Dulmage-Mendelsohn percolation

Random geometry of maximum-matchings, zero modes & Majorana excitations...

Kedar Damle, Tata Institute (TIFR) India

based on: Bhola, Biswas, Islam, KD, PRX 12 021058 (June 2022)

Ritesh Bhola

Sounak Biswas

Mursalin Islam

earlier work: Sanyal, KD, Motrunich, PRL 117, 116806 (2016)

Sambuddha Sanyal

Lesik Motrunich

Some generalities:

- Quenched disorder matters (often)
- Particles scatter and diffuse (may be anomalously...)
- Matter-waves scatter and localise (sometimes weakly...)

Classical transport of fluid

- Simplest setting: Porous random medium
- Random geometry of medium determines fluid transport
- Paradigm of percolation
- More generally: Diffusion in presence of random potentials

Percolation

- Study end-to-end connectivity of a porous medium
- Can you go from one end to other?
- Answer changes as function of porosity
- Simplest model: Randomly diluted regular lattice (graph)

Broadbent and Hammersley, Percolation processes I, Crystals and Mazes (1957)

Precise question about the random geometry

Crossing probability?

- Consider two dimensional square grid or honeycomb net or three dimensional cubic lattice of linear size L
- Remove fraction n_{vac} of sites and delete links to removed sites.
- $P_w(n_{\text{vac}}, L)$: Probability that one can ‘walk’ from left end to right end along existing sites and links.
- How does this behave as a function of n_{vac} and L?

Sharp threshold behaviour

Property of thermodynamic limit

- In $d = 2$ and in $d = 3$, $L \to \infty$ limit characterised by sharp threshold behaviour as function of n_{vac}
- Percolation transition
- Simplest geometric example of a thermodynamic phase transition
- For $n_{\text{vac}} < n_{\text{vac}}^{\text{crit}}$, $P_w \to 1$ as $L \to \infty$
- For $n_{\text{vac}} > n_{\text{vac}}^{\text{crit}}$, $P_w \to 0$ as $L \to \infty$

Approach to thermodynamic limit

Universality and scaling

- $L \to \infty$ limit is approached in interesting way
- $P_w(n_{\mathrm{vac}}, L) = f(\delta L^{1/\nu})$ where $\delta = n_{\mathrm{vac}} - n_{\mathrm{vac}}^{\mathrm{crit}}$
- $f(x)$ is the universal scaling function, ν is a scaling exponent and $n_{\mathrm{vac}}^{\mathrm{crit}}$ is the critical dilution
- $f(x)$ and ν believed to be universal (independent of microscopic-scale details)
- Square lattice and honeycomb net have same $f(x)$ and ν. Cubic lattice different (dimension dependent)

Scale invariance

- Implies different size samples have same P_w for $n_{\mathrm{vac}} = n_{\mathrm{vac}}^{\mathrm{crit}}$
- Scale invariance: Pictures of random geometry look same if we rescale pictures!
- Only true if we ignore lattice scale features, but amazing anyway!

Localization of matter waves

- Anderson localization of electrons in dirty metals
- Localization of quasiparticles in dirty superconductors
- Symmetries of disordered Hamiltonian matter (e.g. in random matrix theory)

Anderson, Ramakrishnan, Abrahams, Thouless, Dyson, Wegner, Mehta...

Simplest lattice model: Disordered tight-binding Hamiltonian

- $\sum_{j \in i} t_{ij}\psi_j + V_i\psi_i = \epsilon\psi_i$ for all i
- ϵ is the energy of the particle described by wave function ψ_i
- t_{ij} are 'hopping amplitudes' for particle to hop from site i to site j
- V_i are values of external potential at sites i
- Allowed ϵ : Eigenvalues of matrix of $t_{ij} + V_i\delta_{ij}$

Vacancy disorder

- Random dilution of the underlying lattice
- Models missing atoms in crystal structure
- Also natural if substitutional impurities correspond to missing orbital (binary alloys)

Quantum percolation

- Anderson localisation meets geometric percolation (Kirkpatrick-Eggarter ’72, Shapir-Aharony-Harris ’82…)
- Vacancy disorder
- No external random potentials
- Can the quantum electron fluid be localised even when the corresponding classical fluid diffuses from end to end?

Simplest case: bipartite lattice with hopping and vacancy disorder

- Particle hopping on a randomly diluted bipartite lattice (binary alloy)
- (Possibly random) hopping amplitudes between nearest neighbour sites
- Bipartite symmetry: State with energy ϵ has partner at energy $-\epsilon$
- Symmetry broken by random potentials, next-neighbour hopping — left out here.

The question

- $\epsilon = 0$ is special
- Does anything interesting happen in the quantum mechanical spectrum of H near $\epsilon = 0$?

More precisely:

- What is the asymptotic low-energy behaviour of $\rho(\epsilon)$?
- Note: No change in symmetries
- Some answers:

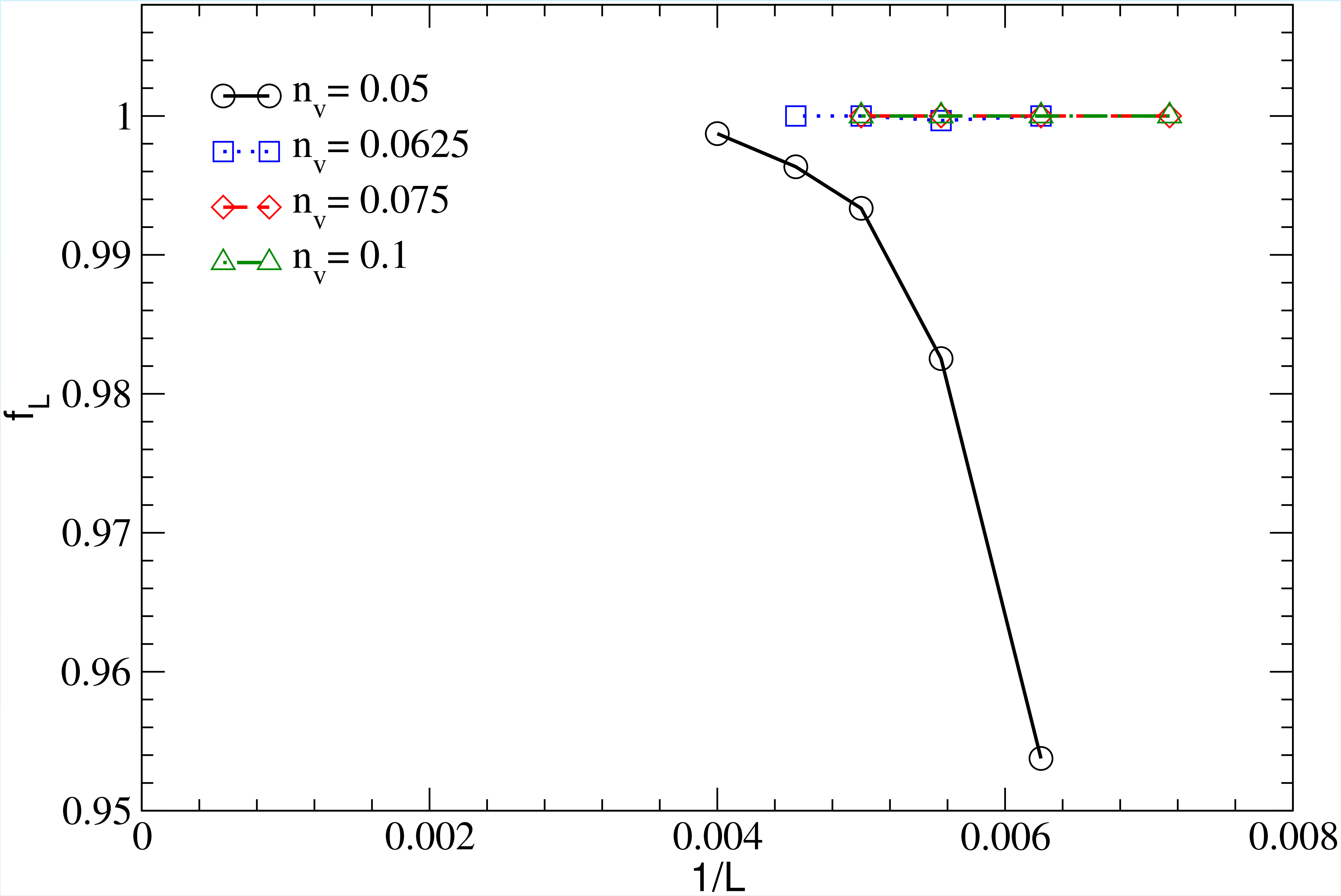
Hopping disorder: Singular tail of low-energy states. DOS has ‘modified Gade-Wegner’ scaling.

(Gade-Wegner ’91, Motrunich, KD, Huse ’02, Mudry-Ryu-Furusaki ’03)

Vacancy disorder: **Very slow crossover** to ‘modified Gade-Wegner’ scaling.

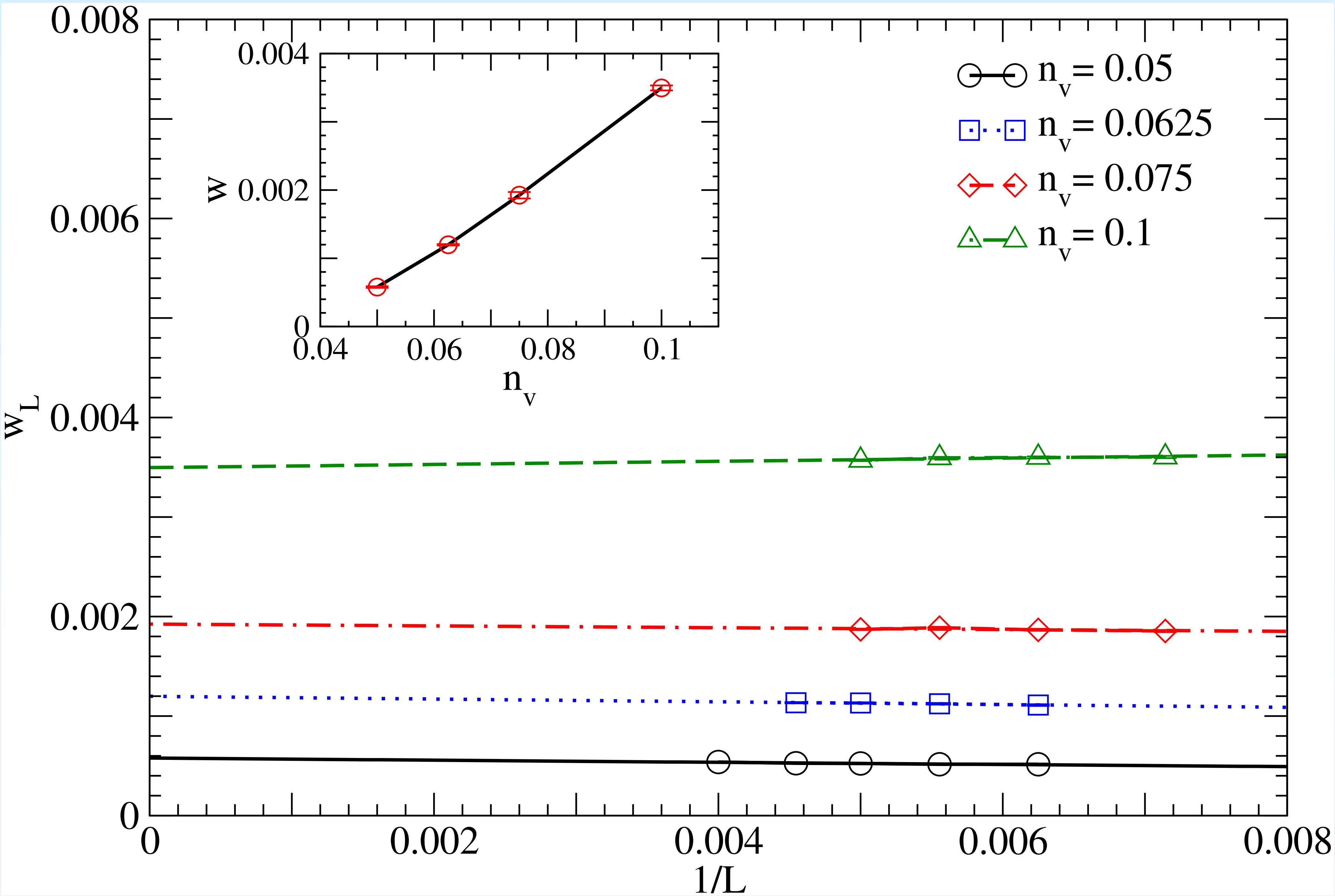
Willans-Chalker-Moessner ’11, Ostrovsky et al ’14, Hefner et al ’14, Sanyal, KD, Motrunich ’16

