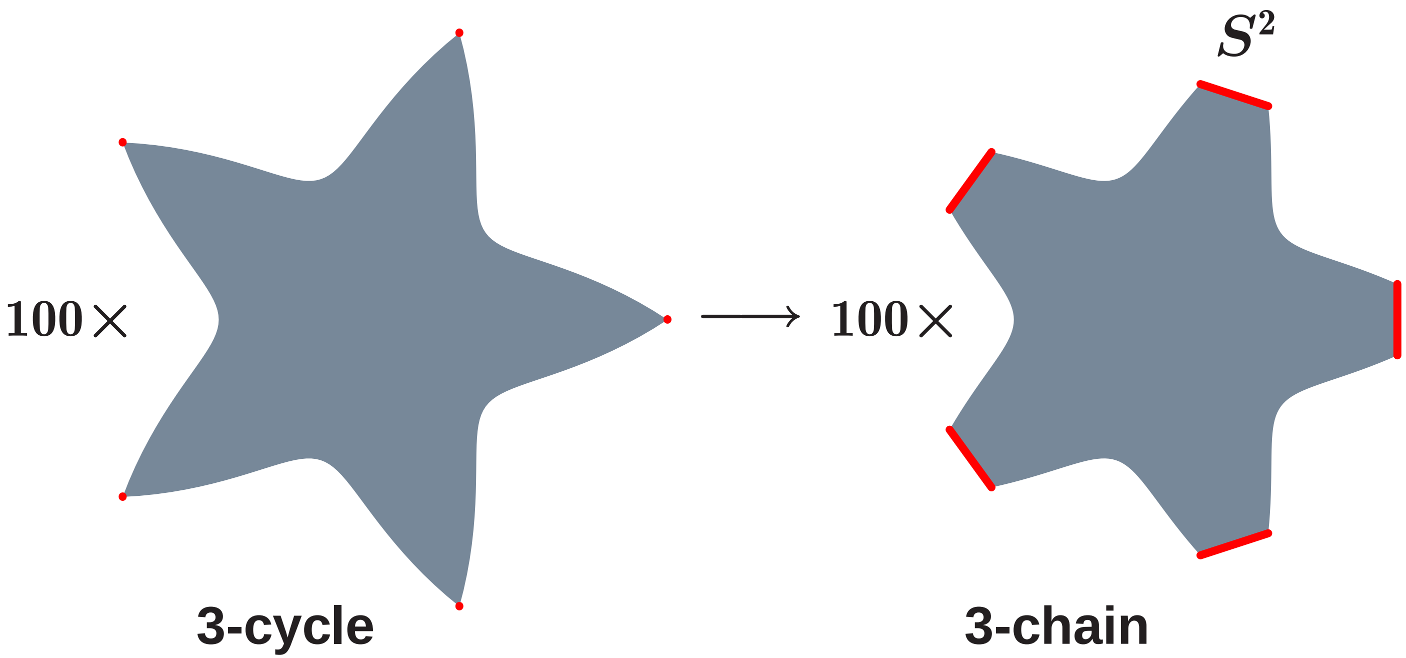
$$\begin{aligned} &= q + q^2 + 7\,q^3 - 7\,q^4 + 7\,q^6 + 6\,q^7 - 15\,q^8 + 22\,q^9 - 43\,q^{11} - 49\,q^{12} \\ &\quad - 28\,q^{13} + 6\,q^{14} + 41\,q^{16} + 91\,q^{17} + 22\,q^{18} - 35\,q^{19} + 42\,q^{21} - 43\,q^{22} \\ &\quad + 162\,q^{23} - 105\,q^{24} - 28\,q^{26} - 35\,q^{27} - 42\,q^{28} + 160\,q^{29} + 42\,q^{31} + \cdots \end{aligned}$$

125 S^3's are blown down but only 101 are independent so 24 4-cycles are created.



$$\zeta(T,1) = \frac{(1 - a_p\,T + p^3T^2)\,(1 - pT)^{100}}{(1-T)(1-p^2T)^{25}(1-p^3T)}$$

Now we resolve 125 nodes with $\mathbb{P}^1$'s, but there are 100 relations so we destroy 100 3-cycles.



$$\zeta(T,1) = \frac{(1 - a_p T + p^3T^2)\,(1-pT)^{100}}{(1-T)(1-pT)^{125}(1-p^2T)^{25}(1-p^3T)}$$

$$= \frac{(1 - a_p T + p^3T^2)}{(1-T)(1-pT)^{25}(1-p^2T)^{25}(1-p^3T)}\,.$$

The ζ-Function and Mirror Symmetry

We now work over $\mathbb{F}_{p^r}$ and let $N_r(\psi)$ denote the number of projective solutions to $P(x,\psi) = 0$.

$$\zeta(T,\psi) \;=\; \exp\left(\sum_{r=1}^{\infty} \frac{N_r(\psi)T^r}{r}\right)$$

As defined the ζ-function does not respect mirror symmetry

$$\zeta(T) \;=\; \frac{\text{Numerator of deg. } 2h^{21}+2 \text{ depending on the cpx. structure of } \mathcal{M}}{\text{Denominator of deg. } 2h^{11}+2}\,.$$

Explicitly for the quintic we have

$$\zeta_{\mathcal{M}}(T,\psi) \;=\; \frac{R_0(T,\psi)\, R_{\mathcal{A}}(p^\rho T^\rho,\psi)^{\frac{20}{\rho}}\, R_{\mathcal{B}}(p^\rho T^\rho,\psi)^{\frac{30}{\rho}}}{(1-T)(1-pT)(1-p^2T)(1-p^3T)}$$

$$\zeta_{\mathcal{W}}(T,\psi) \;=\; \frac{R_0(T,\psi)}{(1-T)(1-pT)^{101}(1-p^2T)^{101}(1-p^3T)}$$

The 5-adic Limit

The desired relations are true in the 5-adic limit. More precisely for all p and ψ

$$R_0(T,\psi) = (1-T)(1-p\,T)(1-p^2T)(1-p^3T) + \mathcal{O}\left(5^2\right)$$

$$R_{\mathcal{A}}(T,\psi)^{20}R_{\mathcal{B}}(T,\psi)^{30} = (1-p\,T)^{100}(1-p^2T)^{100} + \mathcal{O}\left(5^2\right)$$

so that

$$\zeta_{\mathcal{W}} = \frac{1}{\zeta_{\mathcal{M}}} + \mathcal{O}\left(5^2\right)$$

Compare this with the quantum corrections to the classical Yukawa coupling which we write in the form

$$\frac{y_{ttt}}{y_{ttt}^{(0)}} = 1 + \frac{1}{5}\sum_{k=0}^{\infty}\frac{n_k k^3 q^k}{1-q^k} = 1 + \mathcal{O}\left(5^2\right)$$

since Lian and Yau have shown that $5^3 | n_k k^3$ for each k.