The First Law of Thermodynamics For Rotating Black Holes in Anti-de-Sitter Space.

G.W. Gibbons

D.A.M.T.P.

Univ. Cambridge

Cambridge

UK

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based on wk with

H. Lu, D.N. Page, C.N. Pope & M.J. Perry

hep-th/0408217 G.W.G; C.N.P.; M.J.P.

hep-th/0402008 G.W.G, H.L., D.N.P; C.N.P.

Plan

1) Introduction to rotating horizons
& Laws of Thermodynamics

2) Komar Integrals

3) 4-dim rotating holes

4) 5-dim rotating holes

5) Action & Thermodynamics

6) Conformal boundary

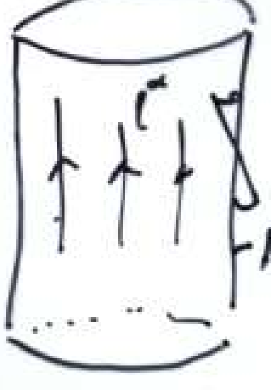
7) recent developments

Event Horizon ≡ Killing Horizon

≡ Stationary Null Surface

i) null surface

(ii) invariant under time translations



(i) ⇒ $u = \text{const.}$ s.t.
$g^{\alpha\beta}\partial_\alpha u \, \partial_\beta u = 0$

$\partial_\alpha u$ co-normal.

Let $l^\alpha = g^{\alpha\beta}\partial_\beta u$ normal

$l^\alpha l^\beta g_{\alpha\beta} = 0$ ⇒

$l^\alpha \partial_\alpha u = 0$.

l^α is both normal & tangent to N!

If $l^\alpha = \frac{dx^\alpha}{d\lambda}$ then

$x^\alpha(\lambda)$ are geodesics with affine parameterization:

$l^\alpha{}_{;\beta} l^\beta = 0$.

These curves are called **NULL GENERATORS**

ii) Let $K^\alpha \frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial t}$ be a

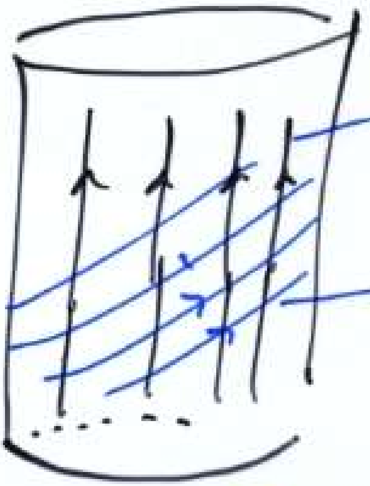
stationary Killing vector so that

K^α is timelike near infinity

(we also want it to be "non-rotating" near infinity)

N is **stationary** if it is left invariant by time translation $\Rightarrow$ $K^\alpha \partial_\alpha h = 0$

$\Rightarrow$ K^α lies in N.

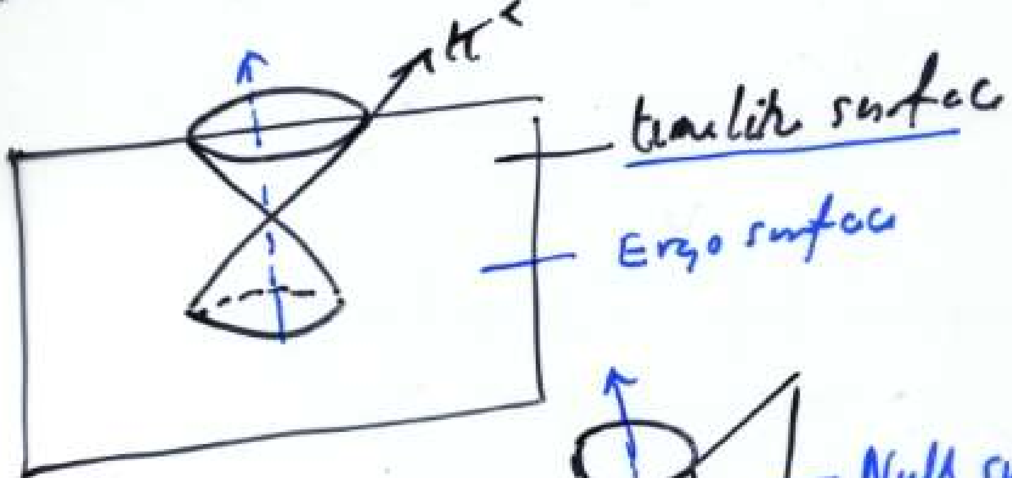


In general K^α is **spacelike** & does not coincide with ℓ^α.

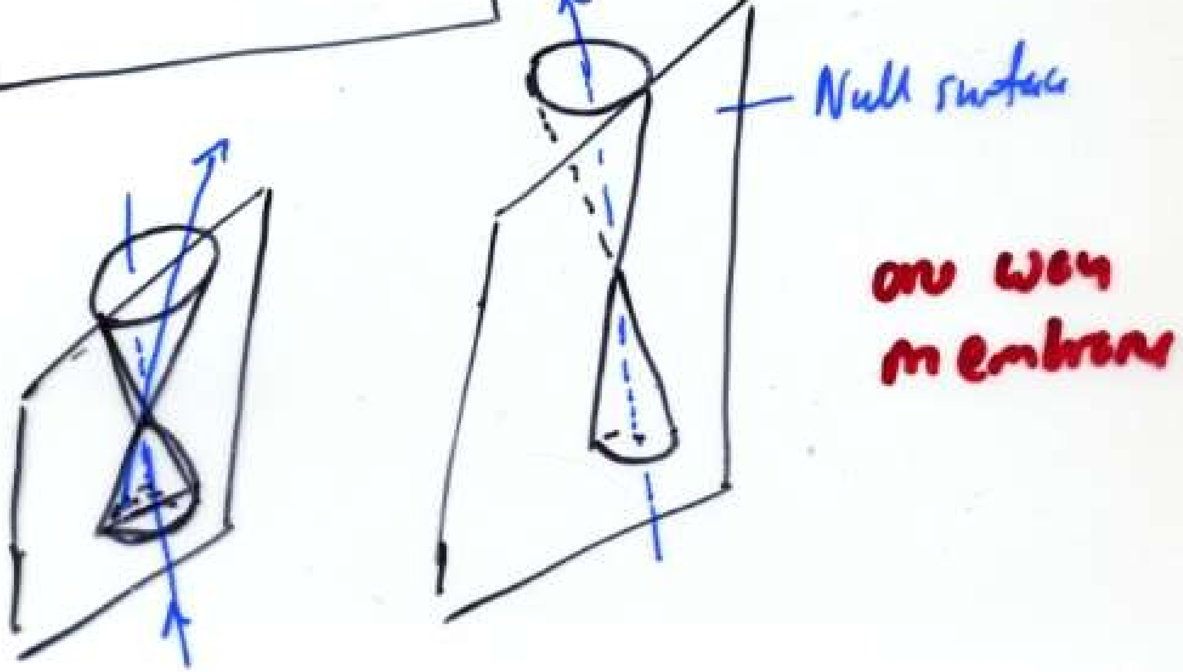
Ergo regions.