New ingredient: Zero modes



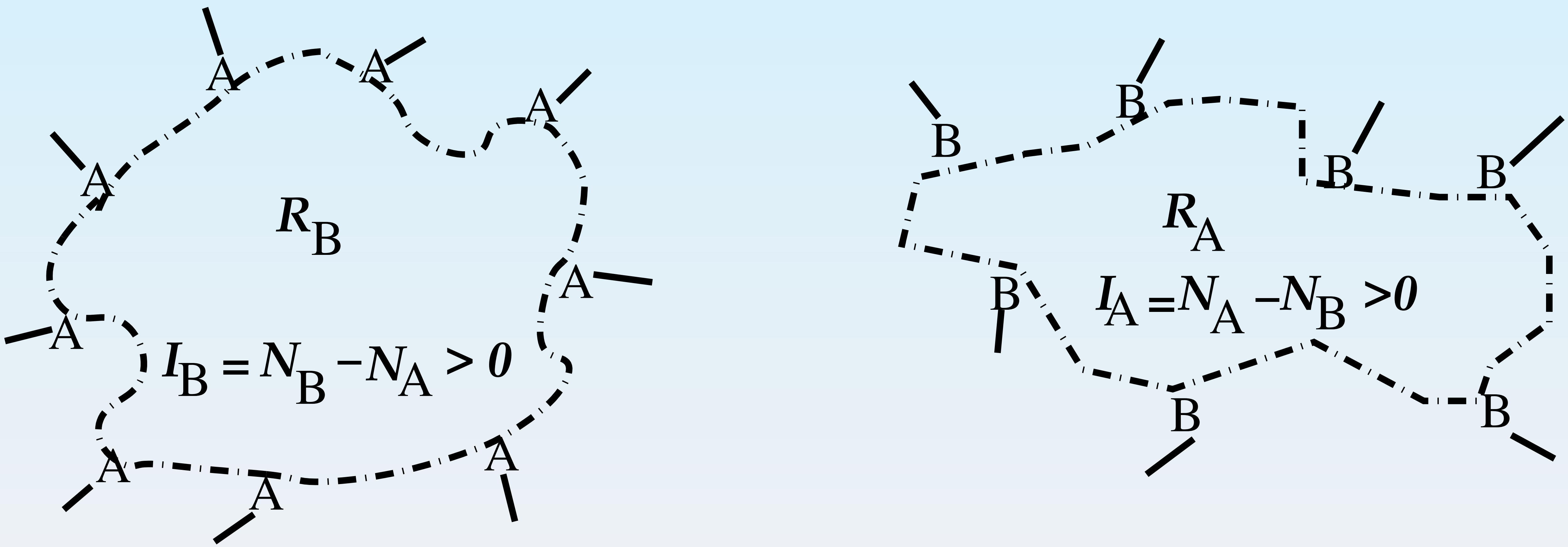
For large enough lattice, almost every sample has at least one zero mode (Sanyal, KD, Motrunich, PRL ’16)

In fact: Nonzero density of zero modes



(Sanyal, KD, Motrunich, PRL ’16)

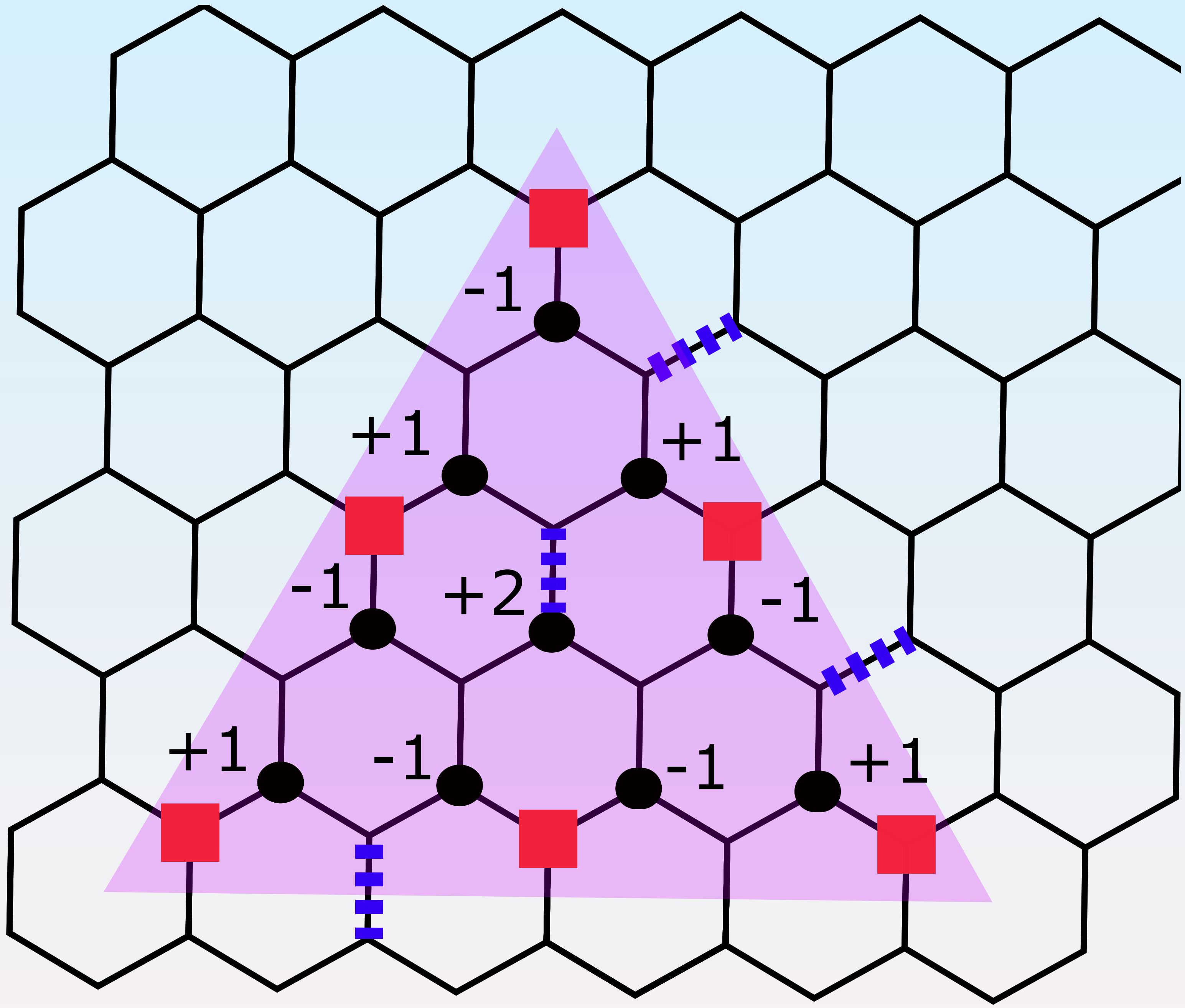
Our idea: R-type regions hosting zero modes



Constraint on zero-energy wavefunction: $\psi_A: \sum_{A \in B_0} t_{AB_0} \psi_A = 0$ $\psi_B: \sum_{B \in A_0} t_{BA_0} \psi_B = 0$

(Sanyal, KD, Motrunich, PRL '16)

Example of R-type region



Rigorous lower bound on density of zero modes on diluted honeycomb lattice
(Sanyal, KD, Motrunich, PRL ’16)

Major puzzle remained:

- Actual density of zero modes much larger than lower bound
- What dominates?

Our approach

- Key idea: Disorder-robust zero modes only depend on connectivity, not hopping strengths.
- R-type regions rely on local imbalance between A and B type site densities.
- Suggests thinking in terms of matchings a.k.a dimer covers
- Places that cannot be covered by dimers host wavefunctions

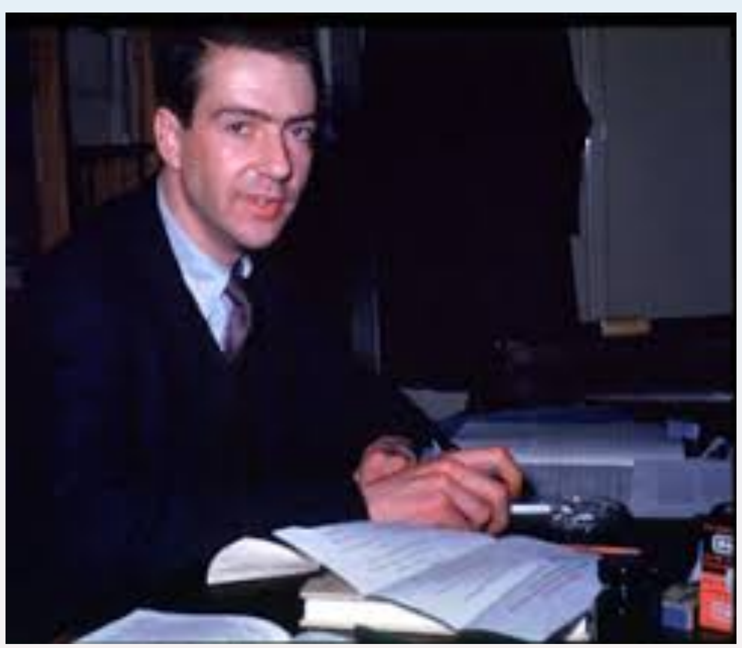
Longuet-Higgins: Counting zero modes from maximum matchings

AIP | The Journal of Chemical Physics

Some Studies in Molecular Orbital Theory I. Resonance Structures and Molecular Orbitals in Unsaturated Hydrocarbons

H. C. Longuet-Higgins

(1950)



Chemist - structure of diborane)

Physicist - (co)advisor of Peter Higgs)

Pioneer in:

Cognitive science and computer vision

‘Machine-intelligence (aka AI!)

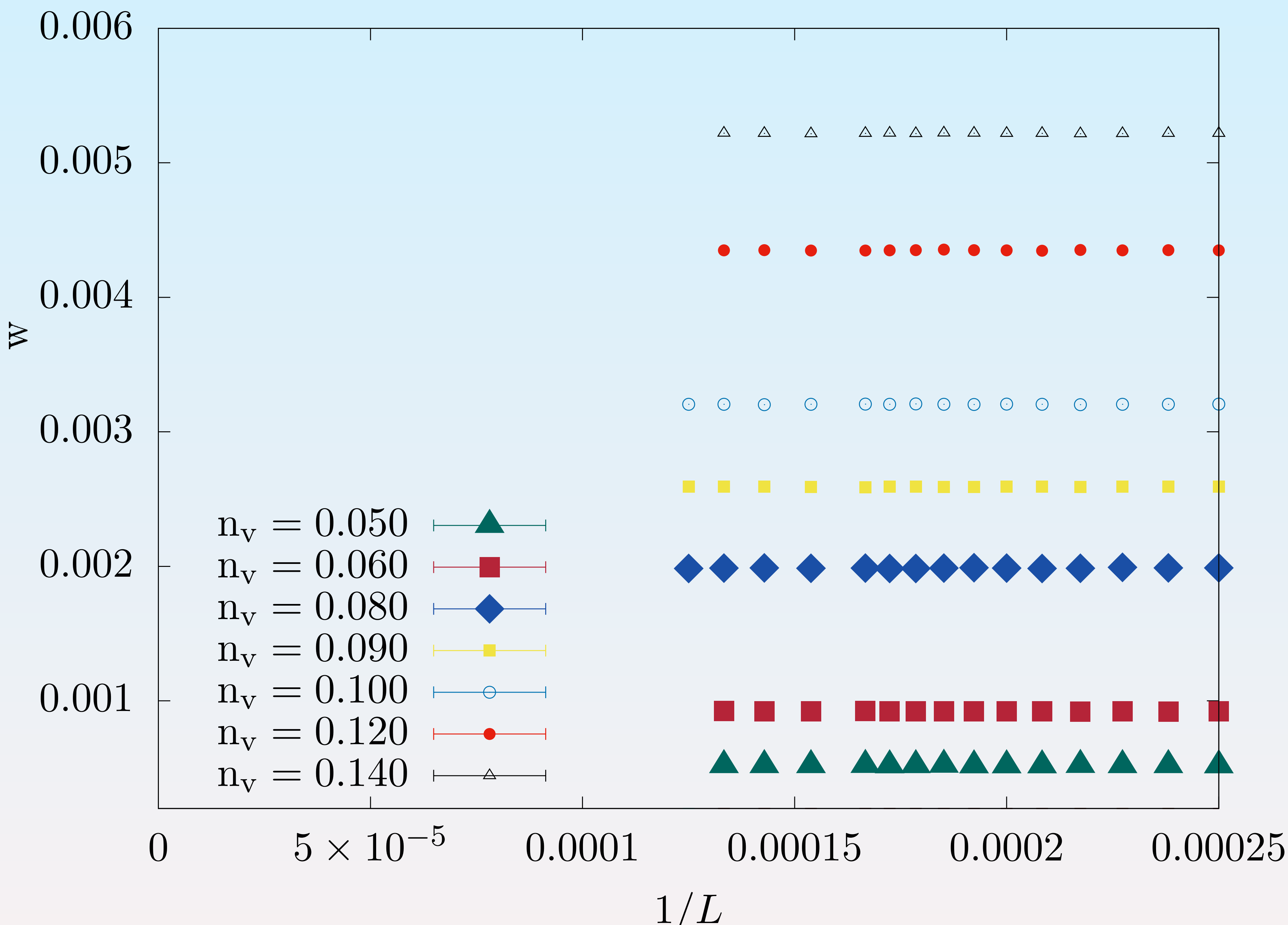
Computer music...

Longuet-Higgins (restated)

- Number of monomers in any maximum matching of bipartite graph gives number of topologically-protected zero modes of corresponding tight-binding model

“nonzero-defect” generalization of “Tutte’s Theorem” for bipartite graphs

Density of disorder-robust zero modes



Equivalent to counting zero modes of a $10^8 \times 10^8$ matrix (!)

Also: independent confirmation at higher dilution by Evers group (Weik et al ’16)

But what do the zero modes “look” like?

- What do we mean by “look” like?
- Our interest: Consequences for basis invariant Green functions and for transport at particle-hole symmetric chemical potential (e.g. undoped graphene)
- What we really want: A general way of identifying “all possible” R-type regions
- In other words: A choice of “maximally localised” basis for the zero-mode subspace

Partial answer from Longuet-Higgins:

- Global statement from Longuet-Higgins: Set of all sites that host a monomer in at least one maximum matching form support of all topologically-protected zero mode wavefunctions
- Clearly, we want more:
- General algorithm for identifying all R-type regions?
- Maximally-localised basis for zero mode subspace?

