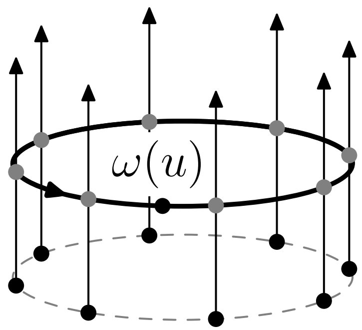
Integrability in $\mathcal{N} = 4$ Super Yang-Mills and Spinning Strings on $AdS_5 \times S^5$



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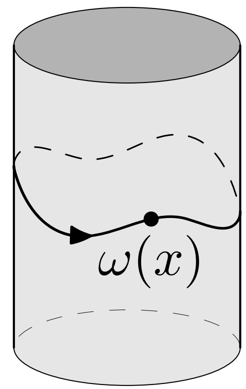


Based on work with V. Dippel, S. Frolov, V. A. Kazakov, C. Kristjansen, J. A. Minahan, K. Sakai, M. Staudacher, A. A. Tseytlin, K. Zarembo: `hep-th/0407277`, ...

Introduction

⋆ **String/gauge dualities**

- Phenomenological: Regge trajectories & stringy colour flux tubes.
- Qualitatively: 't Hooft's large-N limit.
- Concrete: Maldacena's AdS/CFT conjecture.
- Quantitative comparison feasible: BMN-limit, FT spinning strings.
- Or rather not: Three-loop problems.

⋆ **What have we learned?**

- Obtain higher-loop results (the cheap way).
- Integrability in gauge theory (even beyond one-loop) & strings.
- Not only to make mathematicians happy: Bethe Ansätze.
- Exact solution of the quantum planar spectrum within reach!
  Gauge theory: One-loop, higher-loops, all-loops, non-perturbative.
  String theory: Classical, quantum.
  Then compare...

Outline

⋆ **Cast of Characters**

- AdS/CFT, string sigma model, local operators, spin chains, BMN/FT.

⋆ **How to compare, v1: The Sledgehammer Approach**

- Classical spinning strings, scaling dimensions & dilatation operator.

⋆ **How to compare, v2: Classical Hamiltonians**

- Coherent states, light-like strings.

⋆ **How to compare, v3: Classical/Quantum, Action Variables**

- Integrability, Bethe ansätze, Lax monodromy, algebraic curves.

⋆ **Assumptions**

- Planar: No string interactions, strict large-N.
- Spectrum of states: Energies, scaling dimensions.
- Classical bosonic strings, perturbative gauge theory.

AdS/CFT Correspondence

AdS/CFT conjecture claims equivalence of [Maldacena hep-th/9711200][Gubser Klebanov Polyakov][Witten hep-th/9802150]

- IIB string theory on $AdS_5 \times S^5$ background and
- $\mathcal{N} = 4$ superconformal gauge theory.

Symmetry group: $\widetilde{\mathrm{PSU}}(2,2|4)$

- String theory: Isometries of target space.
- Gauge theory: $\mathcal{N} = 4$ superconformal invariance.

Conserved charges: Spins $\mathrm{J}_{1,2,3}$ on S^5, spins $\mathrm{S}_{1,2}$ on AdS_5 and one non-compact generator H/D: Hamiltonian/dilatation generator.

AdS/CFT predicts the agreement of

- the spectrum of H in string theory (energies $\{E\}$) with
- the spectrum of D in gauge theory: (scaling dimensions $\{D\}$).

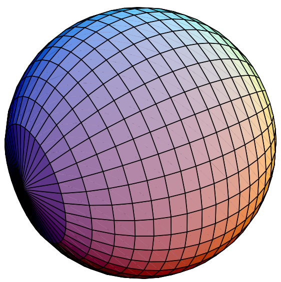
The Target Space

Target space $AdS_5 \times S^5$ is (the universal cover of) the coset

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(1,1)\times\mathrm{Sp}(2)} \simeq \frac{\mathrm{SO}(2,4)}{\mathrm{SO}(1,4)} \times \frac{\mathrm{SO}(6)}{\mathrm{SO}(5)} \times \text{ferm.}$$

Embedding into higher-dimensional flat space

$S^5 \subset \mathbb{R}^6$ $\qquad$ $AdS_5 \subset \mathbb{R}^{2,4}$



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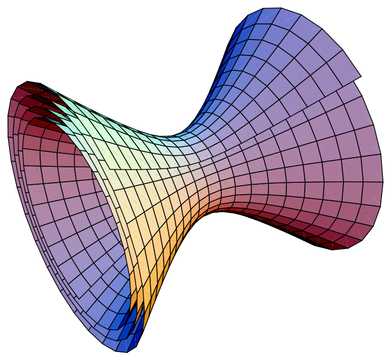


$\vec{X}^2 = \vec{X}^\mathsf{T}\vec{X} = 1.$ $\qquad$ $\vec{Y}^2 = \vec{Y}^\mathsf{T}\eta\vec{Y} = 1.$

String Theory on $AdS_5 \times S^5$

Sigma model action (Green-Schwarz superstring) [Metsaev Tseytlin]

$$S_\sigma = \frac{\sqrt{\lambda}}{2\pi} \int d\tau\, d\sigma \sqrt{-\gamma} \left(\tfrac{1}{2}(\partial_a \vec{X})^2 - \tfrac{1}{2}(\partial_a \vec{Y})^2\right) + \ldots .$$

Equations of motion

$$\partial_+\partial_- \vec{X} + \left(\partial_+ \vec{X} \cdot \partial_- \vec{X}\right)\vec{X} = 0,$$
$$\partial_+\partial_- \vec{Y} + \left(\partial_+ \vec{Y} \cdot \partial_- \vec{Y}\right)\vec{Y} = 0.$$

Virasoro constraint

$$\left(\partial_\pm \vec{X}\right)^2 = \left(\partial_\pm \vec{Y}\right)^2.$$

Symmetry Currents and Global Charges

Conserved currents of $\mathrm{SO}(6)$ and $\mathrm{SO}(2,4)$

$$j = 2\vec{X}d\vec{X}^{\mathsf{T}} - 2d\vec{X}\vec{X}^{\mathsf{T}}, \qquad \tilde{\jmath} = 2\vec{Y}d\vec{Y}^{\mathsf{T}}\eta - 2d\vec{Y}\vec{Y}^{\mathsf{T}}\eta.$$

Flatness

$$dj + j \wedge j = 0, \qquad d\tilde{\jmath} + \tilde{\jmath} \wedge \tilde{\jmath} = 0.$$

Global charges

$$J = \frac{\sqrt{\lambda}}{4\pi} \oint *j + \ldots, \qquad S = \frac{\sqrt{\lambda}}{4\pi} \oint *\tilde{\jmath} + \ldots.$$

EOM (equivalent to conservation) & Virasoro constraints via currents

$$d{*j} = 0, \qquad d{*\tilde{\jmath}} = 0, \qquad \mathrm{Tr}\, j_\pm^2 = \mathrm{Tr}\, \tilde{\jmath}_\pm^2$$

$\mathcal{N}=4$ Gauge Theory

$\mathrm{SU}(N)$ gauge field $\mathcal{A}_\mu$, 4 adjoint fermions $\Psi^a_\alpha, \Psi_{\dot\alpha a}$, 6 adjoint scalars Φ_m

$$S_{\mathcal{N}=4} = N \int \frac{d^4x}{4\pi^2}\,\mathrm{Tr}\Big(\tfrac{1}{4}(\mathcal{F}_{\mu\nu})^2 + \tfrac{1}{2}(\mathcal{D}_\mu \Phi_m)^2 - \tfrac{1}{4}[\Phi_m, \Phi_n]^2 + \ldots\Big).$$

Field theory in position space:

Correlators of local operators $\mathcal{O}_{1,2,\ldots}(x)$

$$\big\langle \mathcal{O}_1(x)\,\mathcal{O}_2(y)\,\mathcal{O}_3(z)\ldots\big\rangle = F_{1,2,3,\ldots}(x,y,z,\ldots)$$

Local operator: Gauge invariant combination of fields.

Perturbation theory at weak coupling: Feynman diagrams.

CFT: Scaling dimension $D_{\mathcal{O}}$ of $\mathcal{O}$ determines two-point function

$$\big\langle \mathcal{O}(x)\,\mathcal{O}(y)\big\rangle \sim \frac{1}{|x-y|^{2D_{\mathcal{O}}(g)}}\,.$$

Local Operators

Local, gauge-invariant combination of the fields, e.g.:

$$\mathcal{O}^a_{\mu m \alpha}(x) = 27\,\mathrm{Tr}\,\Phi_n(x)\,\mathcal{D}^\nu \Psi^a_\alpha(x)\,\mathcal{F}_{\nu\mu}(x)\,\mathrm{Tr}\,\Phi_m(x)\,\Phi_n(x)$$
$$+13g\,\mathrm{Tr}\,\mathcal{D}_\mu \mathcal{D}_\nu \Phi_m(x)\,\Phi_n(x)\,\mathcal{D}^\nu \Phi_n(x)\,\Psi^a_\alpha(x) + \ldots$$

- Position-space representation of QFT.
- Do not identify $\mathcal{O}$ and descendants $\partial_\mu \mathcal{O}$, $\partial_\mu \partial_\nu \mathcal{O}$, ...
- Drop position (x): Local operators as abstract objects.
- Building blocks $\mathcal{W}_A = \{\mathcal{D}^n \Phi, \mathcal{D}^n \Psi, \mathcal{D}^n \mathcal{F}\}$.
  Canonical gauge transformation: $\mathcal{W}_A \mapsto U \mathcal{W}_A U^{-1}$.
- Mixing problem: Lots and lots of similar operators.

Basis of Building Blocks

The building blocks are fields and their derivatives

$$\mathcal{W}_A = \{\mathcal{D}^n \Phi, \mathcal{D}^n \Psi, \mathcal{D}^n \mathcal{F}\}.$$

Consider a scalar Φ. No problem with Φ and $\mathcal{D}_\mu \Phi$, but

$$\mathcal{D}_\mu \mathcal{D}_\nu \Phi = \mathcal{D}_{(\mu} \mathcal{D}_{\nu)} \Phi + \mathcal{D}_{[\mu} \mathcal{D}_{\nu]} \Phi + \tfrac{1}{4} \eta_{\mu\nu} \mathcal{D}^2 \Phi.$$

Jacobi identity

$$\mathcal{D}_{[\mu} \mathcal{D}_{\nu]} \Phi = -ig[\mathcal{F}_{\mu\nu}, \Phi] = \mathcal{O}(g \mathcal{F} \Phi).$$

Equations of motion

$$\mathcal{D}^2 \Phi = \mathcal{O}(g \Psi^2) + \mathcal{O}(g^2 \Phi^3).$$

Only $\mathcal{D}_{(\mu} \mathcal{D}_{\nu)} \Phi$ is elementary; $\mathcal{D}_{[\mu} \mathcal{D}_{\nu]} \Phi$ and $\mathcal{D}^2 \Phi$ are reducible.