In general K^a is spacelike on N & timelike at ∞. Thus $\exists$ a surface (timelike) on which K^a is null ~~but~~

~~K^a not tangent.~~



cf.



Axisymmetric Horizons.

$\frac{\partial}{\partial t}$; $\frac{\partial}{\partial \phi^i}$ Killing fields

K^α m^α_i : $\frac{\partial}{\partial \phi^i}$ have closed orbits

$\frac{\partial}{\partial \phi^i}$ also lie in N.

$$K = \frac{\partial}{\partial t} + \Omega_i \frac{\partial}{\partial \phi_i}$$ is

a Killing field which also lies in N.

Ω_i a constant. If the black hole is rigidly rotating w.r.t. infinity one may choose Ω_i s.t.

$$L^\alpha \propto K^\alpha + \Omega m^\alpha$$

Let $L^\alpha = K^\alpha + \Omega m^\alpha$

$$L^\alpha{}_{;\beta} L^\beta = \kappa L^\alpha$$

(κ — surface gravity)

L^α is tangent to geodesic but not affinely parametrized.

Zeroth Law of Thermodynamics

One may show that under suitable assumptions

$$\kappa = \text{constant on } \mathcal{N}.$$

affine parameter ← group parameter

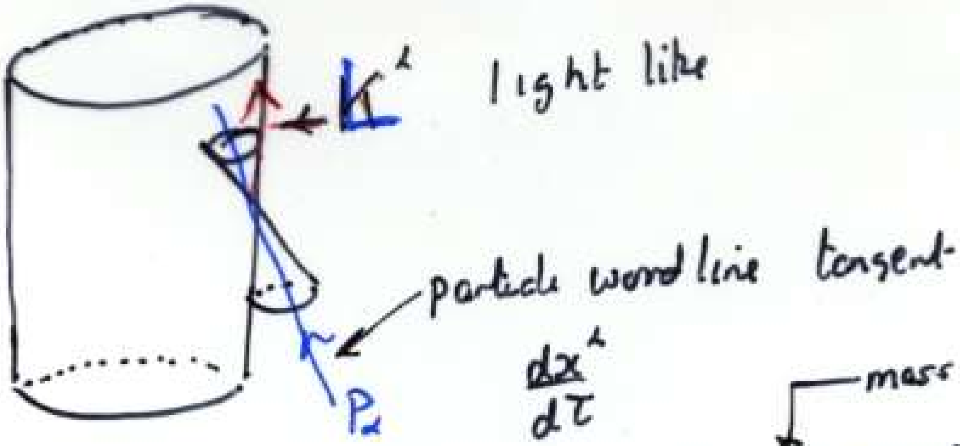
$$\boxed{\lambda = e^{\kappa t}}$$

$$[\text{cf } U = e^{\kappa u}]$$

This exponential reln. between time at infinity & time on the horizon is responsible for Hawking radiation at temperature

$$\boxed{T = \frac{\kappa}{2\pi}}$$

Energetics of Black Holes



mass

momentum $P_\alpha = m\, g_{\alpha\beta} \frac{dx^\beta}{d\tau}$

P_α is future directed timelike

K^α is light-like (& future directed)

$$\Rightarrow \quad P_\alpha K^\alpha \leq 0$$

but $\quad P_\alpha k^\alpha + \Omega P_\alpha m^\alpha = -e + \Omega J$

ϵ = conserved energy $= - k^\alpha P_\alpha$

J = " ang mom $= m^\alpha P_\alpha$

$k^\alpha \partial_\alpha = \frac{\partial}{\partial t} \quad ; \quad m^\alpha \partial_\alpha = \frac{\partial}{\partial \phi}$

$$e - \Omega J \geq 0$$

E is the energy
J ang mom
of the hole.

$$\Rightarrow \quad \boxed{dE - \Omega\, dJ \geq 0.}$$

Penrose Process

If K^α is spacelike

$$e = -K^\alpha p_\alpha \quad , \quad p_\alpha = m\frac{dx^\alpha}{d\tau}$$

can be $\underline{-ve}$ & energy extraction is possible

Note that Ergoregions & spacelike Killing fields cannot occur for BPS black holes because $K^\alpha = \bar{\varepsilon}\gamma^\alpha\varepsilon$, ε Killing spinor must be future directed timelike or null

$$K^0 = \varepsilon^\dagger\varepsilon > 0 \quad \text{in all frames.}$$

First law & Smarr-Gibbs-Duhem reln.

if $\Lambda = 0$ Komar Integrals $\Rightarrow$

$$\boxed{\frac{n-3}{n-2} E = \frac{\kappa A}{8\pi} + \Omega_i J_i} \quad *$$

Dimensional analysis (Euler's Thm l homogeneous functions) $\Rightarrow$

$$\boxed{dE = \frac{\kappa dA}{8\pi} + \Omega_i dJ_i}$$

If $\Lambda \neq 0$ it is no longer true that $*$ (which in any case requires regularization of Komar integrals) & dimensional analysis won't work because Λ has dimensions.