Our key insight: A local statement

Brings into play classic result from graph theory

COVERINGS OF BIPARTITE GRAPHS

A. L. DULMAGE AND N. S. MENDELSOHN

Can. J. Math. 10: 517, 1958

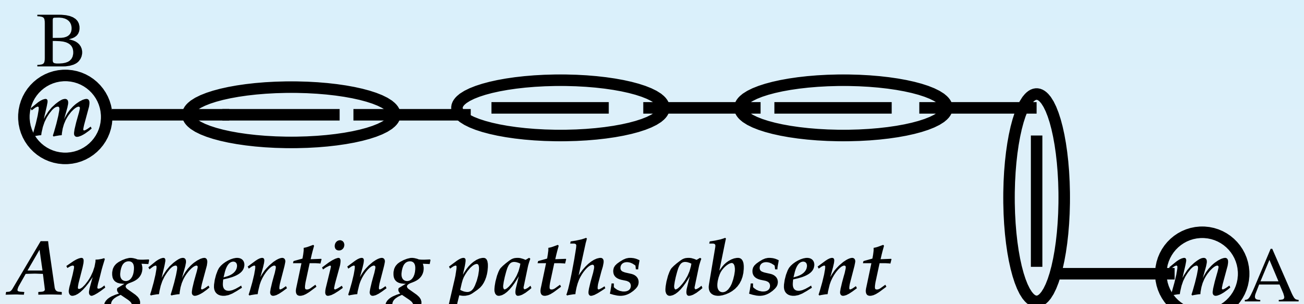
Use structure theory of Dulmage-Mendelsohn to construct non-overlapping 'complete' set of R-type regions.

`R-type' regions of lattice host monomers in maximum matchings and zero modes of a quantum particle

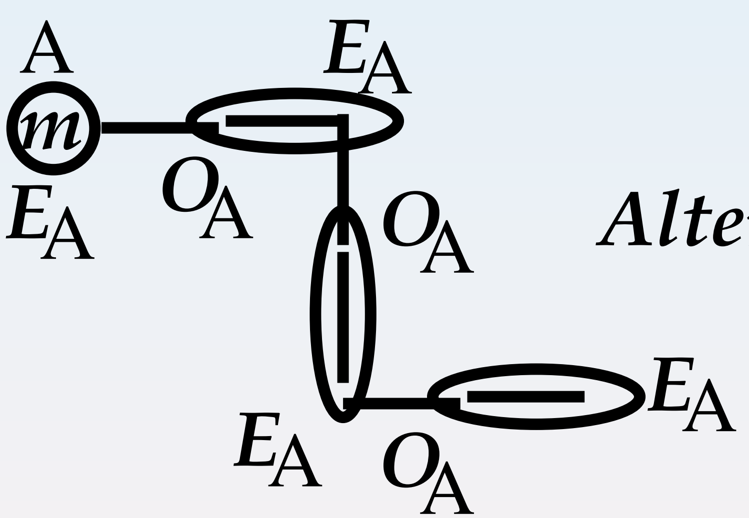
Zero temperature two-terminal conductance is zero if R-type regions don't percolate

Matchings, augmenting paths, and alternating paths

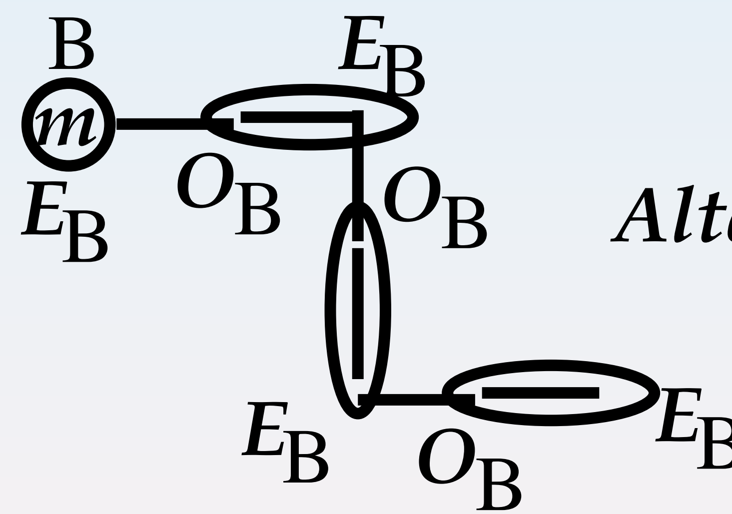
In any maximum matching M:



Augmenting paths absent



Alternating path



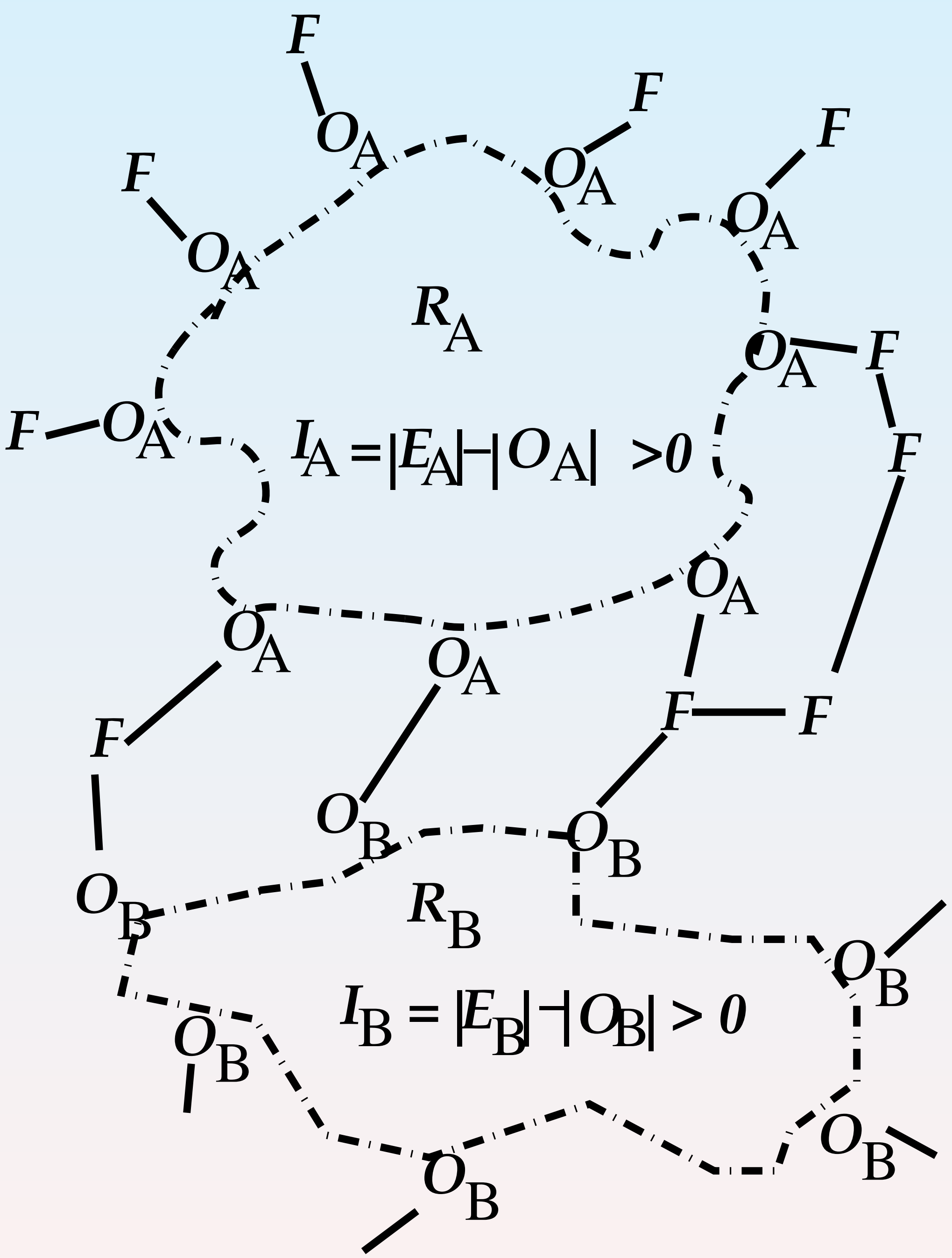
Alternating path

A useful version of the DM decomposition

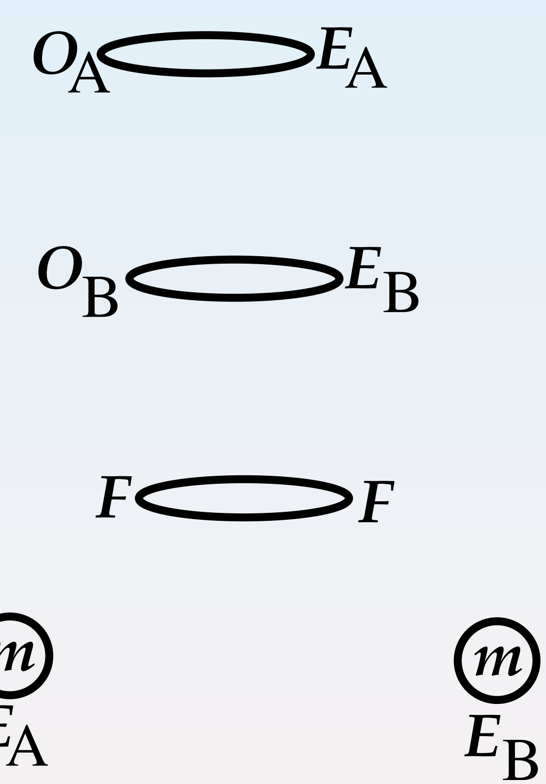
Even, odd and unreachable sites from any one maximum matching M

- Even: Reachable by even-length alternating paths from monomers of M (including length zero)
- Odd: Reachable by odd-length alternating paths from monomers of M
- Unreachable: Not...
- Decomposition:
- $C_A : E_A \cup O_A$
- $C_B : E_B \cup O_B$
- $P : U_A \cup U_B$

Key: Connected components of C_A and C_B



In any maximum matching:



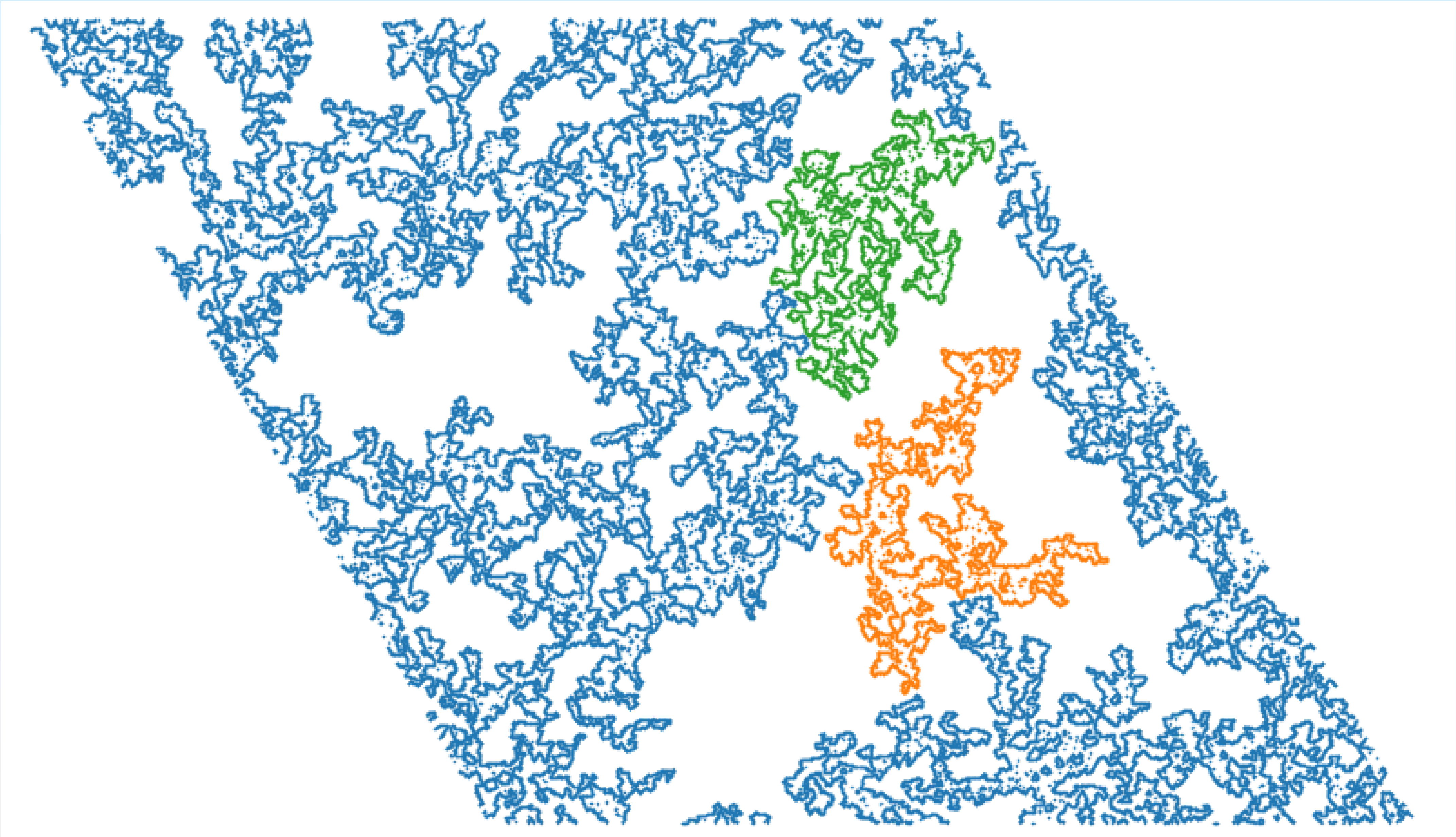
Connected components are R-type regions

- Each R_i^A (R_j^B) hosts I_i^A (I_j^B) topologically-protected zero modes with wavefunctions confined to the region.
- Provides alternate ‘local’ proof for correspondence between monomers of maximum matchings and zero modes of adjacency matrix
- Gives topologically-robust construction of a maximally-localised basis for zero modes
- Standard proofs (Longuet-Higgins, Lovasz) are ‘global’—no information about maximally-localised basis.