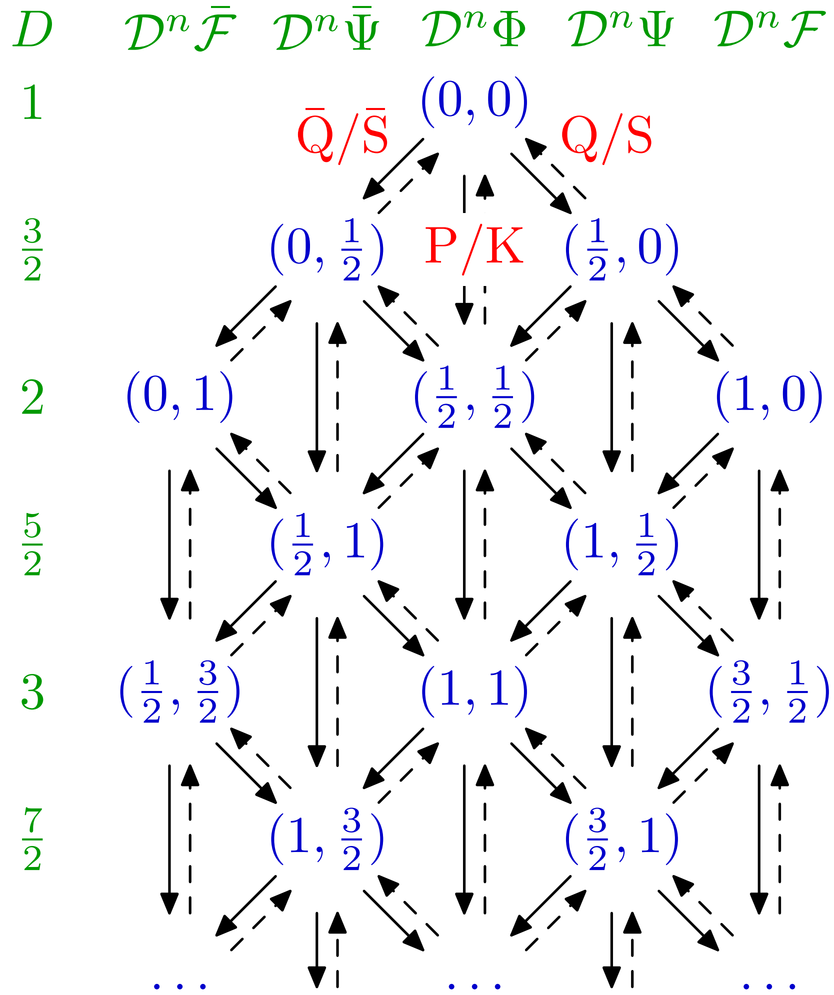
Lorentz Multiplets

D	$\mathcal{D}^n\bar{\mathcal{F}}$	$\mathcal{D}^n\bar{\Psi}$	$\mathcal{D}^n\Phi$	$\mathcal{D}^n\Psi$	$\mathcal{D}^n\mathcal{F}$
1			$(0,0)$		
$\frac{3}{2}$		$(0,\frac{1}{2})$		$(\frac{1}{2},0)$	
2	$(0,1)$		$(\frac{1}{2},\frac{1}{2})$		$(1,0)$
$\frac{5}{2}$		$(\frac{1}{2},1)$		$(1,\frac{1}{2})$	
3	$(\frac{1}{2},\frac{3}{2})$		$(1,1)$		$(\frac{3}{2},\frac{1}{2})$
$\frac{7}{2}$		$(1,\frac{3}{2})$		$(\frac{3}{2},1)$	
	$\ldots$		$\ldots$		$\ldots$
$\mathrm{SL}(2,\mathbb{C})$:	$(\frac{n}{2},\frac{n+2}{2})$	$(\frac{n}{2},\frac{n+1}{2})$	$(\frac{n}{2},\frac{n}{2})$	$(\frac{n+1}{2},\frac{n}{2})$	$(\frac{n+2}{2},\frac{n}{2})$

Poincaré/Conformal Multiplets

D	$\mathcal{D}^n\bar{\mathcal{F}}$	$\mathcal{D}^n\bar\Psi$	$\mathcal{D}^n\Phi$	$\mathcal{D}^n\Psi$	$\mathcal{D}^n\mathcal{F}$
1			$(0,0)$		
$\frac{3}{2}$		$(0,\frac{1}{2})$	P/K	$(\frac{1}{2},0)$	
2	$(0,1)$		$(\frac{1}{2},\frac{1}{2})$		$(1,0)$
$\frac{5}{2}$		$(\frac{1}{2},1)$		$(1,\frac{1}{2})$	
3	$(\frac{1}{2},\frac{3}{2})$		$(1,1)$		$(\frac{3}{2},\frac{1}{2})$
$\frac{7}{2}$		$(1,\frac{3}{2})$		$(\frac{3}{2},1)$	
	$\ldots$		$\ldots$		$\ldots$
SU(2,2):	$[0,-3,2]$	$[0,-2,1]$	$[0,-1,0]$	$[1,-2,0]$	$[2,-3,0]$

$\mathcal{N} = 4$ Supersymmetry/Superconformal Multiplet



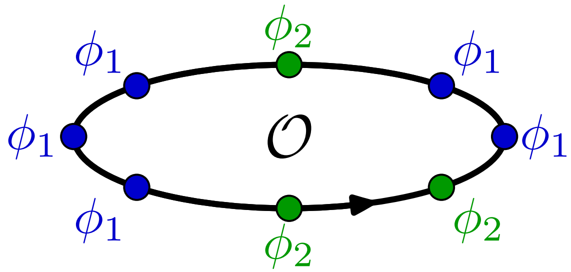
$\mathrm{PSU}(2|4|2)$: $[0;0;0,1,0;0;0]$

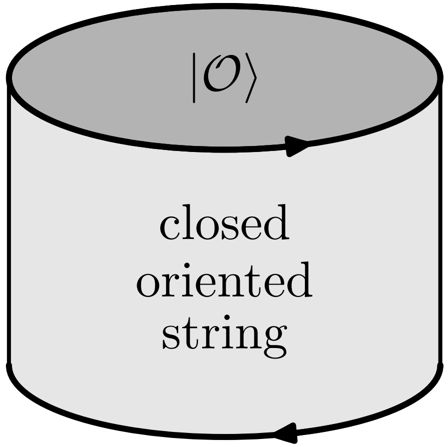
Gauge Theory and Spin Chains

Single trace operator, two complex scalars ϕ_1, ϕ_2 (a.k.a. $\mathcal{Z}, \phi$ or Z, X)

$$\mathcal{O} = \operatorname{Tr} \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2$$

Length L: # of fields

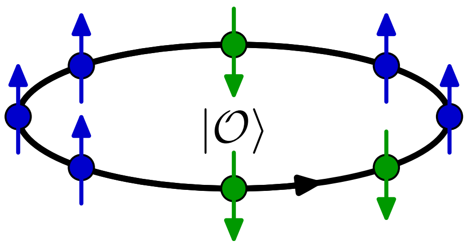




Identify $\phi_1 = |\uparrow\rangle$, $\phi_2 = |\downarrow\rangle$

$$|\mathcal{O}\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$$

Length L: # of sites



Operator mixing, quantum superposition: $|\mathcal{O}\rangle = *|\ldots\rangle + *|\ldots\rangle + \ldots$

How to compare in AdS/CFT?

AdS/CFT: String energies & gauge dimensions match: $\{E\} = \{D\}$.

Supergravity/protected states: $\{E\} = \{D\} = \{2, 3, 4, \ldots\}$. ✓

Generic states: Classical strings vs. e.g. Konishi $\operatorname{Tr} \Phi_m \Phi_m$

- String theory at $\lambda \approx \infty$: $E \sim \sqrt[4]{\lambda}$. [Gubser Klebanov Polyakov]
- Gauge theory at $\lambda \approx 0$: $D = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \ldots$. [Anselmi Grisaru Johansen][Bianchi, Kovacs Rossi, Stanev]

Cannot compare. ×

Large spin S on AdS^5: Classical strings vs. twist-two $\operatorname{Tr}(\mathcal{D}^p \mathcal{W})(\mathcal{D}^{S-p} \mathcal{W})$

- String theory at $\lambda \approx \infty$: $E \sim \sqrt{\lambda} \log S + \mathcal{O}(S^0)$. [Gubser Klebanov Polyakov]
- Gauge theory at $\lambda \approx 0$: $D = f(\lambda) \log S + \mathcal{O}(S^0)$. [Gross Wilczek][Georgi Politzer][Kotikov Lipatov]

Qualitative agreement, cannot compare quantitatively. ✓/×

Dynamical tests prevented by strong/weak nature of the duality.

Large Spin Limits of AdS/CFT

Proposal: Consider states with large spin J on S^5

- BMN limit; non-planar and near $\mathcal{O}(1/J)$ extensions. [Berenstein Maldacena Nastase]

$$\mathcal{O} \sim \operatorname{Tr} \phi_1\phi_1 \ldots \phi_2 \ldots \phi_2 \ldots \phi_2 \ldots \phi_1\phi_1 \longleftrightarrow \text{short quantum strings.}$$

- Semiclassical Spinning Strings. [Frolov Tseytlin]

$$\mathcal{O} \sim \operatorname{Tr} \phi_1 \ldots \phi_1\phi_2 \ldots \phi_2\phi_1 \ldots \phi_1\phi_2 \ldots \phi_2 \longleftrightarrow \text{long classical strings.}$$

Effective coupling constant

$$\lambda' = \frac{\lambda}{J^2}.$$

- String theory: Expansion in λ' and $1/J \sim 1/\sqrt{\lambda}$,
- Gauge theory: ℓ-loop contribution suppressed by (at least) $1/J^{2\ell}$.

Expansion in λ' apparently equivalent to expansion in λ. Compare!