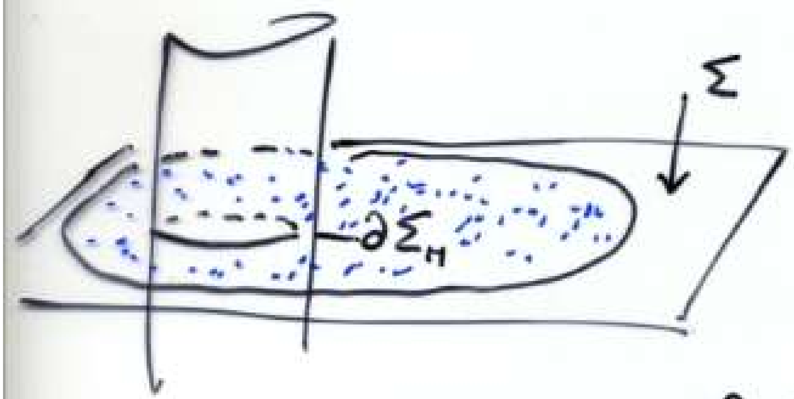
Komar Integrals

One approach to calculating E & J is to use Komar Integrals.

suppose V^α is Killing $\Rightarrow V^{\alpha;\beta} = -V^{\beta;\alpha}$

Ricci identity $\Rightarrow$ $V^{\alpha;\beta}{}_{;\beta} + R^\alpha{}_\beta V^\beta = 0.$

integrate over spacelike surface Σ.



$$\frac{1}{2}\int_{\partial\Sigma_H} V^{\alpha;\beta}\, d\Sigma_{\alpha\beta} - \frac{1}{2}\int_{\partial\Sigma_\infty} V^{\alpha;\beta}\, d\Sigma_{\alpha\beta} + \int_\Sigma R^\alpha{}_\beta V^\beta d\Sigma_\alpha = 0.$$

in the case $R^\alpha{}_\beta = 0$ we may apply this to $k^\alpha \equiv \frac{\partial}{\partial t}$ & $m^\alpha \equiv \frac{\partial}{\partial \phi}$

to identify E & J as surface integrals at ∞

Moreover:

Four Dimensions

Kerr-de Sitter metric (Carter 1968)

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{a}{\Xi}\sin^2\theta\, d\phi\right)^2 + \rho^2\left(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\,dt - \frac{r^2+a^2}{\Xi}\,d\phi\right)^2$$

$$\Delta = (r^2+a^2)(1+r^2/l^2) - 2mr \; ; \quad \Delta_\theta = 1 - a^2/l^2 \cos^2\theta$$

$$\rho^2 = r^2 + a^2\cos^2\theta \quad ; \quad \Xi = 1 - a^2/l^2$$

$$R_{\mu\nu} = -\frac{3}{l^2} g_{\mu\nu} \qquad \Lambda = -3/l^2 < 0$$

Horizon : outer root of $\Delta(r_+) = 0$

Area $\quad A = \dfrac{4\pi(r_+^2+a^2)}{\Xi}$

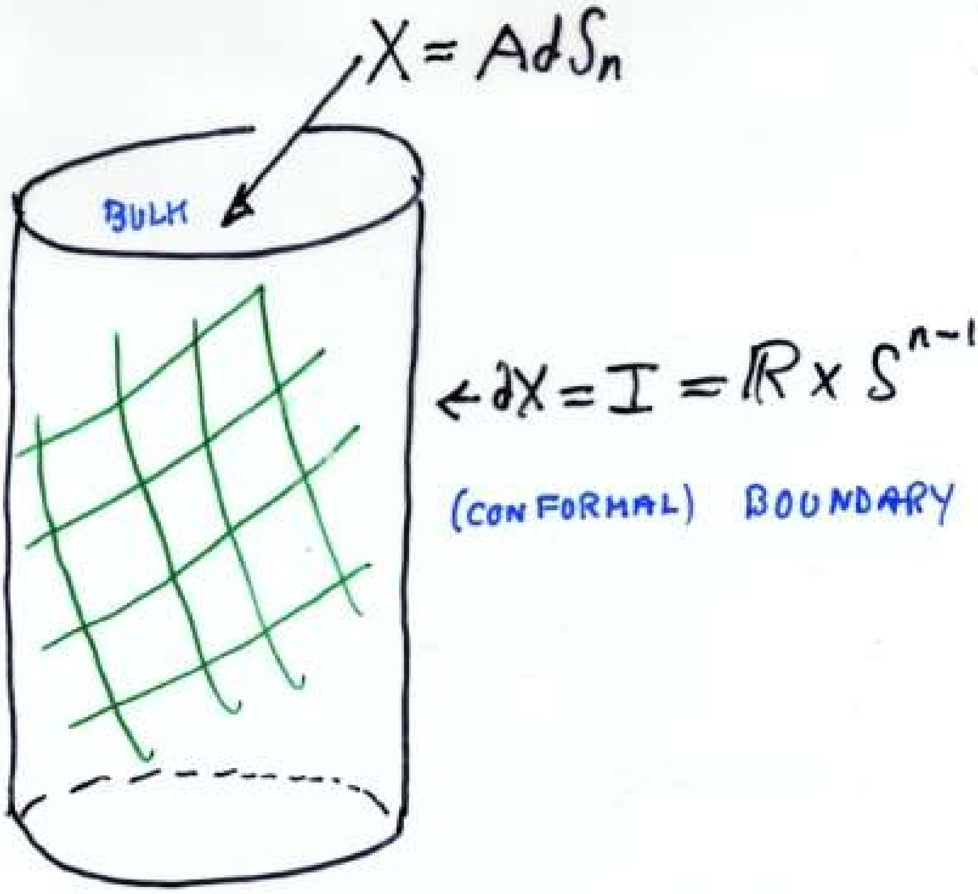
Surface gravity $\quad \dfrac{2\pi}{\kappa} = \dfrac{4\pi(r_+^2+a^2)}{r_+\left(1+\frac{a^2}{l^2}+\frac{3r_+^2}{l^2}-\frac{a^2}{r_+^2}\right)}$

Ang. velo. relative to a non-rotating frame at ∞

$$\Omega = \frac{a(1+r_+^2/l^2)}{r_+^2+a^2}$$

$$\Omega' = \Omega - a/l^2$$

$-a/l^2$ is angular velocity of Boyer-Lindquist frame w.r.t ∞.