Computational strategy

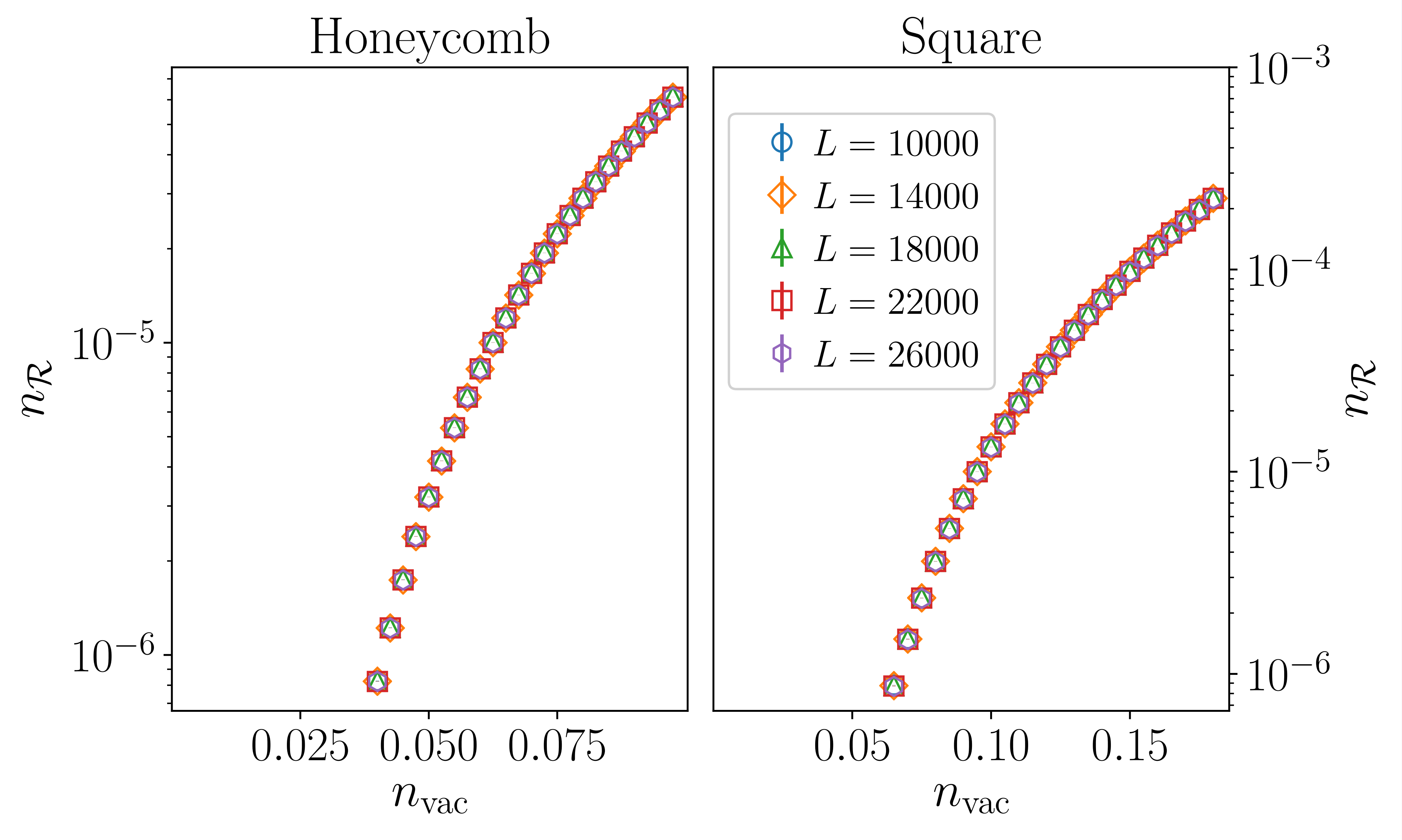
- Compensated disorder ($|A| = |B|$)
- Standard algorithms for finding any one maximum matching
- Alternating path tree from each monomer to obtain DM classification
- Burning algorithm to construct connected components and obtain R-type regions

Basic picture in d=2 (honeycomb lattice)

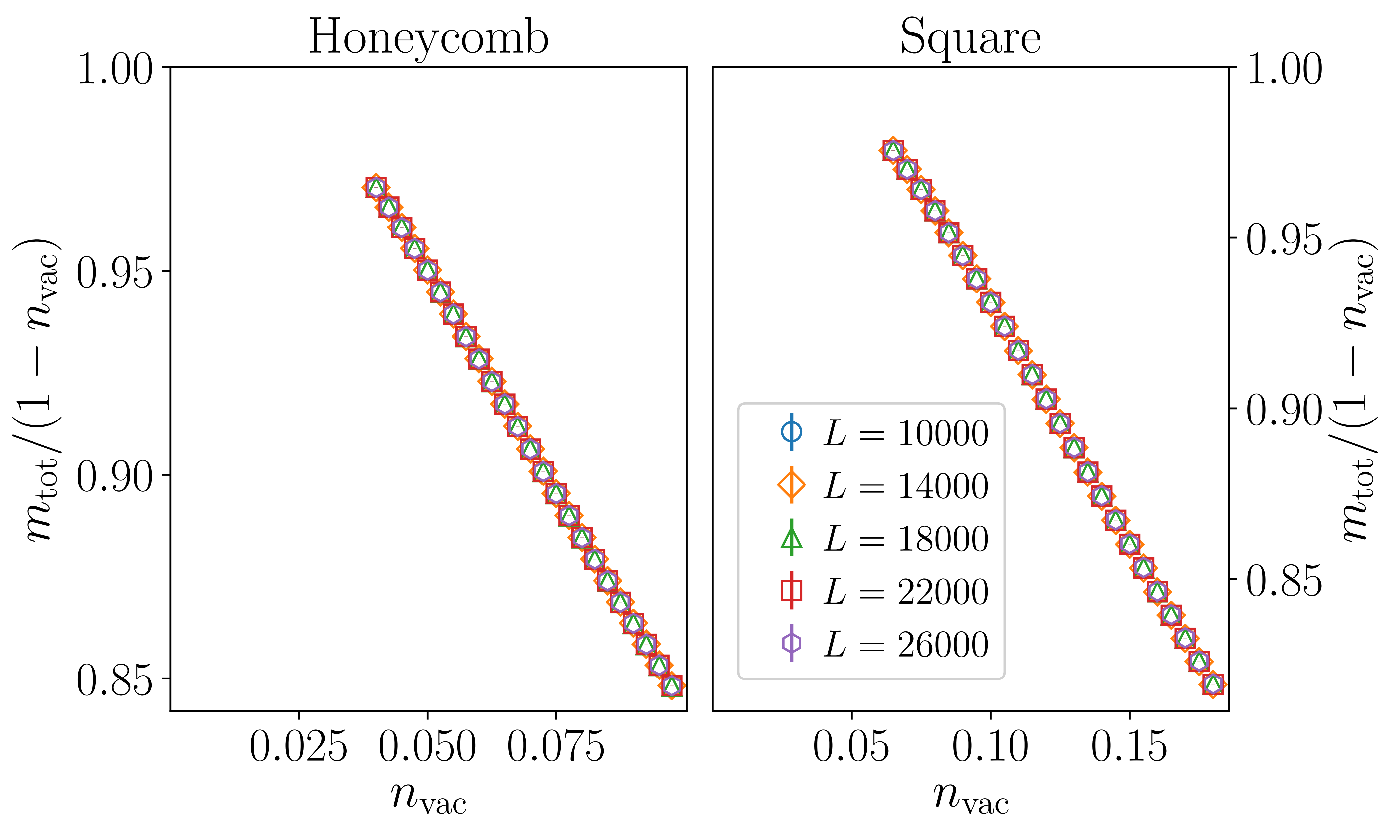


Typical R-type regions are BIG at low (~5%) dilution

Number density of R-type regions

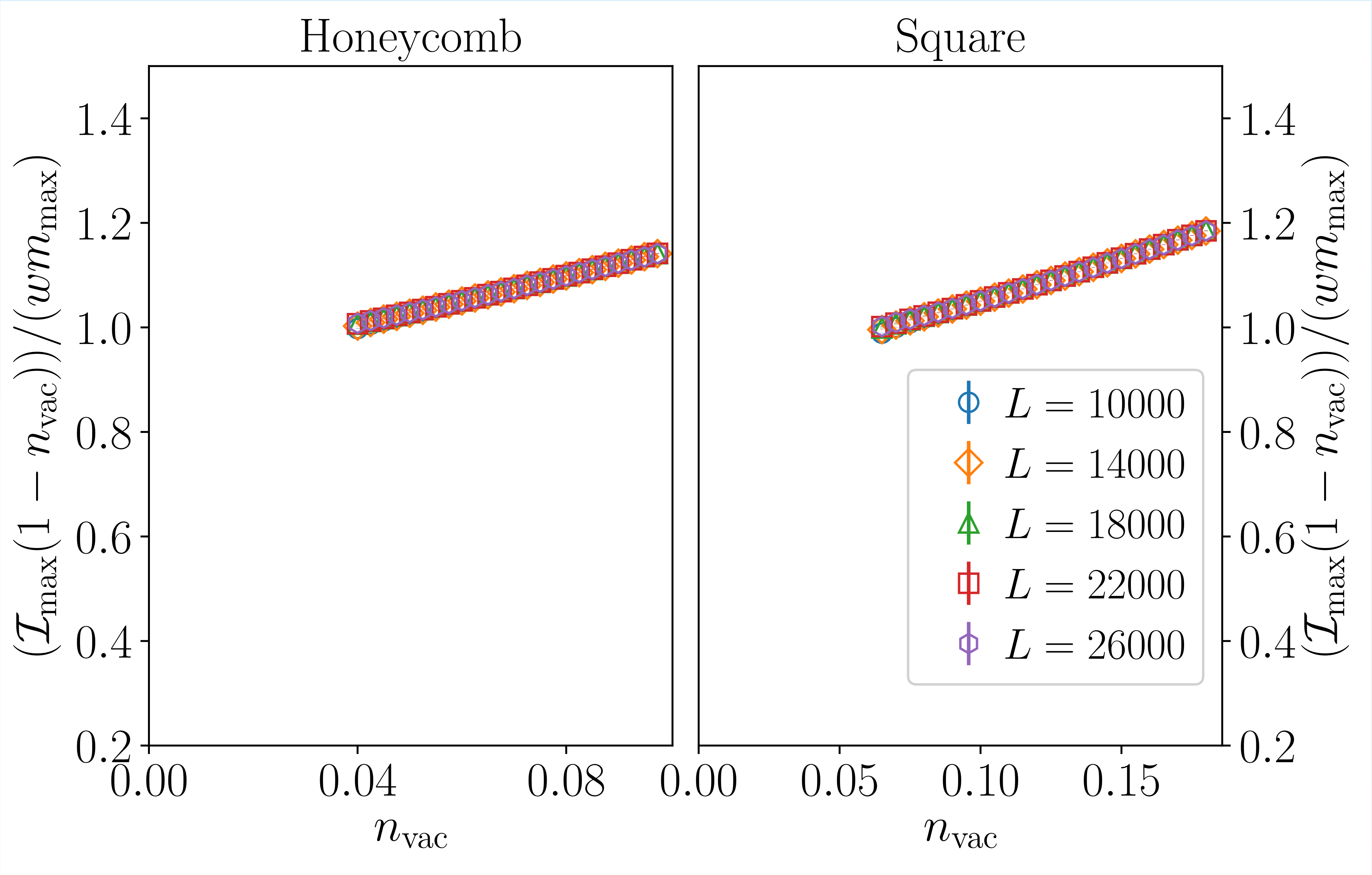


Total mass density of all R-type regions

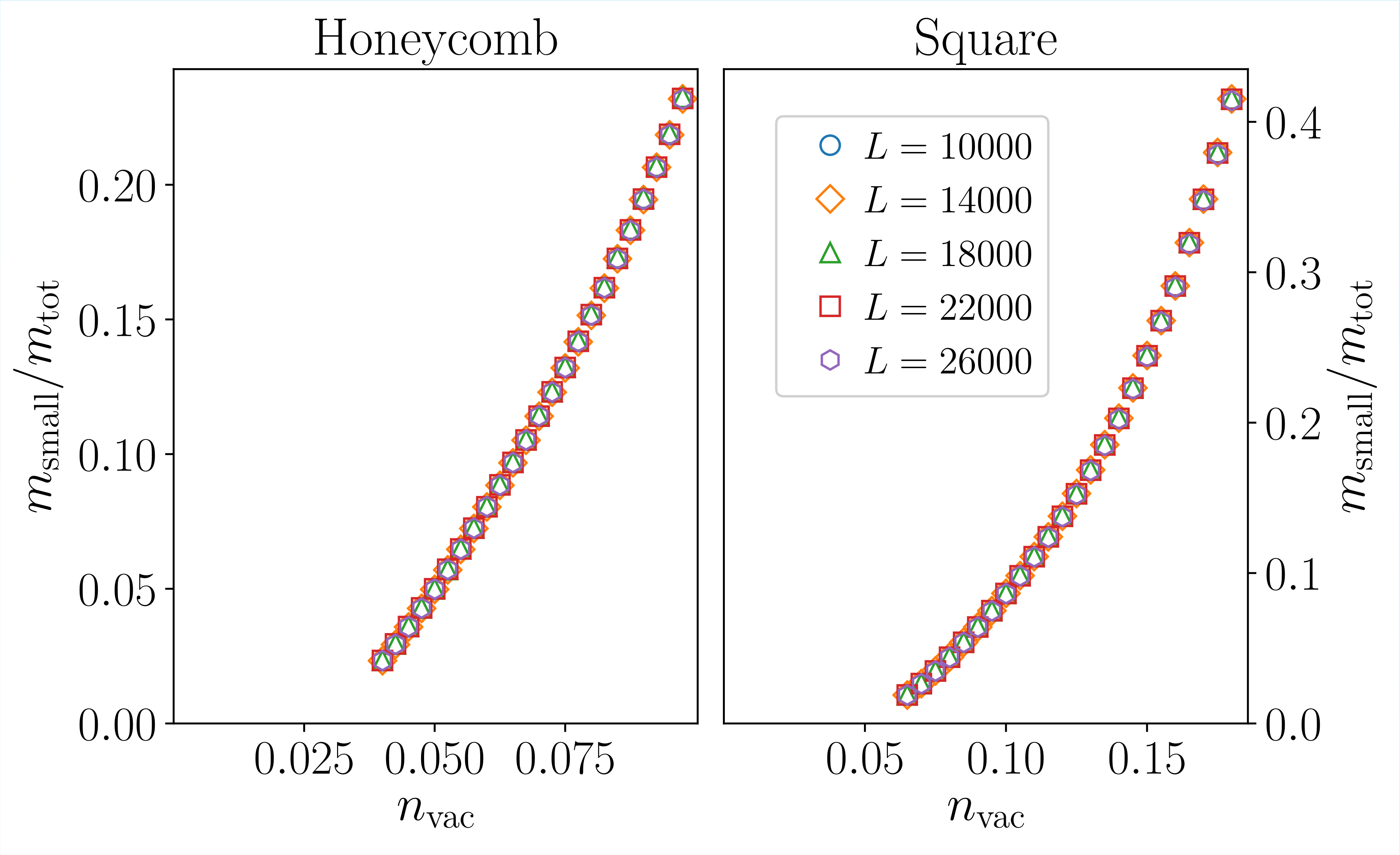


R-type regions take over the lattice in low-dilution limit!