Three-Loop Discrepancies

BMN state with 2 excitations

$$\mathcal{O}_n \approx \sum_{p=0}^{J} \exp \frac{2\pi i n p}{J} \operatorname{Tr} {\phi_1}^p \phi_2 {\phi_1}^{J-p} \phi_2 \qquad D - J \approx 2\sqrt{1 + \frac{\lambda n^2}{J^2}}$$

Gauge theory dimension in near BMN limit $\mathcal{O}(1/J)$ [NB Kristjansen Staudacher]

$$D - J = 2 + \frac{\lambda n^2}{J^2}\left(1 - \frac{2}{J}\right) - \frac{\lambda^2 n^4}{J^4}\left(\frac{1}{4} + \frac{0}{J}\right) + \frac{\lambda^3 n^6}{J^6}\left(\frac{1}{8} + \frac{1}{2J}\right) + \ldots$$

Energy of near plane-wave string [Callan, Lee, McLoughlin Schwarz, Swanson, Wu]

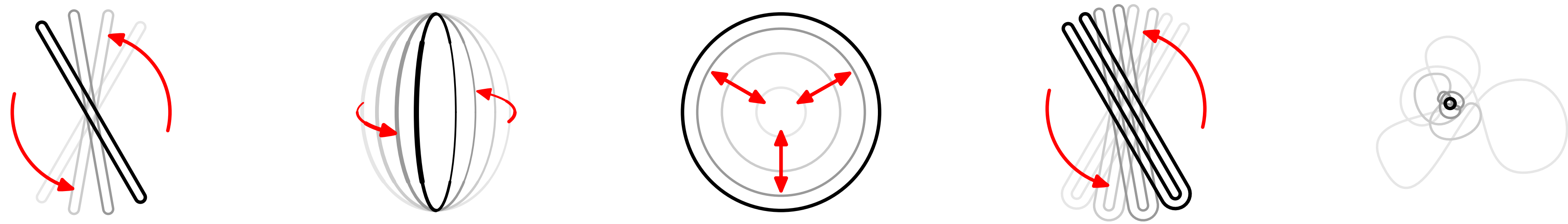
$$E - J = 2 + \lambda' n^2\left(1 - \frac{2}{J}\right) - \lambda'^2 n^4\left(\frac{1}{4} + \frac{0}{J}\right) + \lambda'^3 n^6\left(\frac{1}{8} + \frac{0}{J}\right) + \ldots$$

Three-loop mismatch also for 3 excitations. [Callan McLoughlin Swanson]

Similar disagreement for spinning strings.

Spinning Strings

Many examples investigated: [Gubser Klebanov Polyakov][Frolov Tseytlin][Minahan hep-th/0209047][Frolov Tseytlin]...



Example: Ansatz for Spinning string on $\mathbb{R}_t \times S^2$:

$$t(\tau,\sigma) = \mathcal{E}\,\tau, \quad \vec{X}(\tau,\sigma) = \begin{pmatrix} \sin\vartheta(\sigma)\,\cos\mathcal{J}\tau \\ \sin\vartheta(\sigma)\,\sin\mathcal{J}\tau \\ \cos\vartheta(\sigma) \end{pmatrix}.$$

Equation of motion and Virasoro constraint

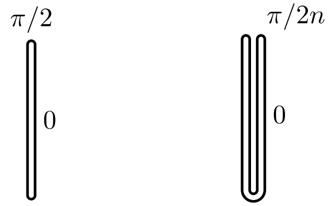
$$\vartheta'' + \mathcal{J}^2 \sin\vartheta\cos\vartheta = 0, \qquad \vartheta'^2 + \mathcal{J}^2\sin^2\vartheta = \mathcal{E}^2.$$

Periodicity

Solve equations of motion and Virasoro constraint

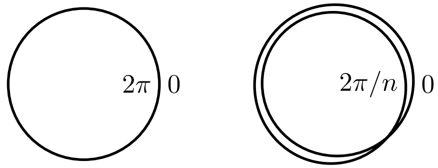
$$\vartheta(\sigma) = \mathrm{am}\big(\mathcal{E}(\sigma - \sigma_0), \eta\big), \qquad \eta = \mathcal{E}/\mathcal{J}.$$

Folded string: $\vartheta(0) = 0$ and $\vartheta'(\pi/2n) = 0$



$$J = \sqrt{\lambda}\,\mathcal{J} = \sqrt{\lambda}\,\frac{2n}{\pi}\,\mathrm{K}(1/\eta), \qquad E = \sqrt{\lambda}\,\mathcal{E} = \sqrt{\lambda}\,\frac{2n}{\eta\pi}\,\mathrm{K}(1/\eta).$$

Circular string: $\vartheta(0) = 0$ and $\vartheta(2\pi/n) = 2\pi$



$$J = \sqrt{\lambda}\,\mathcal{J} = \sqrt{\lambda}\,\frac{2n\eta}{\pi}\,\mathrm{K}(\eta), \qquad E = \sqrt{\lambda}\,\mathcal{E} = \sqrt{\lambda}\,\frac{2n}{\pi}\,\mathrm{K}(\eta).$$

Expansion

Some solutions admit expansion in $\lambda' = \lambda/J^2$. [Frolov Tseytlin]

E.g., particular two-spin solution, $J = J_1 + J_2$

$$J_1 = \sqrt{\lambda}\, \mathcal{J}_1(\eta_{1,2}), \qquad J_2 = \sqrt{\lambda}\, \mathcal{J}_2(\eta_{1,2}), \qquad E = \sqrt{\lambda}\, \mathcal{E}(\eta_{1,2}).$$

Solve J_1, J_2 for η_1, η_2, substitute in energy E and expand around $J = \infty$

$$\begin{aligned} E &= \sqrt{\lambda}\, \mathcal{E}\big(J_1/\sqrt{\lambda}, J_2/\sqrt{\lambda}\big) \\ &= J + \frac{\lambda}{J} \mathcal{E}_1(J_2/J) + \frac{\lambda^2}{J^3} \mathcal{E}_2(J_2/J) + \ldots \end{aligned}$$

Similar to weak coupling expansion in gauge theory. Want to compare!

Scaling Dimensions

Consider a local operator, e.g. Konishi operator: $\mathcal{O} = \operatorname{Tr} \Phi_m \Phi_m$.

Compute two-point function (in $4 + 2\epsilon$ dimensions)

$$\begin{aligned}
\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle &= \frac{12(1-1/N^2)}{|x-y|^{4+4\epsilon}} \left(1 + \ldots + \frac{6g^2}{\epsilon\, \mu^{2\epsilon} |x-y|^{2\epsilon}} + \ldots \right) \\
&\to \frac{12(1-1/N^2)}{\mu^{-12g^2+\ldots}\, |x-y|^4} \left(1 + 6g^2 \log \frac{1}{\mu^2 |x-y|^2} + \ldots \right) \\
&= \frac{12(1-1/N^2)}{|x-y|^{2(2+6g^2+\ldots)}} .
\end{aligned}$$

Scaling dimension of $\mathcal{O}$ [Anselmi, Grisaru, Johansen][Bianchi, Kovacs, Rossi, Stanev]

$$D_{\mathcal{O}} = 2 + 6g^2 - 12g^4 + \ldots = 2 + \frac{3 g_{\mathrm{YM}}^2 N}{4\pi^2} - \frac{3 g_{\mathrm{YM}}^4 N^2}{16\pi^4} + \ldots .$$

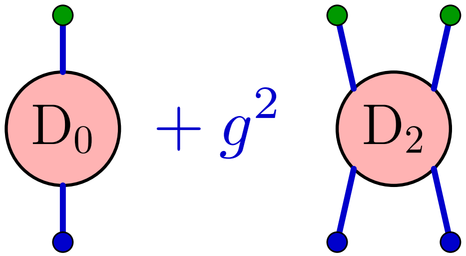
Dilatation Generator

Scaling dimensions $D_{\mathcal{O}}(g)$ as eigenvalues of the dilatation generator $\mathrm{D}(g)$

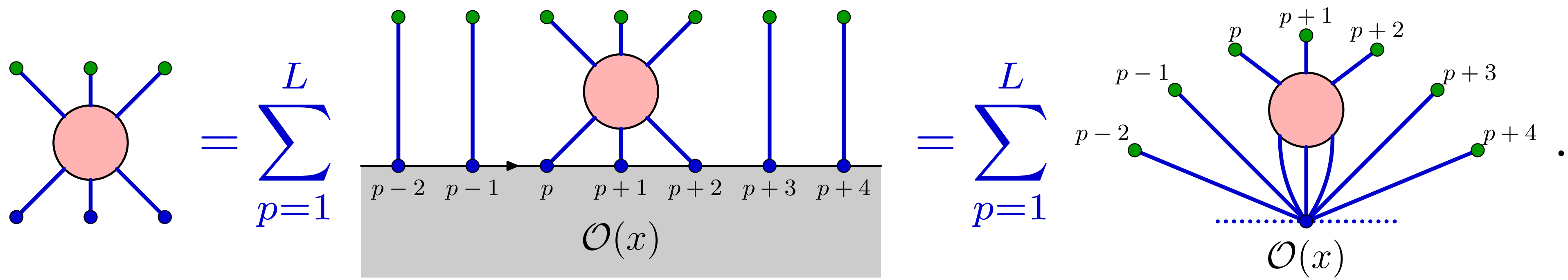
$$\mathrm{D}(g)\,\mathcal{O} = D_{\mathcal{O}}(g)\,\mathcal{O}.$$

Quantum corrections in perturbation theory: $g \sim \sqrt{\lambda}$

$$\mathrm{D}(g) = \mathrm{D}_0 + g^2\,\mathrm{D}_2 + g^3\mathrm{D}_3 + g^4\mathrm{D}_4 + \ldots\,.$$



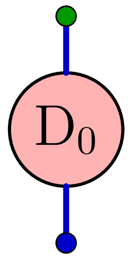
Local action along spin chain (homogeneous). Spin chain Hamiltonian



Tree Level

Classical dilatation acts on one field at a time:

$$\mathrm{D}(0) =$$



Fields have definite classical dimension $\mathrm{D}_0\mathcal{W} = \dim(\mathcal{W})\,\mathcal{W}$ with