Boyer Lindquist Coords rotate w.r.t.
a non-rotating frame at infinity
(even if "mass parameter" $m = 0$)

What are the mass & angular momenta?

Henneaux–Teitelboim : generators of $SO(3,2)$

$$J_{01} = \frac{16\pi m}{\Xi^2} \quad ; \quad J_{23} = -\frac{16\pi m a}{\Xi^2}$$

$$\Rightarrow \quad \boxed{E = \frac{m}{\Xi^2} \quad ; \quad J = \frac{ma}{\Xi^2}}$$

same as Abbott–Deser & Ashtekar–Magnon?

A calculation reveals that

$$\boxed{dE = T\,dS + \Omega\, dJ \quad ; \quad T = \kappa/2\pi \qquad S = \tfrac{1}{4}A}$$

(this establishes that $\tfrac{1}{4}A$ is the entropy)

Moreover

Thermodynamic potential

$$E - TS - \Omega J = \Phi = T\, I_4$$

gravitational action

$$I_4 = -\frac{\pi (r_+^2 + a^2)^2 (r_+^2/l^2 - 1)}{\Xi\, l^2 (3 r_+^4 l^{-2} + (1 + a^2/l^2) r_+^2 - a^2)}$$

Not all authors agree!

Hawking, Hunter, Taylor Robinson

$$E' = \frac{m}{\Xi} \quad ; \quad J = ma/\Xi^2$$

get $E' - TS - \Omega' J = T I_4$; $S = \frac{1}{4}A$

but $T dS + \Omega' dJ$ is not an exact differential!

Caldarelli et al. get our formulae using Brown-York method

Silva agrees with HHT-R but notes that E is required in 1st law.

Five Dimensions

Hawking, Hunter, Taylor-Robinson (1999)

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right)^2 + \Delta_\theta\frac{\sin^2\theta}{\rho^2}\left(a\,dt - \frac{r^2+a^2}{\Xi_a}d\phi\right)^2$$

$$+ \frac{\Delta_\theta\cos^2\theta}{\rho^2}\left(b\,dt - \frac{r^2+b^2}{\Xi_b}d\psi\right)^2 + \rho^2\left(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta}\right)$$

$$\frac{(1+r^2/l^2)}{r^2\rho^2}\left(ab\,dt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2$$

$$\Delta = \frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2/l^2) - 2m$$

$$\Delta_\theta = 1 - a^2/l^2\cos^2\theta - b^2/l^2\sin^2\theta$$

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta$$

$$\Xi_a = 1 - a^2/l^2 \quad ; \quad \Xi_b = 1 - b^2/l^2$$

$$R_{\mu\nu} = -4l^{-2}g_{\mu\nu} \quad ; \quad \Lambda = -4/l^2 < 0$$

$$A = \frac{2\pi^2(r_+^2+a^2)(r_+^2+b^2)}{r_+\,\Xi_a\,\Xi_b}$$

$$K = r_+(1+r_+^2/l^2)\left(\frac{1}{r_+^2+a^2} + \frac{1}{r_+^2+b^2}\right) - \frac{1}{r_+}$$

$$\Omega_a = \frac{a(1+r_+^2/l^2)}{r_+^2+a^2} \quad ; \quad \Omega_b = \frac{b(1+r_+^2/l^2)}{r_+^2+b^2}$$

relative to a non-rotating frame at ∞

We claim that if

$$E = \pi m \frac{(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b)}{4\Xi_a^2\Xi_b^2} \; ; \; J_a = \frac{\pi m a}{2\Xi_a^2\Xi_b} \; ; \; J_b = \frac{\pi m b}{2\Xi_b^2\Xi_a}$$

$$dE = T\,dS + \Omega_a\,dJ_a + \Omega_b\,dJ_b$$

with $S = \frac{1}{4}A$ & moreover

$$E - TS - \Omega_a J_a - \Omega_b J_b = T\, I_5$$

with

$$I_5 = \frac{\pi\beta}{4\Xi_a\Xi_b}\left[m - \frac{1}{l^2}(r_+^2+a^2)(r_+^2+b^2)\right] \; ; \; \beta = 1/T$$

the Euclidean action using the background subtraction method.

The angular momenta agree with Komar expressions

presumably our E agrees with Komar & Abbott-Deser, Hamiltonian mass but this has not been explicitly checked.

But it does agree with Ashtekar mass.

Other authors disagree !