Mode density in large R-type regions looks 'typical'

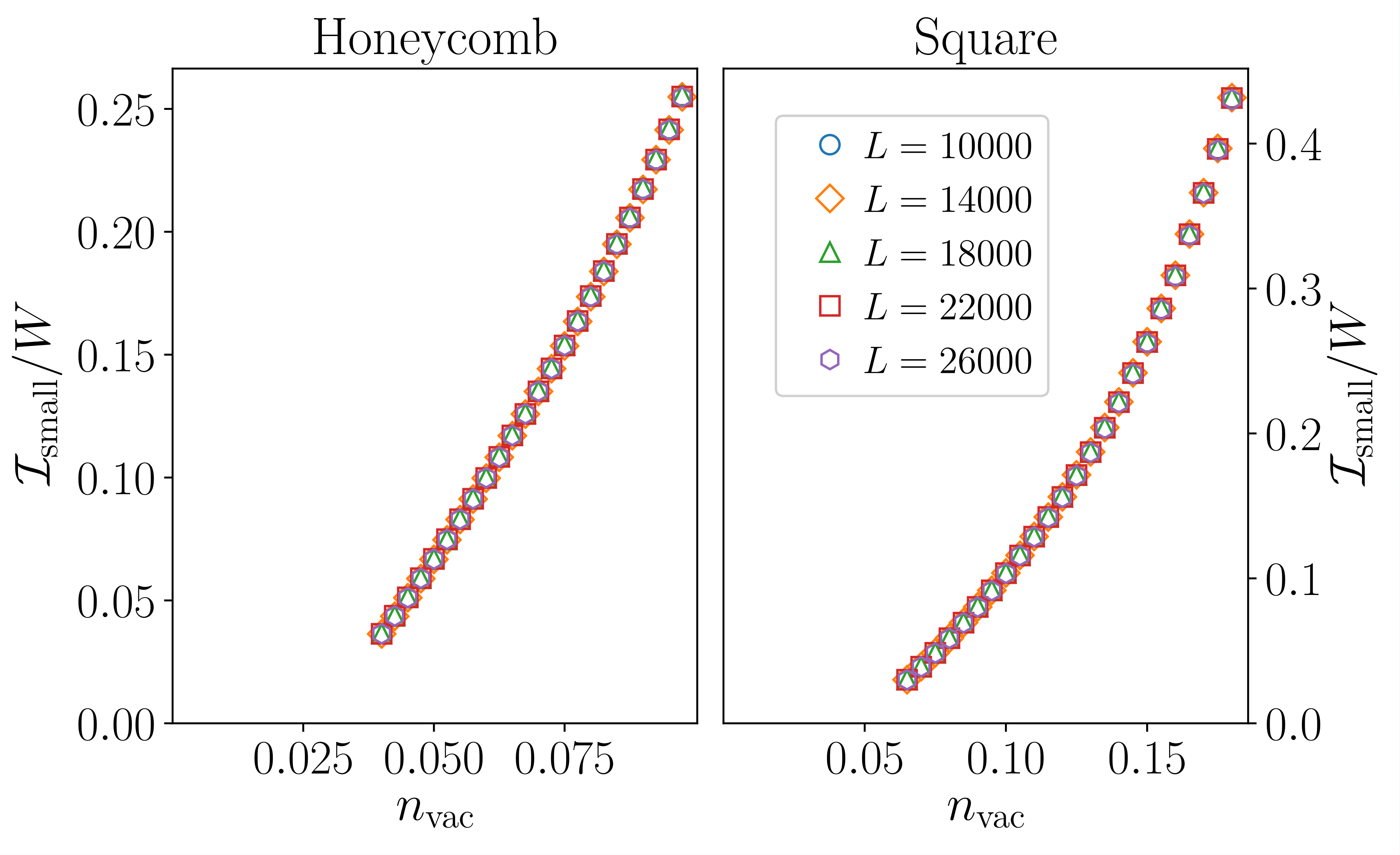


Random geometry dominated by large-scale structures



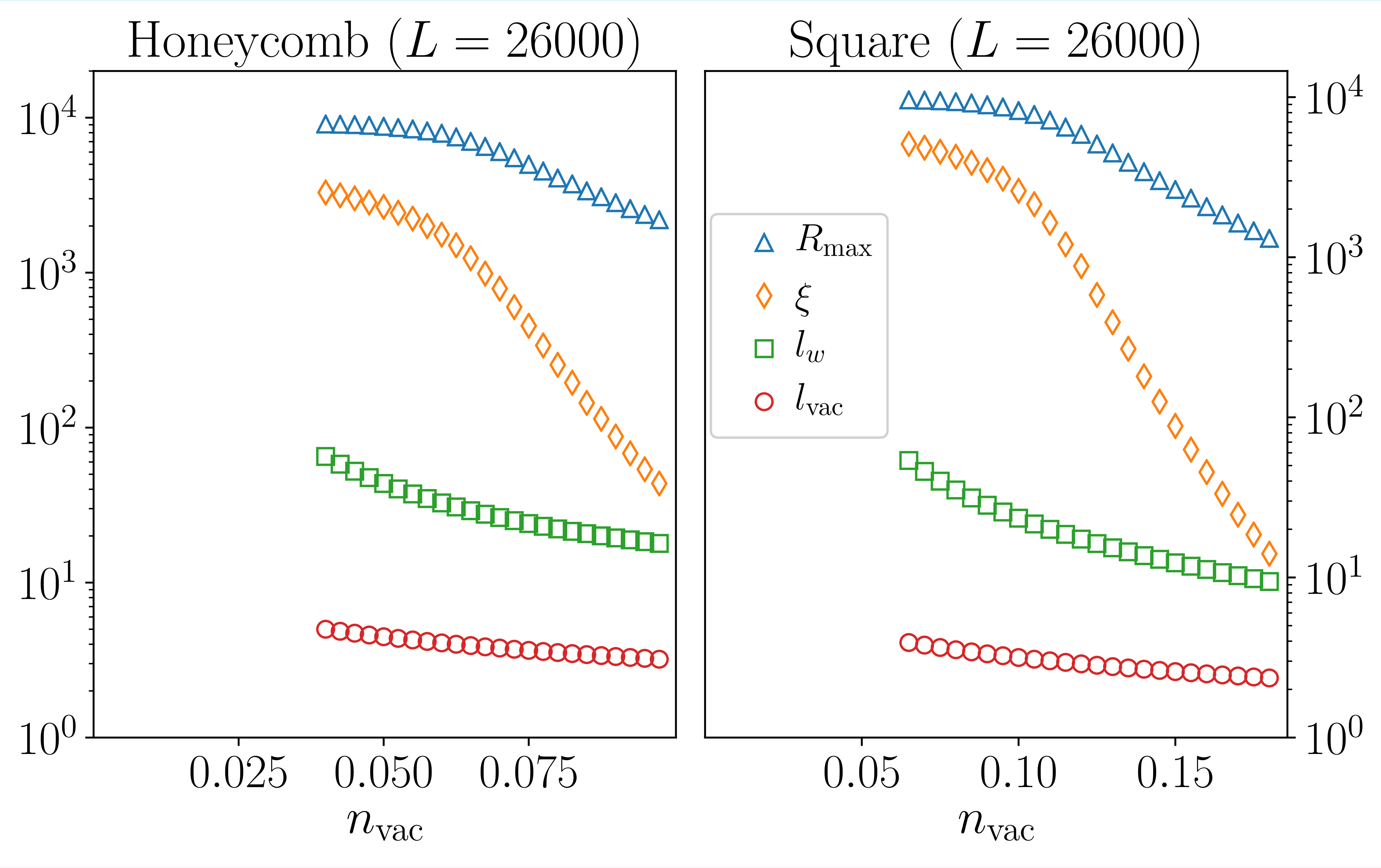
'Small' defined to have $\lesssim 10^4$ vacancies(!)

Zero mode density dominated by large-scale structures



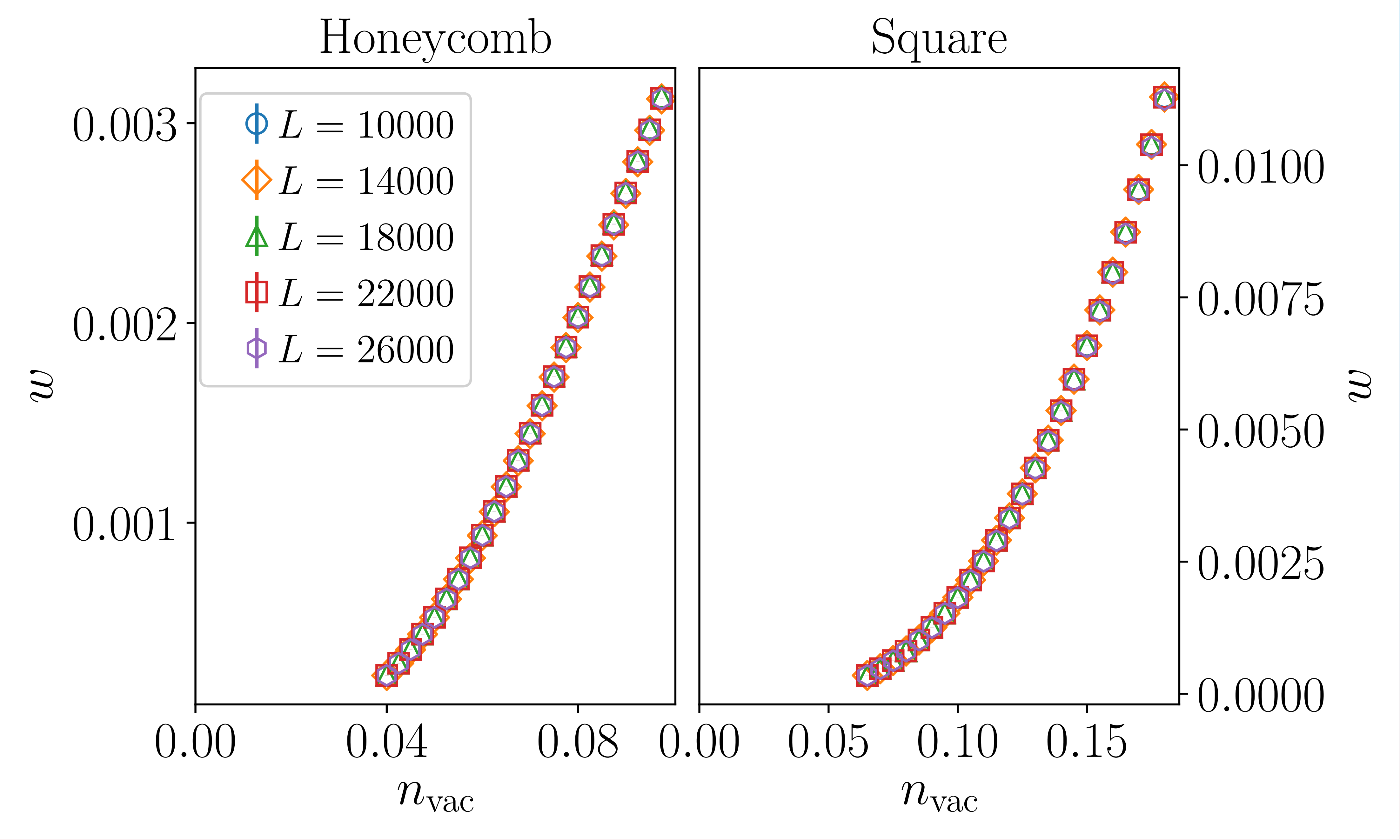
'Small' defined to have $\lesssim 10^4$ vacancies(!)