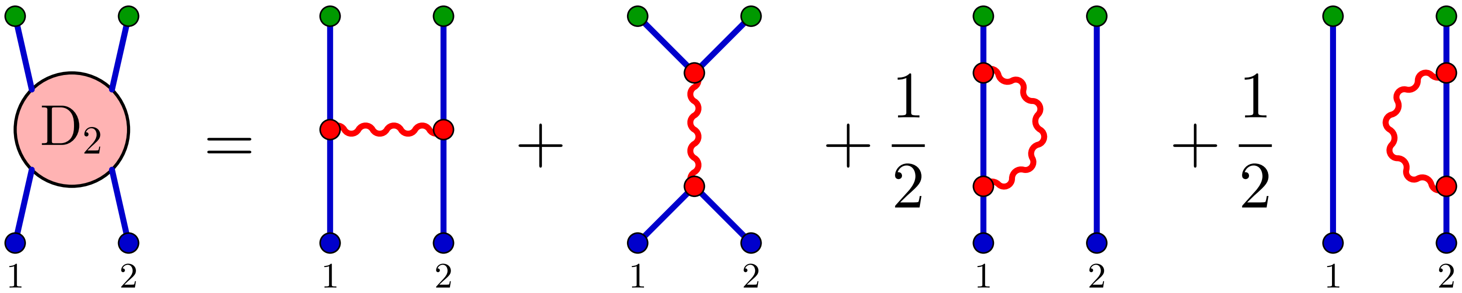
$$\dim(\mathcal{D}^n\Phi) = 1 + n, \quad \dim(\mathcal{D}^n\Psi) = \tfrac{3}{2} + n, \quad \dim(\mathcal{D}^n\mathcal{F}) = 2 + n.$$

For composites: $\mathrm{D}_0\,\mathrm{Tr}\,\mathcal{W}_1\ldots\mathcal{W}_L = \dim(\mathcal{W}_1\ldots\mathcal{W}_L)\,\mathrm{Tr}\,\mathcal{W}_1\ldots\mathcal{W}_L$ with

$$\dim(\mathcal{W}_1\ldots\mathcal{W}_L) = \sum_{p=1}^{L}\dim(\mathcal{W}_p).$$

One-Loop

One-loop $\mathcal{O}(g^2)$ dilatation operator D_2:

$\mathrm{D}_{2(12)} =$ 

Extract logarithmic (divergent) piece of Feynman diagrams.

⋆ Scalars without derivatives: $\mathrm{D}_{2(12)} = \mathcal{I}_{(12)} - \mathcal{P}_{(12)} + \frac{1}{2}\mathcal{K}_{(12)}$. [Minahan Zarembo]

Example: Konishi descendant $\mathrm{Tr}\, \phi_1{}^2 \phi_2{}^2 + \ldots$:

$$\mathrm{D}_2 \,\mathrm{Tr}\, \phi_1\phi_1\phi_2\phi_2 = +2 \,\mathrm{Tr}\, \phi_1\phi_1\phi_2\phi_2 - 2 \,\mathrm{Tr}\, \phi_1\phi_2\phi_1\phi_2,$$

$$\mathrm{D}_2 \,\mathrm{Tr}\, \phi_1\phi_2\phi_1\phi_2 = -4 \,\mathrm{Tr}\, \phi_1\phi_1\phi_2\phi_2 + 4 \,\mathrm{Tr}\, \phi_1\phi_2\phi_1\phi_2.$$

Eigenvalue $D_2 = 0$: Eigenstate: $\mathcal{O} = 2 \,\mathrm{Tr}\, \phi_1\phi_1\phi_2\phi_2 + \mathrm{Tr}\, \phi_1\phi_2\phi_1\phi_2$,
Eigenvalue $D_2 = 6$: Eigenstate: $\mathcal{O} = \mathrm{Tr}\, \phi_1\phi_1\phi_2\phi_2 - \mathrm{Tr}\, \phi_1\phi_2\phi_1\phi_2$.

Tensor Product

$\star$ Find result for generic building blocks: $\{\mathcal{D}^n\Phi, \mathcal{D}^n\Psi, \mathcal{D}^n\mathcal{F}\}$.

Too many field combinations to conveniently compute in field theory.

But: Symmetry implies D_2 classically invariant: $[\mathrm{J}_0, \mathrm{D}_2] = 0$.

Consider tensor product of two spins $\mathbb{V}_{\mathrm{F}} = \langle \mathcal{D}^n\Phi, \mathcal{D}^n\Psi, \mathcal{D}^n\mathcal{F}\rangle$

$$\mathbb{V}_{\mathrm{F}} \otimes \mathbb{V}_{\mathrm{F}} = \sum_{j=0}^{\infty} \mathbb{V}_j.$$

Two-parton multiplets $\mathbb{V}_j$ irreducible and distinct:

$$\mathrm{D}_{2(12)} = \sum_{j=0}^{\infty} D_j \, \mathcal{P}_{(12),j}.$$

Only one sequence of unknown coefficients D_j remains.

Complete One-Loop

⋆ Compute D_j from Feynman diagrams. (Boring!) [NB hep-th/0307015]

⋆ Can do better: Use symmetry. [NB hep-th/0407277]

Outline of algebraic derivation of D_j:

- Superconformal algebra: $\{\mathrm{Q}, \mathrm{S}\} \sim \mathrm{L} + \bar{\mathrm{L}} + \mathrm{R} + \mathrm{D}$.
- For particular states: $\{\mathrm{Q}_1, \mathrm{S}_1\} \sim \mathrm{D}_2$.
- Supercharges $\mathrm{Q}_1, \mathrm{S}_1$ at $\mathcal{O}(g)$ tightly constrained.
- Find $\mathrm{Q}_1, \mathrm{S}_1$ and thus D_2 up to one overall constant (g).

Result:

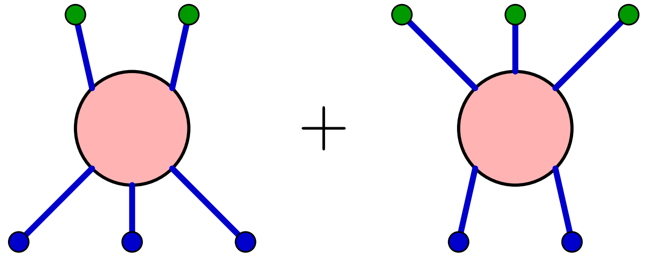
$$D_j = 2h(j), \qquad h(j) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{j}\,.$$

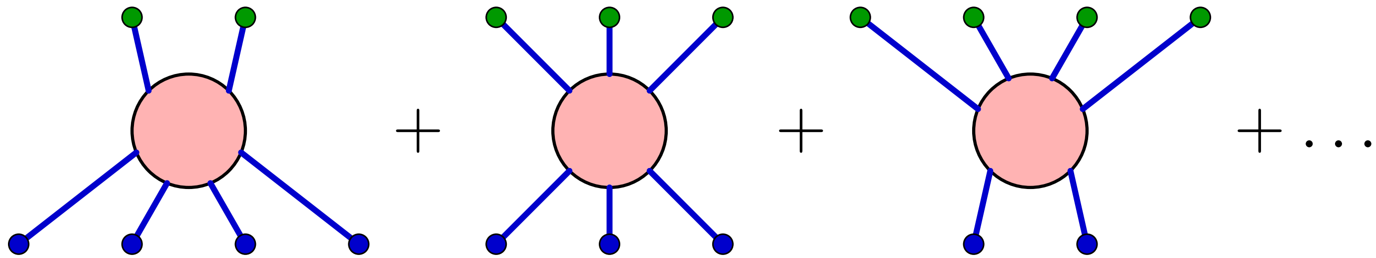
Complete dilatation operator of $\mathcal{N} = 4$ SYM: ($\mathcal{J}_{(12)}$ "total s.c. spin" op.)

$$\mathrm{D}_{2(12)} = \sum_{j=0}^{\infty} 2h(j)\,\mathcal{P}_{(12),j} = 2h(\mathcal{J}_{(12)}).$$

Higher-Loops

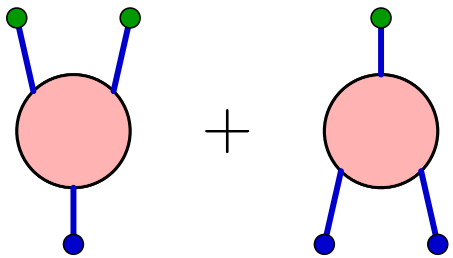
Contribution to dilatation generator at $\mathcal{O}(g^k)$ has (up to) $k+2$ legs

$\mathrm{D}_3 =$ 

$\mathrm{D}_4 =$ 

At higher loops: Length L fluctuates, dynamic spin chain. [NB hep-th/0310252]

Also (super)momenta Q, P & (super)boosts S, K are corrected, e.g.

$\mathrm{P}_1, \mathrm{K}_1, \mathrm{Q}_1, \mathrm{S}_1 =$ 

Algebraic Construction

Assume most general form and use closure of symmetry algebra

$$[\mathrm{J}_A(g), \mathrm{J}_B(g)\} = F_{AB}^{C}\, \mathrm{J}_C(g).$$

Sector of $\phi_{1,2,3}, \psi_{1,2}$: At three loops (using BMN scaling) [NB hep-th/0310252]

- Dimension of Konishi

$$D_{\mathcal{K}} = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} + \ldots$$