Hawking, Hunter, Taylor-Robinson claim

$$E' = \frac{3\pi m}{4\Xi_a\Xi_b} \; ; \; J_a' = \frac{\pi m a}{2\Xi_a^2} \; ; \; J_b' = \frac{\pi m b}{2\Xi_b^2}$$

$$\Omega_a' = \frac{a\,\Xi_a}{r_+^2+a^2} \; ; \; \Omega_b' = \frac{b\,\Xi_b}{r_+^2+b^2}$$

(rotational vel wrt frame which rotates at ∞ !

$$E' - TS - \Omega_a' J_a - \Omega_b' J_b = T I_E$$

$$T dS + \Omega_a' dJ_a + \Omega_b' dJ_b$$

is not an exact differential

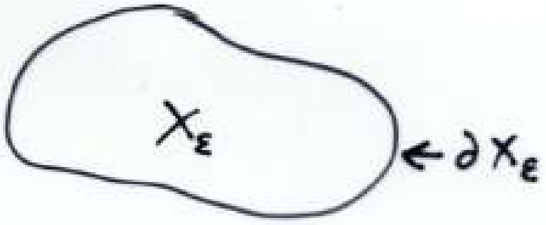
replacing J_a by J_a' gives

$$E' - TS - \Omega_a' J_a' - \Omega_b' J_b' \neq T I_E$$

& $T dS + \Omega_a' dJ_a' + \Omega_b' dJ_b'$ is

not an exact differential

The background-subtraction method



$$I_\varepsilon = \frac{-1}{16\pi}\int_{X_\varepsilon} (R-(n-2)\Lambda)\sqrt{g}\, d^n x - \frac{1}{8\pi}\int_{\partial X_\varepsilon} \mathrm{Tr}\, K \sqrt{h}\, d^{n-1}x$$

Fix bdy & metric $(\partial X_\varepsilon, h)$

1) fill in with fiducial metric
Calculate $I_\varepsilon^{\text{fiducial}}$

1) fill in with actual metric
Calculate $I_\varepsilon^{\text{actual}}$

$$I = \lim_{\partial X_\varepsilon \uparrow \infty} I_\varepsilon^{\text{actual}} - I_\varepsilon^{\text{fiducial}}$$

i.e. calculate action relative to AdS_n

Awad & Johnson get

$$E^{11} = \frac{\pi l^2}{96 \Xi_a \Xi_b} \left(7 \Xi_a \Xi_b + \Xi_a^2 + \Xi_b^2 + \frac{72 m}{l^2} \right)$$

if $a = 0 = b$, $\quad E^{11} = \frac{3\pi l^2}{32} + \frac{3\pi m}{4}$

which does not vanish for pure AdS_5 !

According to Balasubramanian & Kraus

$\frac{3\pi l^2}{32}$ is zero point energy of boundary CFT but what price $SO(4,2)$ invariance (Ashtekar & Das)

Awad & Johnson claim boundary counterterm subtraction procedure yields

$$I_5^{11} = -\frac{\beta \pi l^2}{96 \Xi_b} \Big[12 r_+^2 l^{-2} (1 - \Xi_a - \Xi_b) + \Xi_a^2 + \Xi_b^2 + \Xi_a \Xi_b + 12 r_+^4 l^{-4} - 2 (a^4 + b^4) l^{-4} - 4 a^2 b^2 l^{-4} (3 r_+^{-2} l^2 - 1) - 12 \Big]$$

but this disagrees with HHT-R & with us.

Awad & Johnson & Thermodynamics

They find $\quad E'' - TS - \Omega_a' J_a - \Omega_b' J_b = T I_5''$

but

$$T dS + \Omega_a' dJ_a + \Omega_b' dJ_b$$

is not an exact differential !

moreover

$$\frac{\partial I_5''}{\partial \beta} - \frac{\Omega_a}{\beta} \frac{\partial I_5''}{\partial \Omega_a} - \frac{\Omega_b}{\beta} \frac{\partial I}{\partial \Omega_b} \neq E''$$

$$\frac{A}{4} \neq \beta \frac{\partial I_5''}{\partial \beta} - I_5'' \; ; \; J_a \neq -\frac{1}{\beta} \frac{\partial I_5''}{\partial \Omega_a} \; ; J_b \neq -\frac{1}{\beta} \frac{\partial I_5'}{\partial \Omega_b}$$

!

Conformal Boundary geometry

$\bar{X} = X \sqcup \partial X$

$\bar{g} = \Omega^2 g \quad ; \; d\Omega \neq 0 \text{ on } \partial X$

$\bar{h} = \bar{g}|_{\partial X}$ metric on boundary

is defined only up to a conformal rescaling

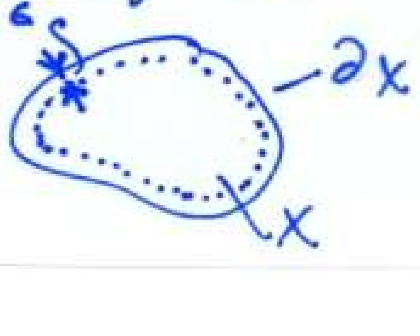
$\Omega \Rightarrow f\Omega \Rightarrow \bar{h} \Rightarrow f^2 \bar{h}$

$f \neq$ on ∂X.

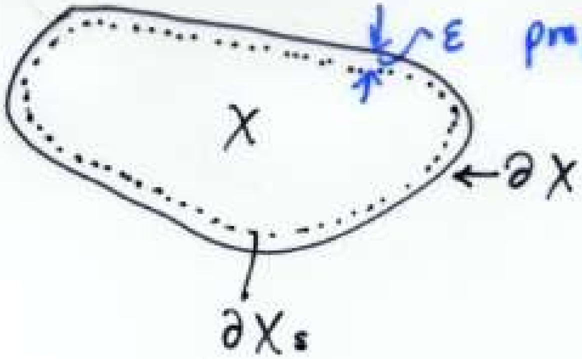
In odd bulk dimension it is known that the boundary counter term regulated action* & the regulated stress tensor (from which an "energy" & "angular momentum" can be calculated) depend on the conformal factor because of "anomalies".