Comparison of length-scales

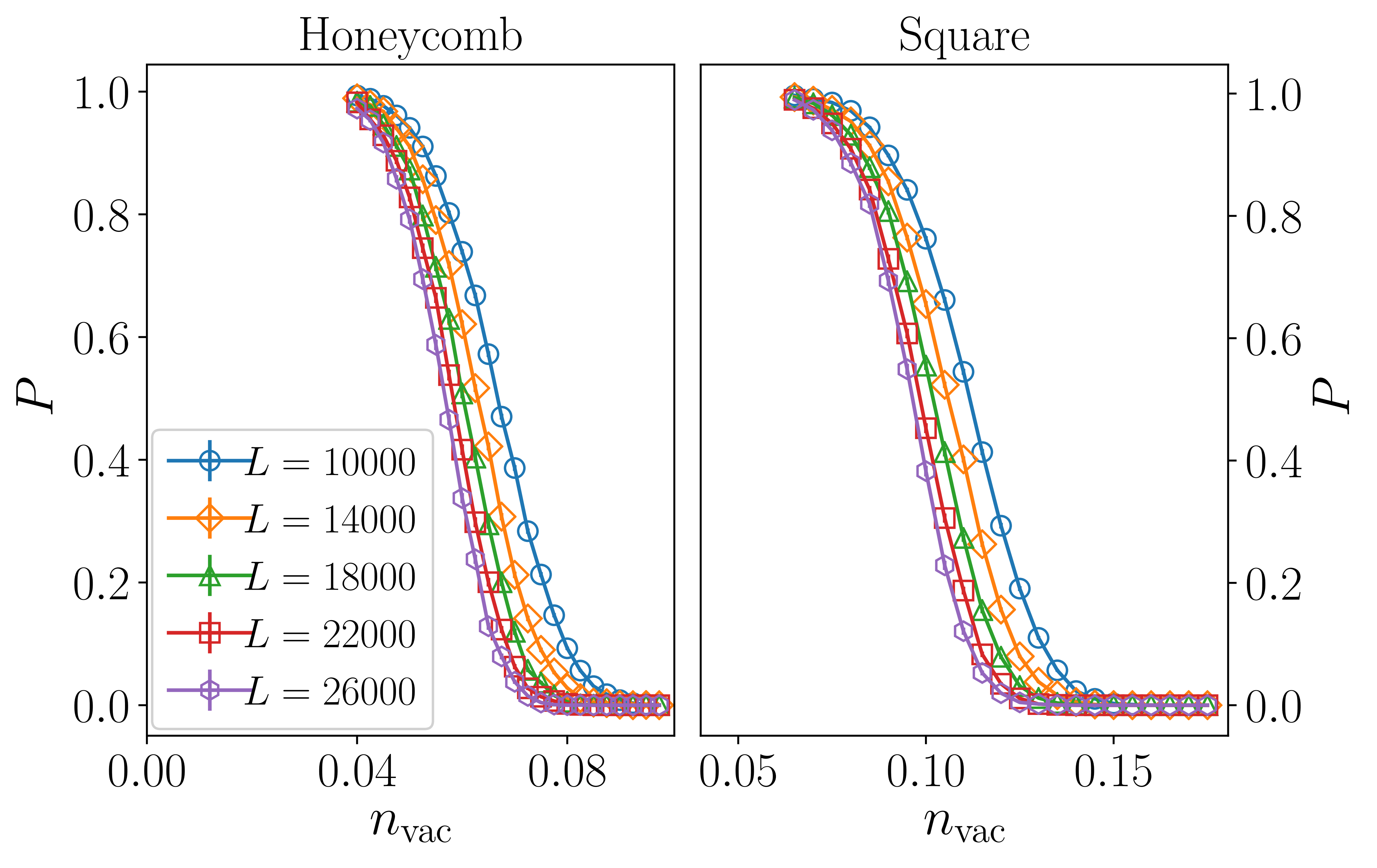


Size of largest R-type region is nearly system-size limited at low dilution

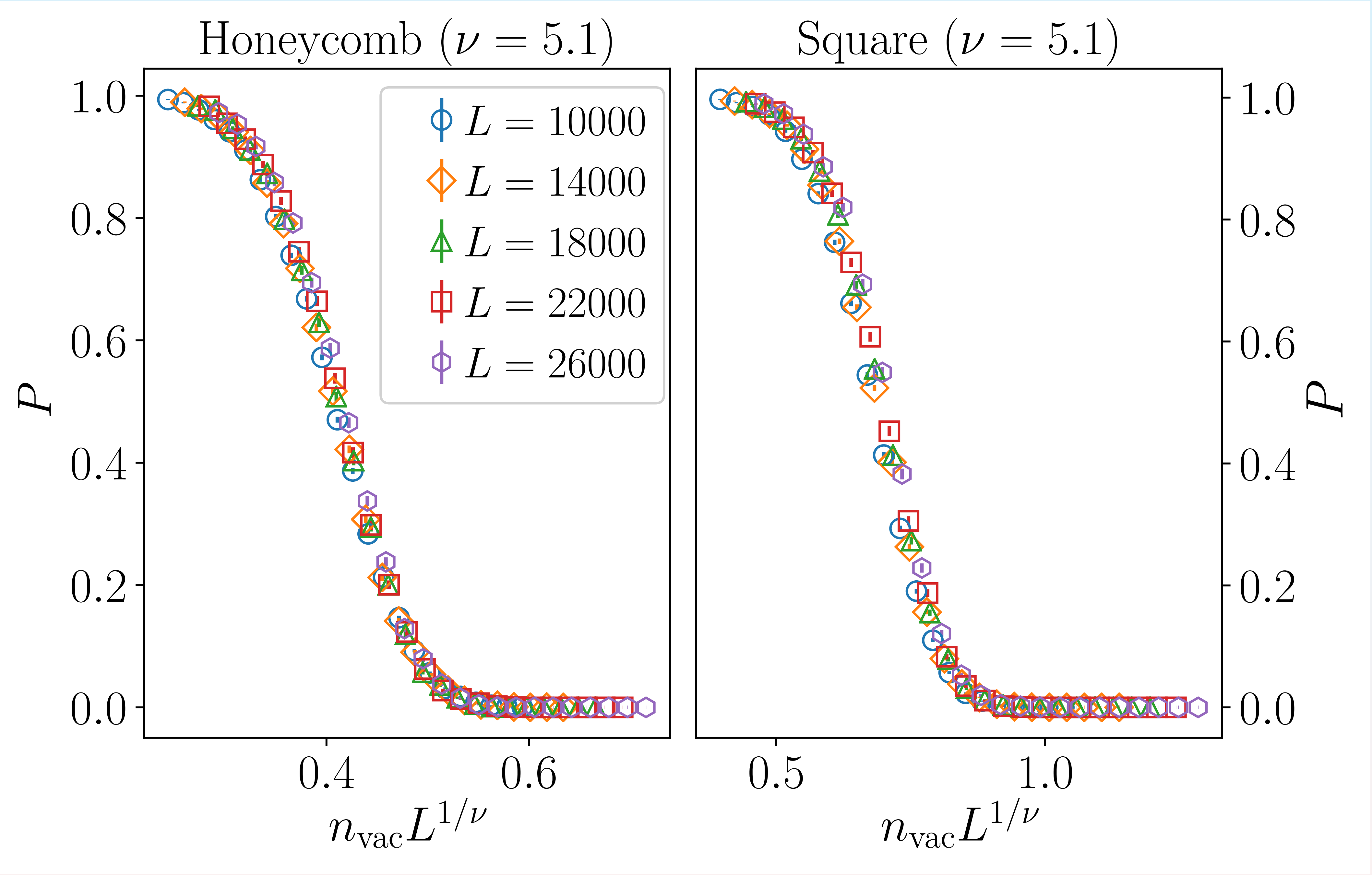
Zero-mode density: featureless



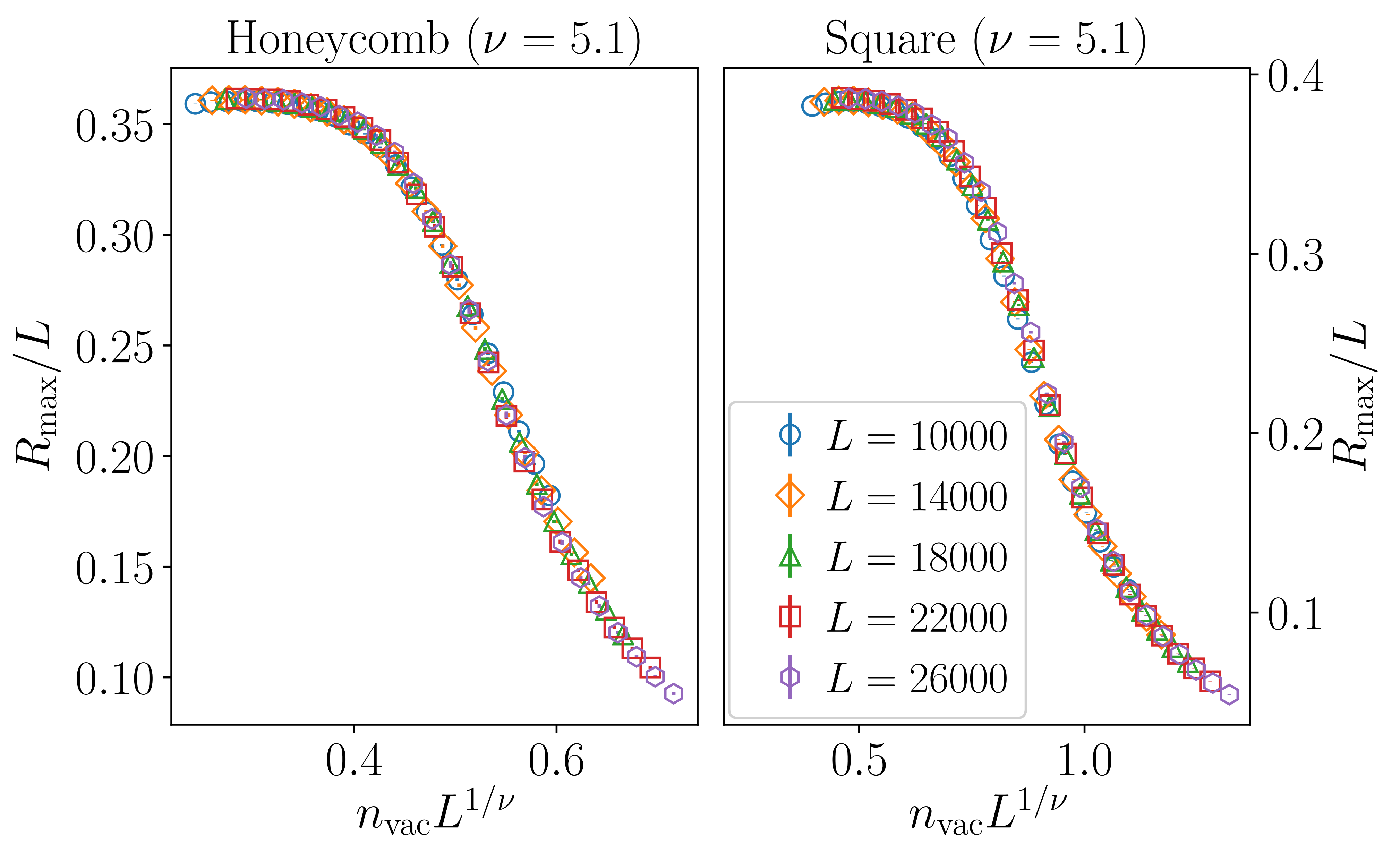
Incipient percolation: wrapping probabilities



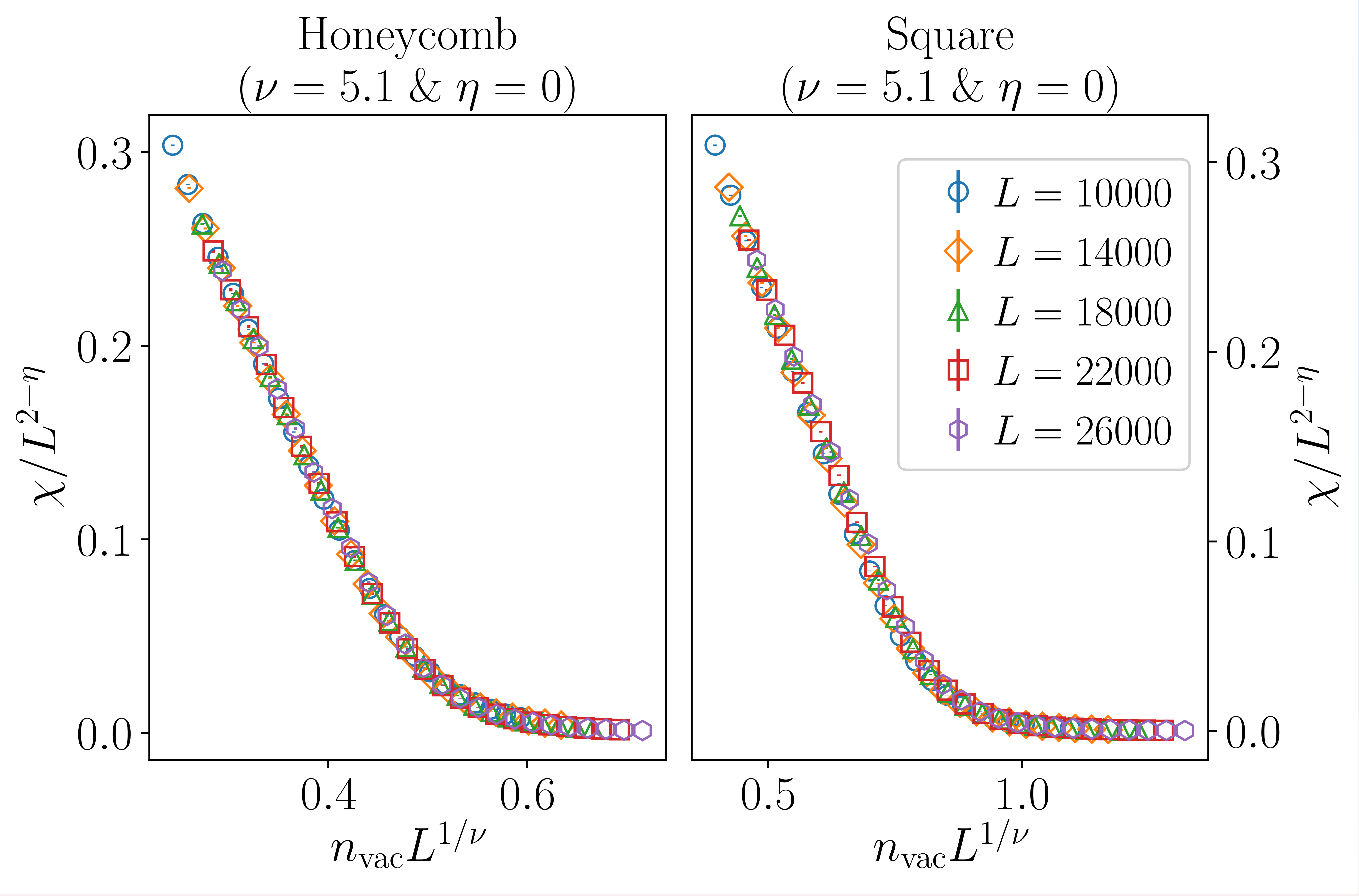
Universal scaling at zero-dilution critical point