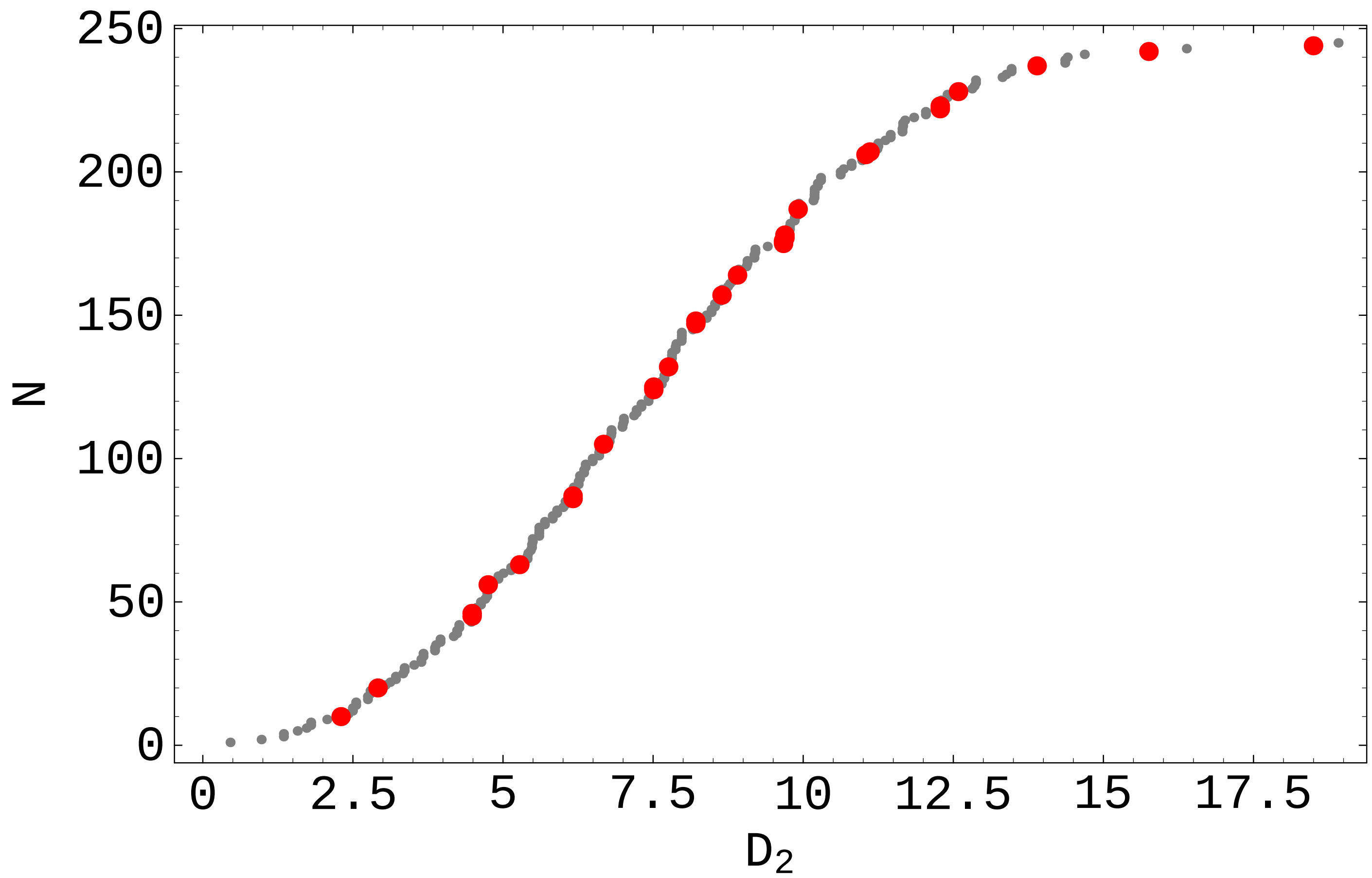
- Computation in QCD and lift to $\mathcal{N} = 4$ SYM. [Moch Vermaseren Vogt][Kotikov, Lipatov Onishchenko Velizhanin]
- Two-loop computation and lift by multiplet shortening. [Eden, Jarczak Sokatchev]
- BMN matrix model. [Klose Plefka]

Sector of $\phi_{1,2}$: At five loops (using BMN scaling & integrability)

- Reproduces BMN energy formula $D - J = \sum_k \sqrt{1 + \lambda' n_k^2}$. [NB, Dippel Staudacher]

Stringing Spins

Spectrum of $\operatorname{Tr} \phi_1{}^{14-K} \phi_2{}^K$, $K = 0, \ldots, 6$, red: $K = 7$.



How to identify states? Lowest state for $K = 7$ okay.

Comparison of Hamiltonians

Direct comparison works fine up to two loops; but. . . [NB, Frolov Staudacher Tseytlin]. . .

- Need to find suitable ansätze for string theory.
- Can solve only a few analytically.
- Extremely hard to identify gauge theory states.
- Diagonalisation tedious.
- Numerical accuracy low at small length $L \approx 20$.
- No proof. . .

Idea: Do not compare spectrum, but Hamiltonian [Kruczenski hep-th/0311203]

Problems:

- Classical vs. quantum Hamiltonian.
- Analytic in λ' vs. first few loop orders.

Coherent States

Want to reconstruct classical wave function $\vec{X}(\tau, \sigma)$ from spin chain.

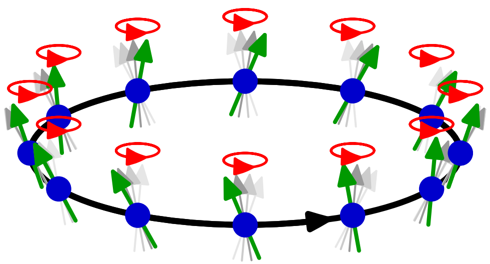
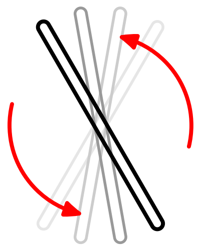
- Quantum spin $\frac{1}{2}$: Up $|\uparrow\rangle = |\phi_1\rangle$, down $|\downarrow\rangle = |\phi_2\rangle$ or superposition.
- Classical spin on S^2: ϑ (angle), φ (phase).

Quantum state has no phase information (Heisenberg).
Cannot reconstruct $\vec{X}$!

Coherent spin $\frac{1}{2}$: (over-complete basis) [Kruczenski hep-th/0311203]

$$|\vartheta, \varphi\rangle = \exp(-\tfrac{i}{2}\varphi)\cos(\tfrac{1}{2}\vartheta)\,|\uparrow\rangle + \exp(+\tfrac{i}{2}\varphi)\sin(\tfrac{1}{2}\vartheta)\,|\downarrow\rangle$$

Spins $|\vartheta, \varphi\rangle$ map directly to S^2

 $\longrightarrow$ 

Coherent Hamiltonian

Spin chain for discretised wave-function $\vec{X}(\tau, \frac{2\pi}{L}s) = \vec{X}_s = \vec{X}(\vartheta_s, \varphi_s)$.

$$|\vec{X}\rangle = |\vartheta_1, \varphi_1\rangle \cdots |\vartheta_L, \varphi_L\rangle.$$

Classical Hamiltonian (off-diagonal elements suppressed)

$$\langle\vec{X}|g^2 \mathrm{D}_2|\vec{X}\rangle = \frac{g^2}{4}\sum_{s=1}^{L}\bigl(\vec{X}_s - \vec{X}_{s+1}\bigr)^2 \to \frac{g^2}{4L}\int_0^{2\pi} d\sigma \bigl(\partial_\sigma \vec{X}\bigr)^2 = H[\vec{X}]$$

Canonical brackets for spin $\frac{1}{2}$ model

$$\{X_{s,k}, X_{s',l}\} = \tfrac{1}{2}\delta_{s,s'}\,\varepsilon_{klm}X_m.$$

Equations of motion

$$\partial_\tau \vec{X} = \{H, \vec{X}\} = \frac{g^2}{4L}\,(\partial_\sigma\partial_\sigma \vec{X}) \times \vec{X}.$$

Relativistic Expansion

Expansion of string on $\mathbb{R}_t \times S^3$ around "light-cone"

$$t(\tau,\sigma) = \mathcal{E}\,\tau, \quad \vec{X}(\tau,\sigma) = \begin{pmatrix} \cos(\frac{1}{2}\vartheta)\,\cos(\mathcal{E}\tau + \phi + \frac{1}{2}\varphi) \\ \cos(\frac{1}{2}\vartheta)\,\sin(\mathcal{E}\tau + \phi + \frac{1}{2}\varphi) \\ \sin(\frac{1}{2}\vartheta)\,\cos(\mathcal{E}\tau + \phi - \frac{1}{2}\varphi) \\ \sin(\frac{1}{2}\vartheta)\,\sin(\mathcal{E}\tau + \phi - \frac{1}{2}\varphi) \end{pmatrix}.$$

Relativistic limit $\mathcal{E} \to \infty$ while $\dot{\vartheta}, \dot{\varphi}, \dot{\phi} \sim 1/\mathcal{E}$.
Can solve for EOM for ϕ (Hopf fibration) and get ($\vec{X} \in S^2$ as before)

$$\partial_\tau \vec{X} = \frac{g^2}{4\mathcal{E}}(\partial_\sigma\partial_\sigma\vec{X}) \times \vec{X} + \ldots, \qquad H[\vec{X}] = \frac{g^2}{4\mathcal{E}}\int_0^{2\pi} d\sigma\,(\partial_\sigma\vec{X})^2 + \ldots$$

In leading order $L = \mathcal{E}$ and we get generic agreement.

Integrability

Coherent approach works fine in the investigated cases, but. . . [Kruczenski Ryzhov, Tseytlin]. . .

- Relies on perturbation theory. Complexity increases with order.
- Needs explicit form of dilatation operator as input.
  Gets very complicated beyond one-loop or for larger sectors.
- Relativistic limit subtle beyond one-loop.

Idea: Compare integrable structure. [Kazakov, Marshakov Minahan, Zarembo]

- Gauge theory: Integrability at one loop and higher loops.
- Bethe ansätze.
- Thermodynamic limit.
- String theory Lax connection and monodromy.
- Analytic properties of the monodromy.
- Algebraic curve.

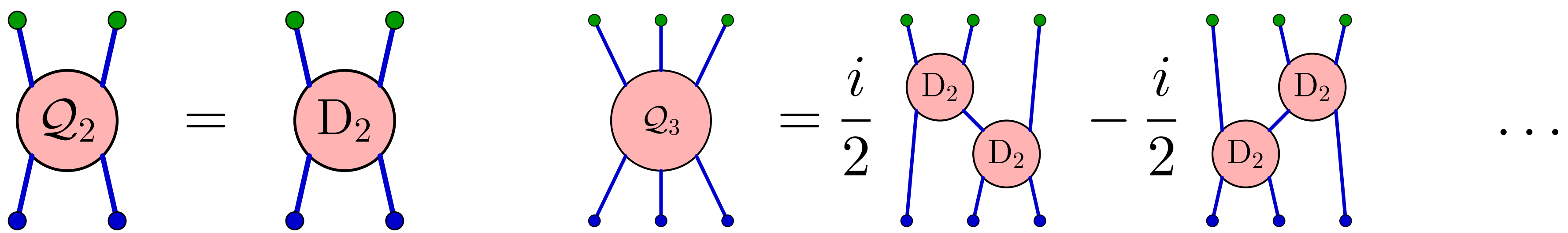
One-Loop Integrability

Only in planar limit!