* obtained by Taylor expanding in proper distance from the boundary w.r.t. the metric $\bar{g}$

study limit $\epsilon \downarrow 0$



Boundary Counter-term Renormalization



$$I_\varepsilon = -\frac{1}{16\pi}\int_{X_\varepsilon}(R-(n-2)\Lambda)\sqrt{\bar{g}}\,d^{n-1}x - \frac{1}{8\pi}\int_{\partial X_\varepsilon}\mathrm{Tr}\,K\sqrt{h}\,d^{n-1}x$$

$$\boxed{\sqrt{h}\,T^{\varepsilon}_{ij} = -2\,\frac{\delta I_\varepsilon}{\delta(\sqrt{h}h^{ij})}}$$

BOUNDARY STRESS TENSOR

Both I_ε & T^ε_{ij} divergent as $\varepsilon \downarrow 0$

To regulate expand in powers of ε & subtract off certain universal counter terms. (expressed entirely in terms of geometry of bdy & its local embedding)

• This procedure gives a non-zero energy of AdS_n ("zero point energy" of boundary theory)

$$\partial X = \mathbb{R}\times S$$

$$\tilde{E}[K^i] = \int_S T^0_{ij}K^i\,d\Sigma^j$$

Killing vector of boundary

Potential Problems

- The construction depends on choice of conformal representative on boundary in odd dimensions conformal anomalies $\Rightarrow$ I_0 & T^0_{ij} depend on this choice

- The implementation depends on identifying correctly a geodesic normal coordinate system w.r.t. conformal boundary ∂X

- It is not obvious that

$$\sqrt{h}\, T^0_{ij} = -2 \frac{\delta I_0}{\delta(\sqrt{h}\, h^{ij})}$$

(interchange of limits!)

- It is not obvious that

$\tilde{E}[K^i]$ is actually

← time translation

the thermodynamic mean energy

(Tolman redshifting etc)

Boyer-Lindquist coordinates are rotating & ellipsoidal at ∞.

Asymptotical spherical & non-rotating coordinates are given by

$$y^2 \sin^2\hat{\theta} = (r^2+a^2)\sin^2\theta$$
$$y^2 \cos^2\hat{\theta} = (r^2+b^2)\cos^2\theta$$
$$\hat{\phi} = \phi + a l^{-2} t$$
$$\hat{\psi} = \psi + b l^{-2} t$$

$$ds^2 = -\left(1+\frac{y^2}{l^2}\right)dt^2 + \frac{dy^2}{1+\frac{y^2}{l^2}-\frac{2m}{\Delta_{\hat{\theta}} y^2}} + y^2 d\hat{\Omega}_3^2$$
$$+ \frac{2m}{\Delta_{\hat{\theta}}^3 y^2}\left(dt - a\sin^2\hat{\theta}\,d\hat{\phi} - b\cos^2\hat{\theta}\,d\hat{\psi}\right)^2 + \ldots$$

$$\Delta_{\hat{\theta}} = 1 - \frac{a^2}{l^2}\sin^2\theta - \frac{b^2}{l^2}\cos^2\theta$$

$$d\hat{\Omega}_3^2 = d\hat{\theta}^2 + \sin^2\hat{\theta}\,d\hat{\phi}^2 + \cos^2\hat{\theta}\,d\hat{\psi}^2$$

$\Omega = \frac{l}{y}$ $\qquad ds^2\big|_{\partial X} = -dt^2 + l^2 d\hat{\Omega}_3^2$ $\qquad$ HHTR

$\Omega = \frac{l}{r}$ $\qquad ds^2\big|_{\partial X} = \frac{1}{\Delta_{\theta}}\left(-dt^2 + l^2 d\Omega_3\right)^2$ $\qquad$ A-J.

Tolman redshifting : temperature is space dependent

$$T(\hat{\theta}) = T_0\sqrt{\Delta_{\hat{\theta}}}$$

Higher Dimensions

- HH TR found higher dim'l. Kerr-Anti-de Sitter metric with just one rotation parameter turned on
- They left it as a problem to find the general case.
- This problem was solved by GWG H. Lü, D.N. Page & C.N. Pope using:
    1) ellipsoidal coords
    2) Kerr-Schild ansatz

$$ds^2 = d\bar{s}_0^2 + (k_\mu dx^\mu)^2$$

Anti-de-Sitter

$$k^\mu k_\mu = 0$$

& tgt. to null geodesic congruence.

The main advantage of this ansatz is that if

$$g_{\mu\nu} = g^{0}_{\mu\nu} + h_{\mu\nu} \; ; \; h^{\mu}{}_{\nu} = g^{\mu\lambda} h_{\lambda\nu}$$

then The field eqns linearize !

$h^{\mu}{}_{\nu}$ satisfies linearized Einstein eqns about background $g^{0}_{\mu\nu}$.

The solution is its own linearized approximation

Using this & guessing the higher dimensional form of $k_{\mu} dX^{\mu}$ we were able to check (using Mathematica) our ansatz in all dims ≤ 11.

The solution depends on

$\left[\frac{n-1}{2}\right]$ rotation parameters a_i in n spacetime dimensions

$\Xi_i = 1 - a_i^2 / l^2$: $\Lambda = -\frac{n-1}{l^2} < 0$

and a mass parameter m

$$\boxed{J_i = \frac{m a_i \mathcal{A}_{n-2}}{4\pi \Xi_i \left(\prod_j \Xi_j\right)}}$$ Komar

$$\mathcal{A}_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} = \mathrm{vol}\left(S^{n-2}\right)$$

$$A = \mathcal{A}_{n-2} \prod_i \frac{(r_+^2 + a_i^2)}{\Xi_i} ; \quad \Omega_i = \frac{\left(1 + \frac{r_+^2}{l^2}\right) a_i}{r_+^2 + a_i^2}$$

if

$$E = \frac{m \mathcal{A}_{n-2}}{4\pi\left(\prod_j \Xi_j\right)} \left(\sum_{i=1}^{N} \frac{1}{\Xi_i} - \frac{1}{2} \right) \quad n \text{ odd}$$

$$E = \frac{m \mathcal{A}_{n-2}}{4\pi\left(\prod_j \Xi_j\right)} \left(\sum_{i=1}^{N} \frac{1}{\Xi_i} \right) \quad n \text{ even}$$