Universal scaling at zero-dilution critical point

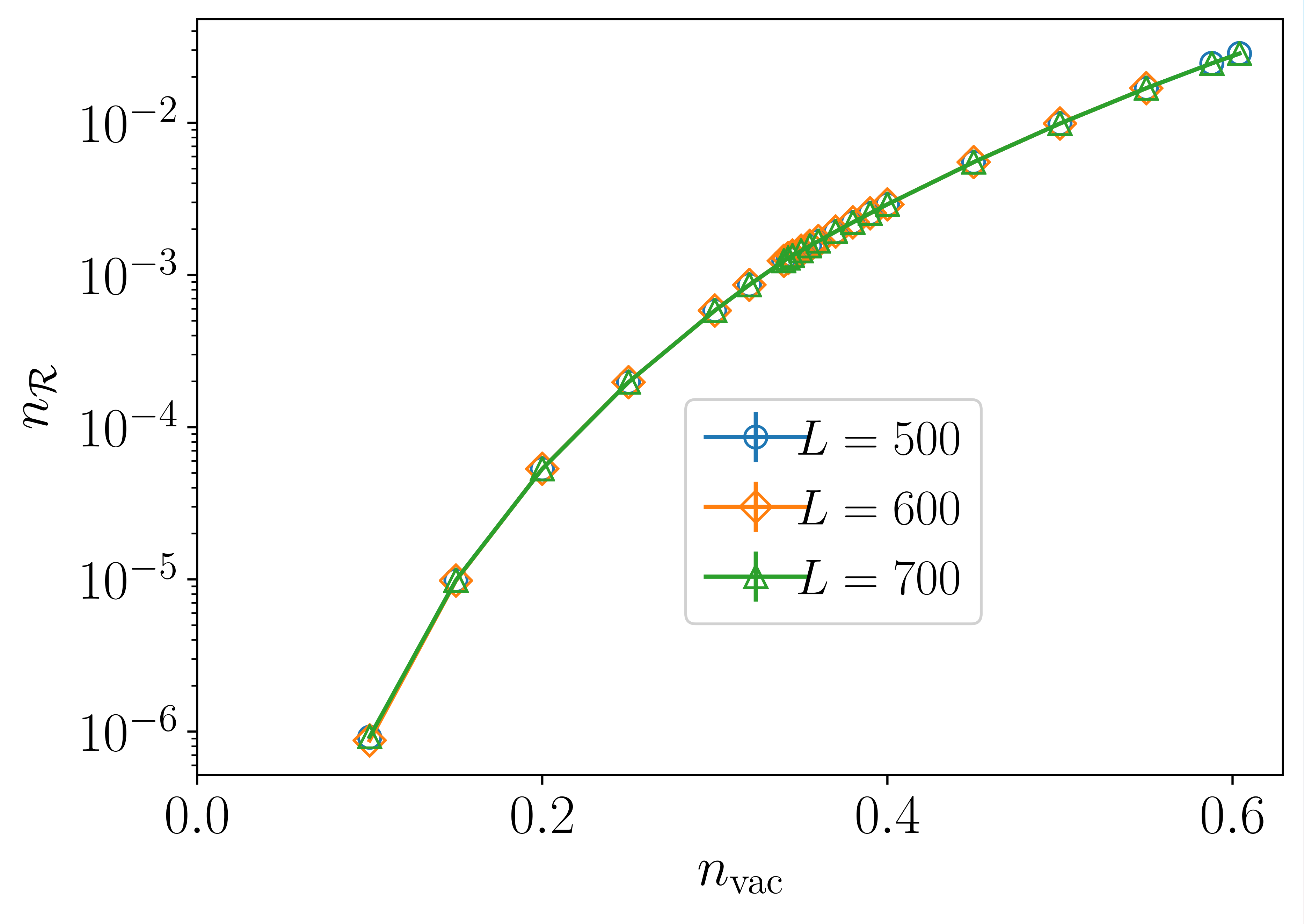


Universal scaling of mass susceptibility $\chi = \langle m^2 \rangle / L^2$

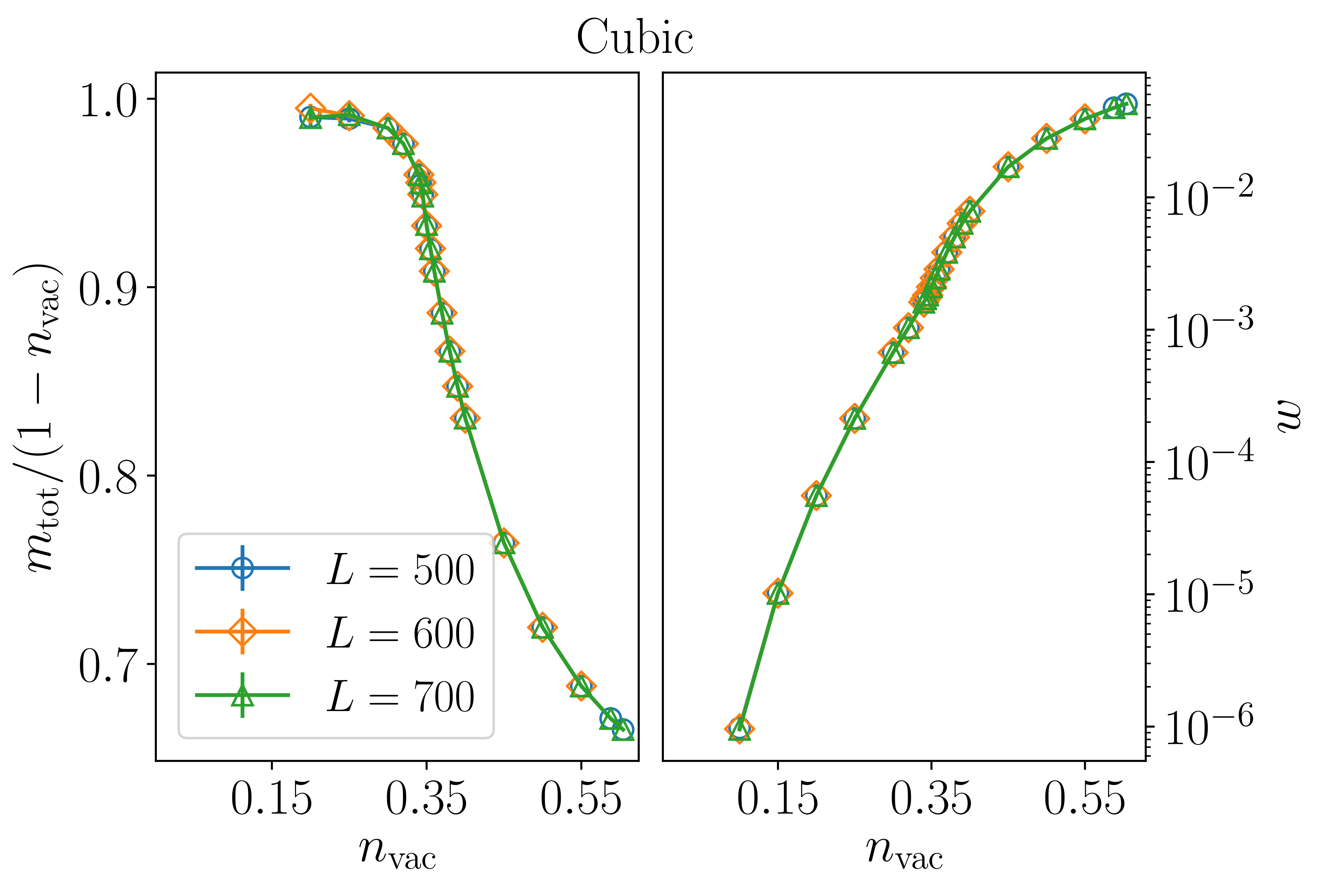


Anomalous exponent η indistinguishable from 0

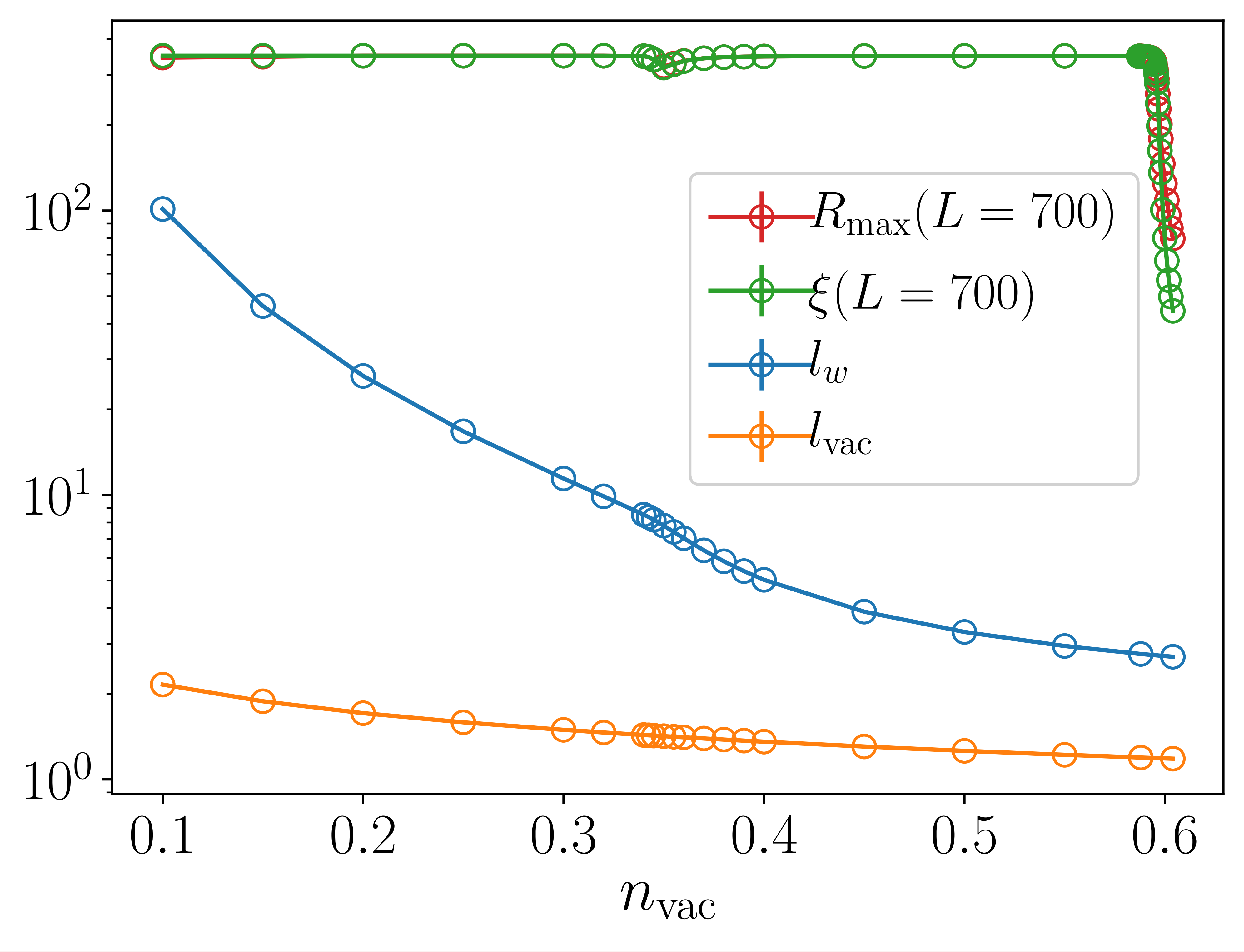
Cubic lattice: Number density of R-type regions