Existence of higher charges $\mathcal{Q}_{2,3,4,\ldots}$ (scalar, commuting),

$$[\mathrm{J}_0, \mathcal{Q}_r] = [\mathcal{Q}_r, \mathcal{Q}_s] = 0, \qquad \mathrm{D}_2 = \mathcal{Q}_2.$$

Structure of charges



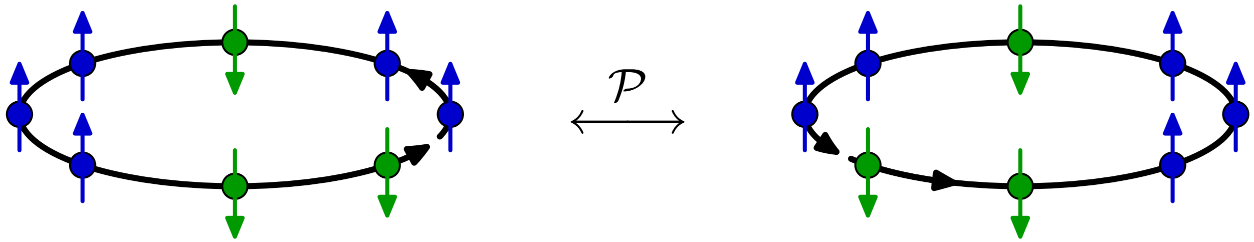
One-loop integrability found for

- **Sector of scalars** $\{\Phi_m\}$**:** $\mathfrak{so}(6)$ spin chain. [Minahan Zarembo]
- **Complete** $\mathcal{N} = 4$ **SYM:** $\mathfrak{psu}(2,2|4)$ super spin chain. [NB Staudacher]
- Sectors of $\mathcal{N} < 4$ theories. [Braun Derkachov Manashov] [Braun, Derkachov Korchemsky Manashov] [Belitsky hep-ph/9907420] [NB, Ferretti Heise, Zarembo] [Wang Wu] . . .

Test for Integrability

Consider parity $\mathcal{P}$ (charge conjugation/spin chain/world sheet):

$$\operatorname{Tr} \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2 \stackrel{\mathcal{P}}{\longleftrightarrow} \operatorname{Tr} \phi_2\phi_2\phi_1\phi_1\phi_1\phi_2\phi_1\phi_1$$



Even/odd charges have even/odd parity

$$\mathcal{P}\mathcal{Q}_r\mathcal{P}^{-1} = (-1)^r\,\mathcal{Q}_r.$$

Implies degenerate pairs of opposite parity: (only in planar limit!)

$$D_+ = D_-.$$

Test for integrability. [Grabowski Mathieu]

Higher-Loop Integrability

No formalism yet (R-matrix, Yang-Baxter equation, ...).

Existence of higher charges $\mathcal{Q}_r(g)$, [NB Kristjansen Staudacher]

$$[\mathrm{J}(g), \mathcal{Q}_r(g)] = [\mathcal{Q}_r(g), \mathcal{Q}_s(g)] = 0, \qquad \mathrm{D}(g) = \mathrm{D}_0 + g^2 \mathcal{Q}_2(g).$$

Test: Planar parity pairs preserved at higher loops [NB Kristjansen Staudacher]

$$D_+(g) = D_-(g).$$

Higher-loop integrability for

- **Sector of scalars $\{\Phi_m\}$:** Not closed at higher loops due to mixing.
- **Sector of $\{\phi_{1,2,3}, \psi_{1,2}\}$:** Observed at three-loops (pairs). [NB hep-th/0310252]
  Even through length fluctuates.
- **Sector of $\{\phi_{1,2}\}$:** Construct five-loops dilop. via integrability. [NB, Dippel Staudacher]

Bethe Ansatz

Bethe equations for sector of $\{\phi_1, \phi_2\}$. Bethe roots: $u_k \in \mathbb{C}$. [Minahan Zarembo]

$$\prod_{\substack{j=1 \\ j\neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i} = \left(\frac{u_k + i/2}{u_k - i/2}\right)^L .$$

Momentum constraint and one-loop scaling dimension:

$$\prod_{j=1}^{K} \frac{u_j - i/2}{u_j + i/2} = 1, \qquad D_2 = \sum_{j=1}^{K} \frac{1}{{u_j}^2 + 1/4}.$$

Example: Konishi $\mathcal{O} = \mathrm{Tr}\, {\phi_1}^2 {\phi_2}^2 + \ldots$, $L = 4$, $K = 2$. Find

$$u_{1,2} = \pm\frac{1}{\sqrt{12}}, \qquad D_2 = \frac{1}{1/12 + 1/4} + \frac{1}{1/12 + 1/4} = 6$$

Higher-Loop Bethe Ansatz

Higher-loops for sector of $\{\phi_1, \phi_2\}$. Modified Inozemtsev. [Serban Staudacher][NB, Dippel Staudacher]

$$\prod_{\substack{j=1\\j\neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i} = \frac{x(u_k + i/2)^L}{x(u_k - i/2)^L}, \qquad x(u) = \tfrac{1}{2}u + \tfrac{1}{2}\sqrt{u^2 - 2g^2}\,.$$

Momentum constraint and one-loop scaling dimension:

$$\prod_{j=1}^{K} \frac{x(u_j - i/2)}{x(u_j + i/2)} = 1, \quad D = L + g^2 \sum_{j=1}^{K} \left(\frac{i}{x(u_j + i/2)} - \frac{i}{x(u_j - i/2)} \right).$$

Example: Konishi $L = 4$, $K = 2$. Find

$$u_{1,2} = \pm \frac{1}{\sqrt{12}} \left(1 + 4g^2 - 5g^4 + \ldots\right), \quad D = 4 + 6g^2 - 12g^4 + 42g^6 + \ldots$$

Bethe Ansätze

⋆ **For conformal $\mathcal{N} = 4$ gauge theory**

- Complete at one loop: Seven types of Bethe roots. ✓ [NB Staudacher]
- Sector of ϕ_1, ϕ_2: Up to very high loop order. ✓ [NB, Dippel Staudacher]
- Sector of ϕ_1, ψ_1: Up to three loops. ✓ [Staudacher hep-th/0412188]
- Sector of Φ_m: Higher loops in the thermodynamic limit. ✓ [Minahan hep-th/0405243]
- States with fluctuating length L. ×
- Higher rank algebras L beyond one loop. ×

⋆ **For related models**

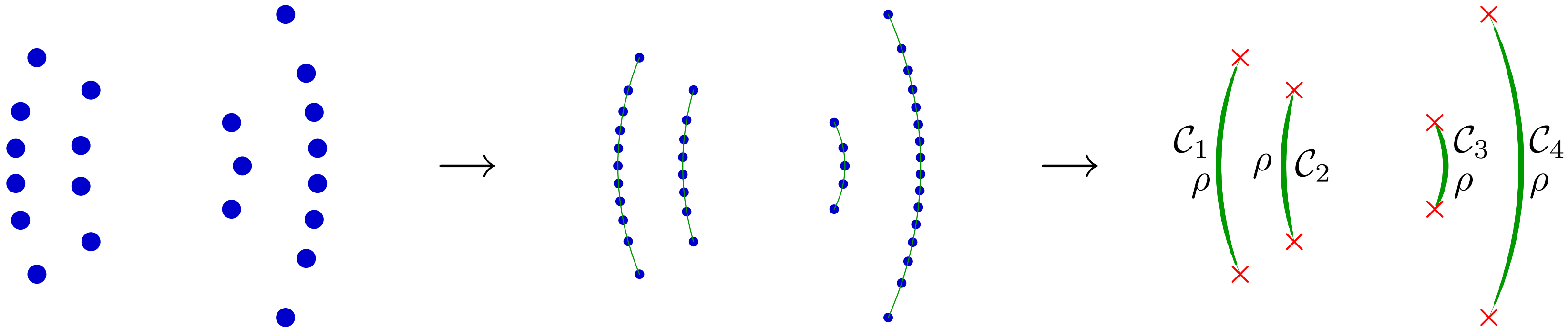
- Near-BMN strings: Up to $\mathcal{O}(1/J)$ and possibly beyond? [Arutyunov Frolov Staudacher]
  Reproduces $\sqrt[4]{\lambda}$ for generic states.
- Large N_c QCD, closed & open chains. [NB, Ferretti Heise, Zarembo]
- Sectors of $\mathcal{N} = 1$, $\mathcal{N} = 2$ gauge theory, etc.. [Chen Wang Wu][DeWolfe Mann][Di Vecchia Tanzini]

Thermodynamic Limit

[NB, Minahan Staudacher Zarembo]

- Long spin chains, $L \to \infty$.
- Large number of Bethe roots $K \sim L$.
- Low energy, $D_2 \sim 1/L$.

Roots u_k condense on (disconnected) contours $\mathcal{C} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_A$:



Discrete sums turn into integrals ($\prod = \exp \sum \log$) with density $\rho(u)$

$$\sum_{k=1}^{K} f(u_k) \to \int_{\mathcal{C}} du\, \rho(u)\, f(u).$$

Bethe Equations in the Thermodynamic Limit

Bethe equations in thermodynamic limit become integral equations

$$2 \int_{\mathcal{C}} \frac{dv\, \rho(v)}{v-u} + \frac{1}{u} = 2\pi n_a \quad \text{for } u \in \mathcal{C}_a$$

Momentum constraint and one-loop scaling dimension:

$$\int_{\mathcal{C}} \frac{du\, \rho(u)}{u} = 2\pi n_0\,, \qquad D_2 = \frac{2}{L} \int_{\mathcal{C}} \frac{du\, \rho(u)}{u^2}\,.$$

Example: Two-cut solution $n_{1,2} = \pm n$ [NB, Minahan Staudacher Zarembo]

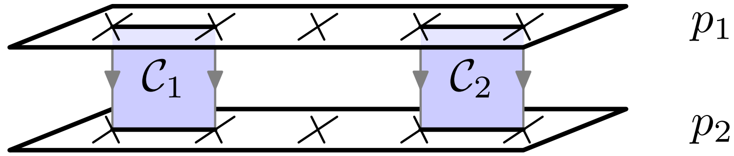
$$D = L + \frac{2\lambda n^2}{\pi^2 L} \mathrm{K}(t)\big(\mathrm{E}(t) + (t-1)\mathrm{K}(t)\big) + \ldots, \qquad \frac{K}{L} = 1 - \frac{\mathrm{E}(t)}{\mathrm{K}(t)}\,.$$

Agrees with folded string at one loop (two loops) [NB, Frolov Staudacher Tseytlin][Serban Staudacher]

Algebraic Curve

Define a function $p(u)$ (quasi-momentum) with two sheets $p_k(u)$

$$p_{1,2}(u) = \pm \int_{\mathcal{C}} \frac{dv\, \rho(v)}{v-u} \pm \frac{1}{2u}\,,$$



Bethe equations: $\not{p}(u) := \frac{1}{2}p(u+\epsilon) + \frac{1}{2}p(u-\epsilon)$

$$\not{p}_1(u) - \not{p}_2(u) = 2\pi n_a \quad \text{for } u \in \mathcal{C}_a$$

Quasi-momentum $p(u)$ is continuous (modulo 2π).