then

$$\boxed{dE = T\,dS + \sum_i \Omega_i \, dJ_i}$$

$n = 2N + 1$

or $n = 2N + 2$

Moreover, the action is

$$I_n = \frac{\beta A_{n-2}}{8\pi\left(\prod_j \Xi_j\right)} \left(m - \frac{1}{l^2} \prod_{i=1}^{N} \left(r_+^2 + a_i^2 \right) \right)$$

n odd

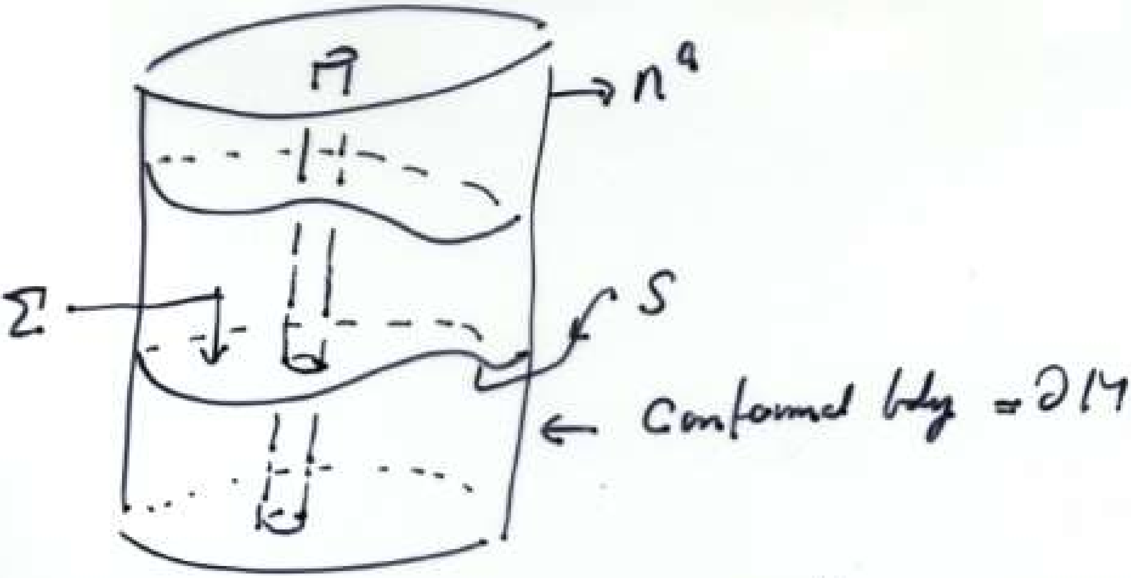
$$I_n = \frac{\beta A_{n-2}}{8\pi\left(\prod_j \Xi_j\right)} \left(m - \frac{r_+}{l^2} \prod_{i=1}^{N} \left(r_+^2 + a_i^2 \right) \right)$$

n odd.

& then

$$E - TS - \sum_i \Omega_i J_i = T I_n$$

Ashtekar Conformal mass



$$E_{ab} = \frac{l^2}{\Omega^{n-2}} C_{eafb} n^a n^b \quad \text{on } \partial M$$

$$E_{ab}{}^{:b} = 0 \quad \Rightarrow (E_{ab} K^b)^{:a} = 0$$

$$Q[K] = \frac{1}{8\pi} \frac{l}{n-3} \int_S E_{ab} K^b d\sigma^a$$ independent of S

K^a : Killing vector on the bdy

$Q[K^a]$ transform under conformal gp correctly & do not depend on conformal factor.

They are "moment maps" or "canonical generators" for Hamiltonian on Σ.

The conformal masses were calculated by Das & Mann (hep-th/0008028) in the case of just one rotation parameter. (i.e. $b=0$ in 5-dimensions).

They found

$$\frac{3m\pi}{4\Xi}$$

This does not agree with our expressions but they used a frame & hence a Killing vector field K^c which is rotating at infinity.

The obvious question is what happens if we use a non-rotating Killing vector?

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We checked that if one corrects the Das-Mann values by passing to a non-rotating frame:

The non rotating Killing vector is

$$\frac{\partial}{\partial t} + \frac{a}{l^2}\frac{\partial}{\partial \phi}$$

$$Q\left[\frac{\partial}{\partial t} + \frac{a}{l^2}\frac{\partial}{\partial \phi}\right] = Q\left[\frac{\partial}{\partial t}\right] + \frac{a}{l^2} Q\left[\frac{\partial}{\partial \phi}\right]$$

Das-Mann value ... and man J.

$$Q\left[\frac{\partial}{\partial t} + \frac{a}{l^2}\frac{\partial}{\partial \phi}\right] = E = E^{\text{Dass Mann}} + \frac{a}{l^2} J$$

We find complete agreement with our values.

Moreover we can extend the calculation to multiple rotn. parameters & again get agreement

$$\boxed{\frac{\partial}{\partial t} + \sum_{i=1}^{N} \frac{a_i}{l^2}\frac{\partial}{\partial \phi_i}}$$
~~~~

Katz – Bicak – Lynden-Bell

Procedure

These authors (Phys Rev D55 (1997) 5759) formulate a superpotential method for calculating energy & momentum using Noether's thm.

Deruelle & Katz (gr-qc/0410135) have used that formalism to calculate their values for the Kerr-Anti-de Sitter metric. The results confirm the values obtained by GWG, ~~HBP~~ & CN.P.

Gauss-Bonnet Gravity

(Deruelle & Morisawa gr-qc/0411135)

$$L = -2\Lambda + R + \alpha R^{\mu\nu\rho\sigma} P_{\mu\nu\rho\sigma}$$

(Lovelock Lagrangian)

$$P_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - 2(R_{\mu[\rho} g_{\sigma]\nu} - R_{\nu[\rho} g_{\sigma]\mu}) + R\, g_{\mu[\rho} g_{\sigma]\nu}$$

$$\Rightarrow R^{\mu}_{\ \nu} + 2\alpha R^{\mu\lambda\rho\sigma} P_{\nu\lambda\rho\sigma} = \tfrac{1}{2}\delta^{\mu}_{\nu} L$$

Linearize around AdS

Claim linearized eqns same as linearized Einstein eqns as long as

$$\Lambda \to \Lambda \left(\frac{l}{\mathcal{L}}\right)^2$$

$$\mathcal{L} = \frac{1}{2\tilde{\alpha}}\left(1 \pm \sqrt{1 - \frac{4\tilde{\alpha}}{l}}\right) \qquad \tilde{\alpha} = (D-3)(D-4)\alpha$$

$$l^2 = \frac{(D-1)(D-2)}{2\Lambda}$$

Now, using the Kerr-Schild ansatz

$$\bar{g}_{\mu\nu} + k_\mu k_\nu$$

one can compute the linearized Gauss-Bonnet solns: they are just a multiple of the Einstein ones & hence find the masses & angular momenta in the Gauss-Bonnet theory.

Deruelle & Morisawa find::

$$J_{AB}^{\text{Gauss-Bonnet}} = \sqrt{1 - \frac{4\tilde{\alpha}}{L^2}}\, J_{AB}^{\text{Einstein}}$$

~~Presumably $S^{\text{Gauss-Bonnet}} = \sqrt{1 - \frac{4\tilde{\alpha}}{L^2}}\, S^{\text{Einstein}}$~~

Generalized Smarr Relation

Barnich & Compère gr-qc/0412029

Using cohomological methods obtain

Smarr rel. directly:

$$E - \Omega J = \frac{\kappa A^{\text{spheroid}}}{8\pi} + \frac{A^{\text{spheroid}}}{8\pi}\left(M - \frac{r_+}{l^2}\pi\left(r_+^2 + a^2\right)\right)$$

which also agrees with GWG, MJP & DNP.

Conclusion

1) The situation with vacuum rotating black hole is under control: solutions are known & energetics well understood.

2) Some progress has been made on charged black holes but not all solns known.

3) More needs to be done to understand boundary stress tensor, Hawking Page transition etc.

4) Uniqueness of rotating black holes has not been proved & may not be true (black rings)

5) Higher derivative corrections can be studied

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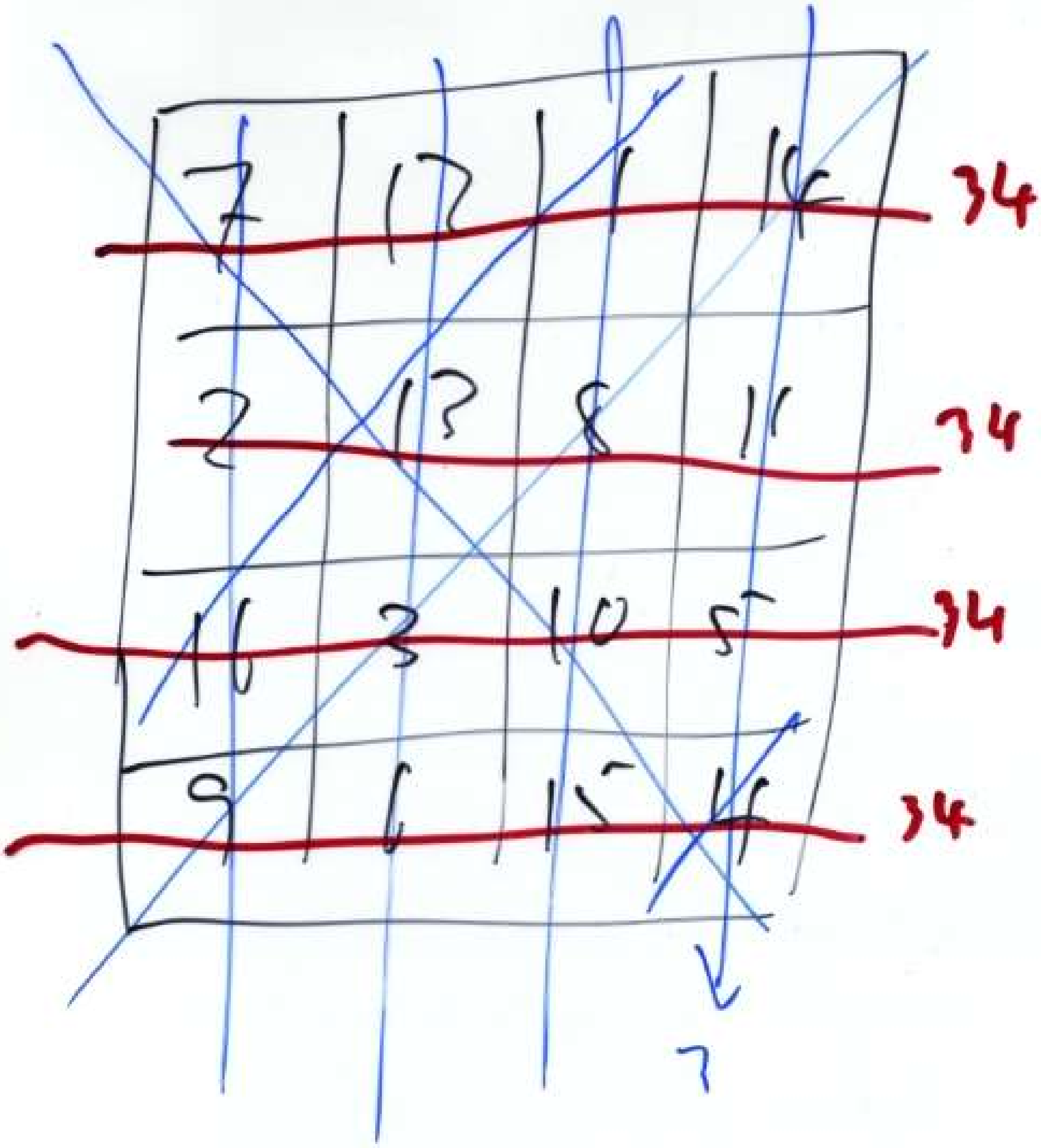
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2 13 8 11
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