Derivative $p'(u)$ has the following properties [Kazakov, Marshakov Minahan, Zarembo]

- Analytic and single-valued. ✓
- Finitely many branch cuts. ✓
- Double-pole at $u = 0$. ✓

Algebraic curve: $F(u, p') = P_2(u)p'^2 + P_0(u) = 0$.

Bosonic String on $\mathbb{R} \times S^3$

S^3 is group manifold of $\mathrm{SU}(2)$.
Coordinates $h(\sigma,\tau) \in \mathrm{SU}(2)$ with $h^\dagger h = 1$, $\det h = 1$ and $t(\sigma,\tau) \in \mathbb{R}$.
Standard sigma model action

$$S_\sigma = \frac{\sqrt{\lambda}}{4\pi} \int \big(-dt \wedge {*dt} + \mathrm{Tr}\, dh^\dagger \wedge {*dh} + \Lambda\,(\det h - 1)\big).$$

Equations of motion

$$h^{-1} d{*dh} - h^{-1} dh \wedge h^{-1} {*dh} = 0, \quad \det h = 1, \quad d{*dt} = 0.$$

Gauge-fix time $t = E\tau/\sqrt{\lambda}$. Virasoro constraint

$$\mathrm{Tr}(h^{-1}\partial_\pm h)^2 = -2(\partial_\pm t)^2 = -2E^2/\lambda.$$

Lax Pair

Introduce left $\mathfrak{su}(2)$ current

$$j = h^{-1} dh.$$

Flatness, conservation (equations of motion) & Virasoro constraints

$$dj + j \wedge j = 0, \quad d{*}j = 0, \qquad \operatorname{Tr} j_\pm^2 = -2E^2/\lambda.$$

Lax pair: Family of flat connections ($\leadsto$ integrability)

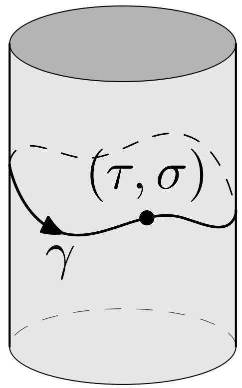
$$a(x) = \frac{1}{1-x^2}\, j + \frac{x}{1-x^2}\, {*}j.$$

Flatness of $a(x)$ for all x equivalent to flatness and conservation of j:

$$da(x) + a(x) \wedge a(x) = 0.$$

Monodromy

Monodromy of Lax connection around closed string



$$\omega(x) = \mathrm{P}\exp \oint_\gamma \big(-a(x)\big).$$

Independent of path γ, but not of end point $\gamma(0) = \gamma(2\pi) = (\tau, \sigma)$

$$d\omega(x) + \big[a(x), \omega(x)\big] = 0.$$

Shift generates similarity transformation. Eigenvalues preserved

$$\omega(x) \simeq \mathrm{diag}\big(e^{+ip(x)}, e^{-ip(x)}\big).$$

The quasi-momentum $p(x)$ is a conserved, gauge-invariant quantity.
Complete set of action variables in Hamilton-Jacobi formalism.

Global Charges

Expansion of Lax connection at $x = \infty$: [Kazakov, Marshakov Minahan, Zarembo]

$$a(x) = -\frac{1}{x} *j + \mathcal{O}(1/x^2), \qquad J = \frac{\sqrt{\lambda}}{2\pi} \oint *j.$$

Global $\mathfrak{su}(2)_\mathrm{R}$ charges J_R can be read off from monodromy at $x = \infty$

$$\omega(x) = I + \frac{1}{x} \frac{2\pi J_\mathrm{R}}{\sqrt{\lambda}} + \mathcal{O}(1/x^2), \qquad p(x) = \frac{1}{x} \frac{2\pi J_{\mathrm{R},3}}{\sqrt{\lambda}} + \mathcal{O}(1/x^2).$$

Fix $p(\infty) = 0$.

Global $\mathfrak{su}(2)_\mathrm{L}$ charges J_L can be read off from monodromy at $x = 0$

$$g\omega(x)g^{-1} = I - x \frac{2\pi J_\mathrm{L}}{\sqrt{\lambda}} + \mathcal{O}(x^2), \quad p(x) = 2\pi n_0 - x \frac{2\pi J_{\mathrm{L},3}}{\sqrt{\lambda}} + \mathcal{O}(x^2).$$

Winding number n_0.

Local Charges

Poles at $x=\pm 1$: Can diagonalise Lax connection perturbatively

$$u(x,\sigma)\big(\partial_\sigma + a_\sigma(x,\sigma)\big)u(x,\sigma)^{-1} = \partial_\sigma - i\sum_{k=-1}^{\infty}(x\mp 1)^k\,\delta Q_k^\pm(\sigma).$$

Leading charge density $\delta Q_{-1}^\pm(\sigma)$ related to current $j_\pm$

$$\delta Q_{-1}^\pm \simeq -\tfrac{i}{2} j_\pm \simeq \tfrac{1}{2}(E/\sqrt{\lambda})\,\mathrm{diag}\big(+1,-1\big).$$

Diagonalised current gives conserved local charges $Q_k^\pm$

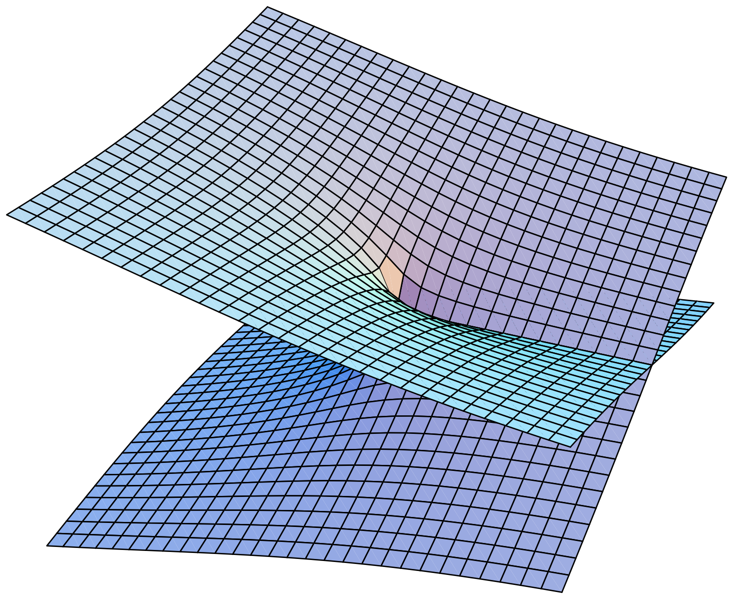
$$p(x) = \sum_{k=-1}^{\infty}(x\mp 1)^k\,Q_k^\pm, \qquad Q_k^\pm = \oint_0^{2\pi}\delta Q_{k,11}^\pm(\sigma).$$

Residue of p at $x=\pm 1$ equals $Q_{-1}^\pm = \pi E/\sqrt{\lambda}$ by Virasoro constraint.

Analyticity

Monodromy $\omega(x)$ is analytic in x except at $x = \pm 1$: $\omega(x) \sim \exp \frac{iQ_{-1}^{\pm}}{x \mp 1}$.

Diagonalisation introduces new singularities $\{x_a^*\}$ (eigenvalue crossing)



One full turn around x_a^* interchanges eigenvectors/values (labelling).

Generic behaviour at degenerate eigenvalues $e^{+ip(x_a^*)} = e^{-ip(x_a^*)}$:

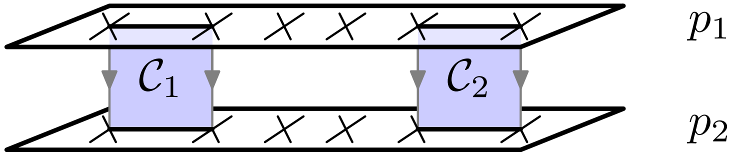
$$e^{+ip(x)}, e^{-ip(x)} = e^{ip(x_a^*)} \left(1 \pm \alpha \sqrt{x - x_a^*} + \mathcal{O}(x - x_a^*)\right).$$

Algebraic Curve

Two eigenvalues $p_{1,2}(x)$ (defined modulo 2π) of $\omega(x)$ [Kazakov, Marshakov Minahan, Zarembo]

$$\omega(x) \simeq \mathrm{diag}(e^{ip_1(x)}, e^{ip_2(x)})$$

as one function p on a Riemann surface $\mathbb{M}_\mathbf{2}$ with two sheets



- Single poles at $x = \pm 1$.
- Branch points are square-root singularities $\sqrt{x - x_a^*}$.
- Function continued analytically across cuts on other sheet (modulo 2π).
- Assume finitely many $(2A)$ branch points. Other solutions as $A \to \infty$.

Derivative $p'(x)$ defines algebraic curve $F(x, p') = P_2(x)p'^2 + P_0(x) = 0$.

String configuration $h(\tau, \sigma)$ $\longrightarrow$ Algebraic curve $F(x, p') = 0$.

Admissible Curves

Not all algebraic curves can arise from the sigma model.

$$F(x,p') = P_2(x)\,p'^2 + P_1(x)\,p' + P_0(x).$$

Constraints:

- Sum of solutions vanishes $p'_1 + p'_2 = 0$.
- Asymptotics $p'(x) \sim 1/x^2$ at $x = \infty$ and $p'(x) \sim 1$ at $x = 0$.
- Double poles in $p'(x)$ at $x = \pm 1$. No residues.
- Physical branch points $p'(x) \sim 1/\sqrt{x - x^*_a}$.
- No unphysical branch points $p'(x) \sim \sqrt{x - x^\times}$.
- $p(x) = \int_\infty^x p'(x')\,dx'$ must be single-valued (modulo 2π).

Count moduli of admissible curves.

Coefficients

Ansatz for derivative of quasi-momentum

$$p'_{1,2}(x) = \pm \frac{c_{A+2}\, x^{A+2} + \ldots + c_0\, x^0}{(x^2-1)^2 \sqrt{(x-x^*_1)\cdots(x-x^*_{2A})}}\,.$$

The ansatz satisfies already:

- $3A+3$ free coefficients, $c_0, \ldots c_{A+2}, x^*_1, \ldots, x^*_{2A}$.
- Asymptotics $p'(x) \sim 1/x^2$ at $x=\infty$ and $p'(x) \sim 1$ at $x=0$.
- Physical branch points at $x = x^*_a$.
- No unphysical branch cuts: No square root in numerator.

Further constraints:

- Double poles at $x=\pm 1$ with equal coefficient: 1 constraint.
- No residue at $x = \pm 1$: 2 constraints.

Remaining coefficients: $3A$.

Cycles

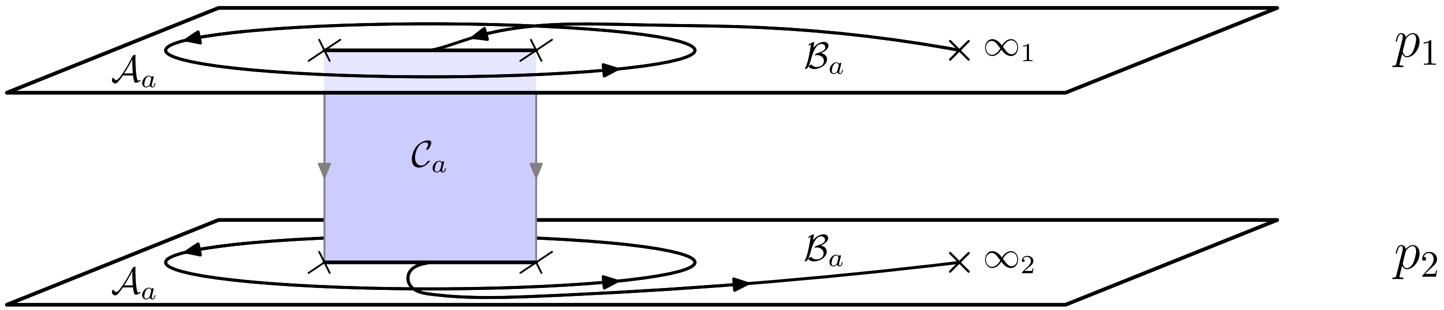
Single-valuedness: All closed cycles must be integer

$$\oint dp \in 2\pi\mathbb{Z} \qquad \text{as well as} \qquad \int_{\infty_1}^{\infty_2} dp \in 2\pi\mathbb{Z} \quad \text{and} \quad \int_{\infty}^{0} dp \in 2\pi\mathbb{Z}.$$

Can arrange cuts $\mathcal{C}_a$ such that

$$\oint_{\mathcal{A}_a} dp = 0, \qquad \int_{\mathcal{B}_a} dp = 2\pi n_a$$

with $\mathcal{A}, \mathcal{B}$-cycles defined as in



From $\mathcal{A}, \mathcal{B}$-cycles: $A-1$ and $A+1$ constraints. Remaining: A.

String Moduli

Precisely A moduli remain. Conclusions:

One "mode number" $n_a \in \mathbb{Z}$ and one "amplitude" $K_a \in \mathbb{R}$ for each cut

$$n_a = \frac{1}{2\pi} \int_{\mathcal{B}_a} dp, \qquad K_a = -\frac{1}{2\pi i} \oint_{\mathcal{A}_a} \left(1 - \frac{1}{x^2}\right) p(x)\, dx.$$

One constraint among $\{(n_a, K_a)\}$: "level matching".

One additional parameter $L = J_{\mathrm{L},3}$: "length".

Solutions classified by $\{(n_a, K_a)\}$ and L or if $n_a \neq n_b$

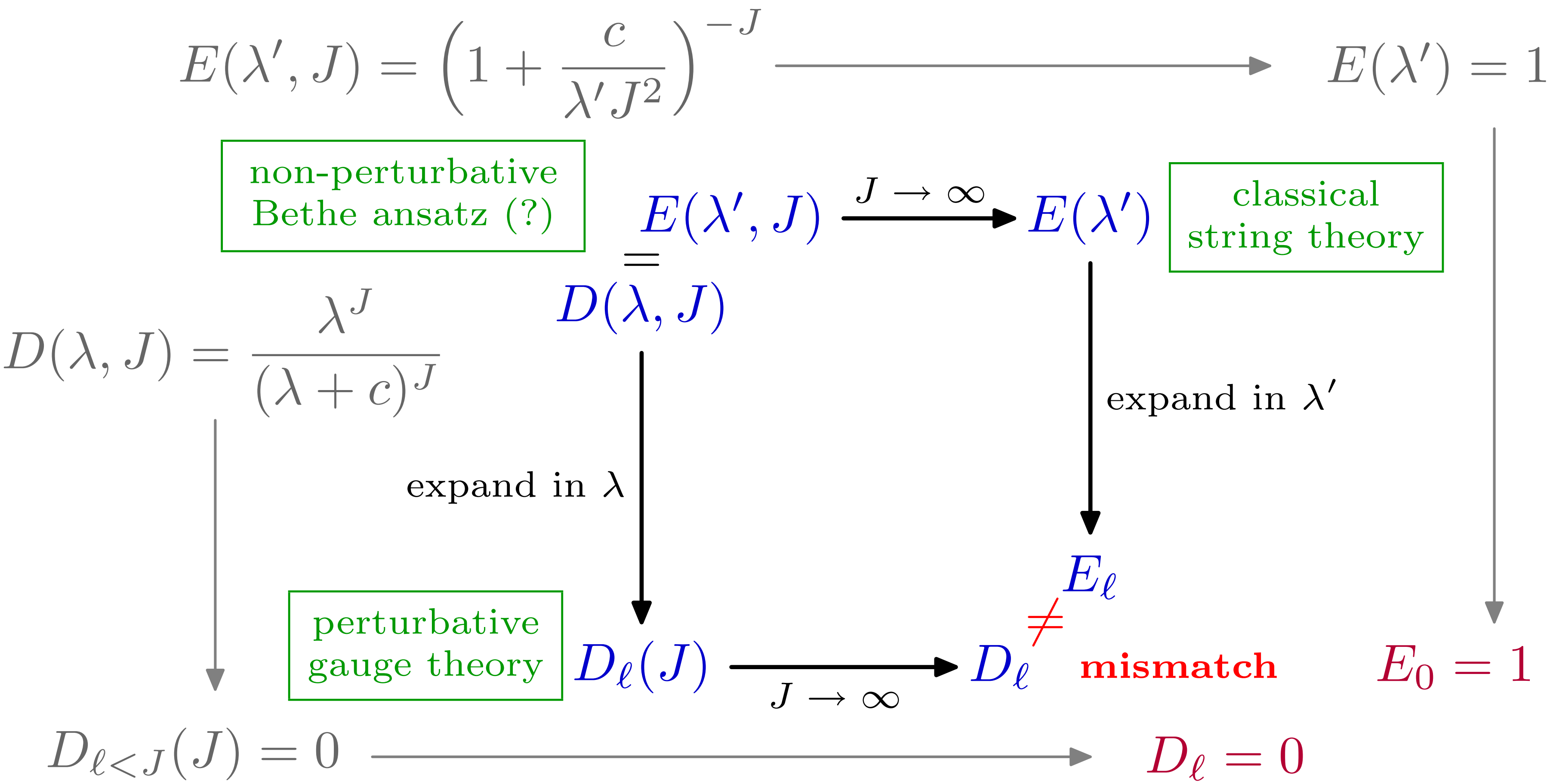
Solutions classified by $\{K_n\}$ and L.

Similar to gauge theory:

- Need to rescale $u \sim x/L$ to move poles at $x = \pm 1$ to $u = 0$.

Agreement up to two loops. Clearly different at three loops. [Kazakov, Marshakov Minahan, Zarembo]

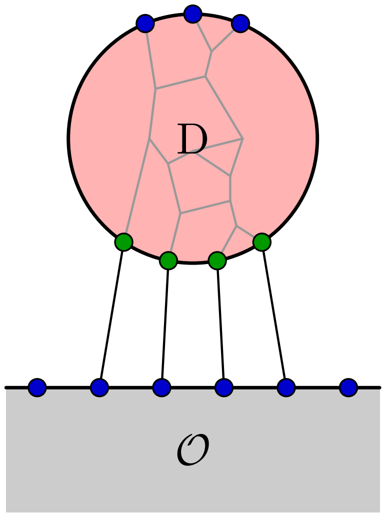
Order of Limits



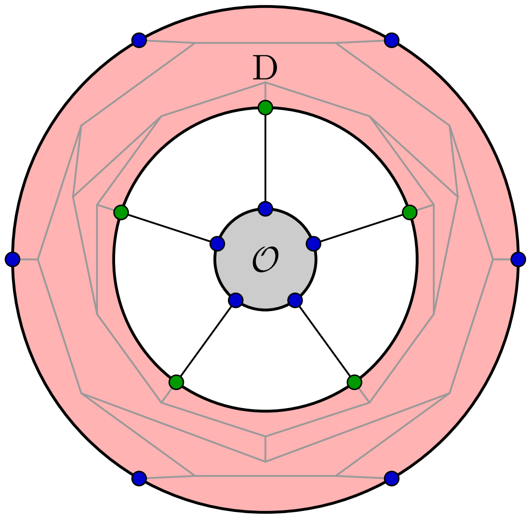
- Cannot compare in perturbation theory. [NB, Dippel Staudacher]
- Spinning strings and near-BMN proposals do not quite work.
- Integrability might lead to the exact solution of either theory.

Wrapping Interactions

At higher loop orders there is an additional type of planar interaction



regular



wrapping

- Starts contributing at $\mathcal{O}(\lambda^L)$.
  May nevertheless repair discrepancy (see example).
- Asymptotic Bethe ansatz does not incorporate wrappings.
- Algebraic construction apparently not useful here.

[NB, Dippel Staudacher]

Conclusions

⋆ **Strings on $AdS_5 \times S^5$**

- Classical Spinning Strings Solutions.
- Integrability, Lax connection, Monodromy.
- Construction of an algebraic curve and classification.

⋆ **$\mathcal{N} = 4$ Gauge Theory**

- Construction of dilatation operator.
- Coherent states in the thermodynamic limit.
- One-loop/higher-loop integrability, Bethe Ansätze.
- Algebraic curve in the thermodynamic limit.

⋆ **Comparison for AdS/CFT, BMN & Spinning Strings**

- Direct, Hamiltonian, Integrable Structures.
- Agreement up to two loops.
- Problems at three loops. BMN & Spinning Strings does not work.

Open Questions

- Will this talk ever end?

⋆ **Near-BMN & Spinning Strings**

- Understand the problems at three loops. Wrappings?
- Better question: Why did it work up to two loops in the first place?

⋆ **Various Directions**

- Theories with less supersymmetry. Phenomenology?
- Can we use integrability for non-planar/stringy interactions?

⋆ **Exact Bethe ansätze**

- Gauge theory at higher loops: λ-dependence.
  Inspiration from classical string theory.
- Quantum strings: Discretised curve, Bethe roots.
  Inspiration from one-loop gauge theory.
- Compare & see whether AdS/CFT really works.