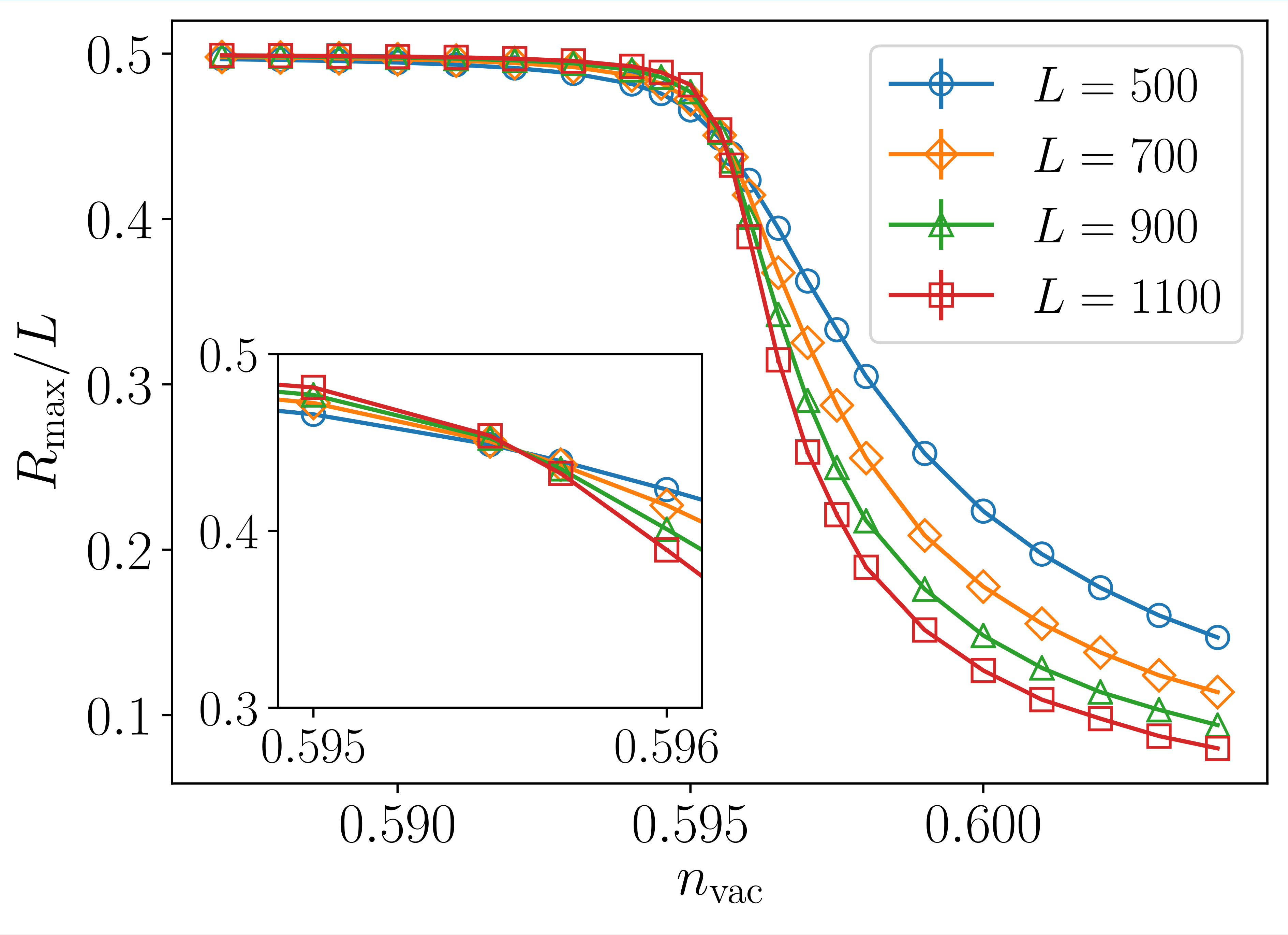
R-type regions take over lattice at low dilution



Emergence of a large length scale

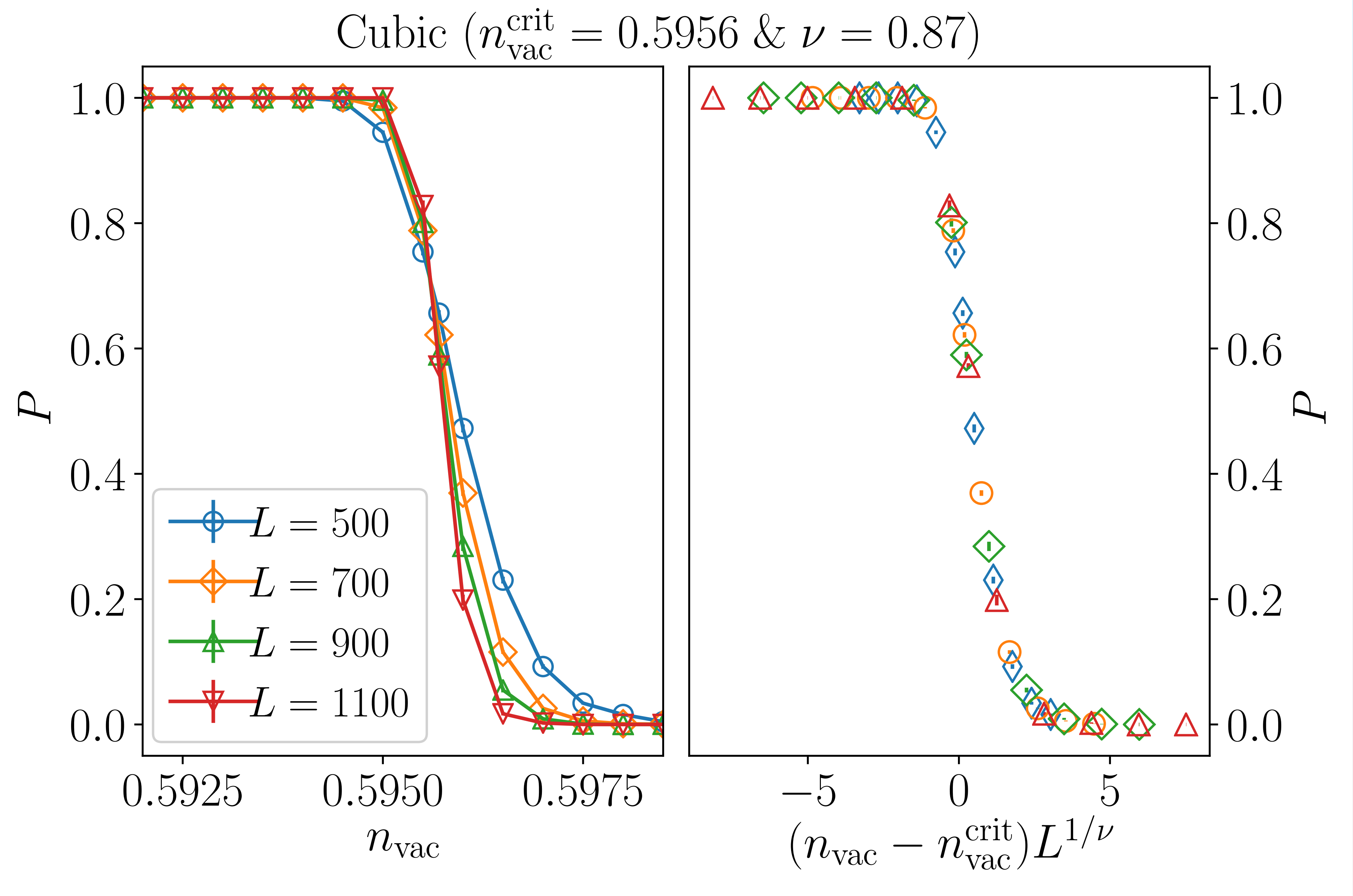


Percolation transition



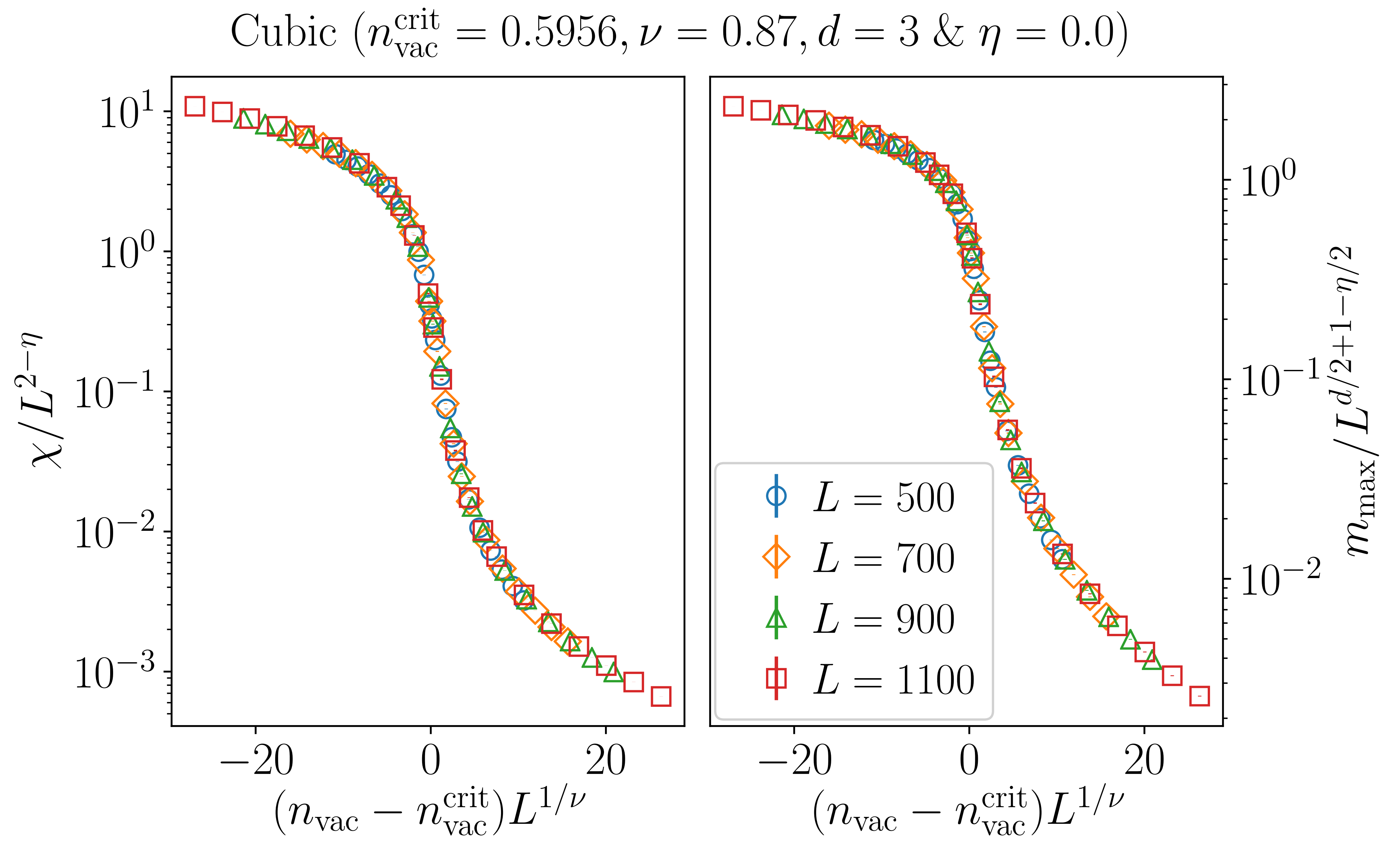
Unconnected with geometric percolation transition of lattice itself

Percolation transition: scaling



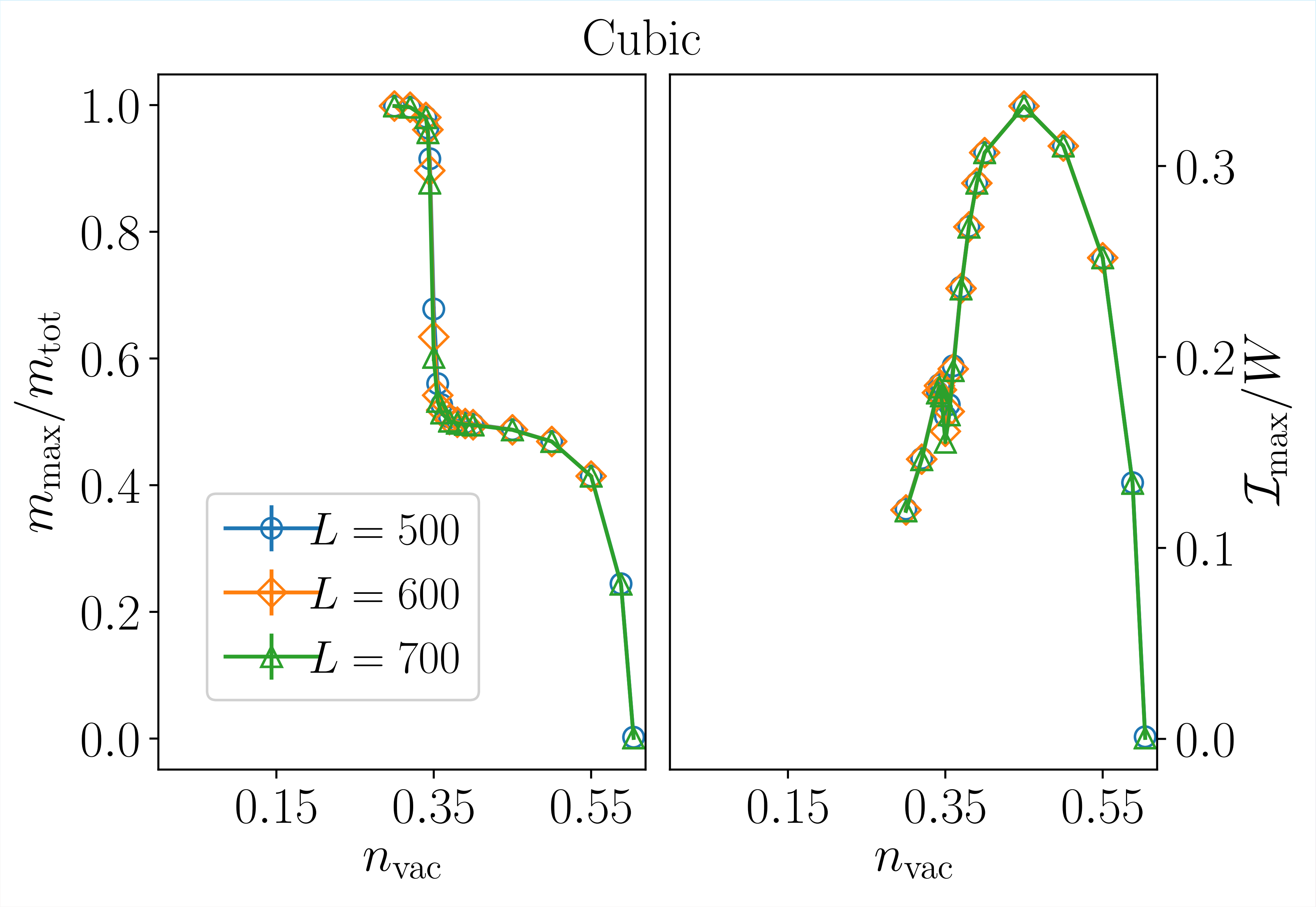
Unconnected with geometric percolation transition of lattice itself

Percolation transition: scaling

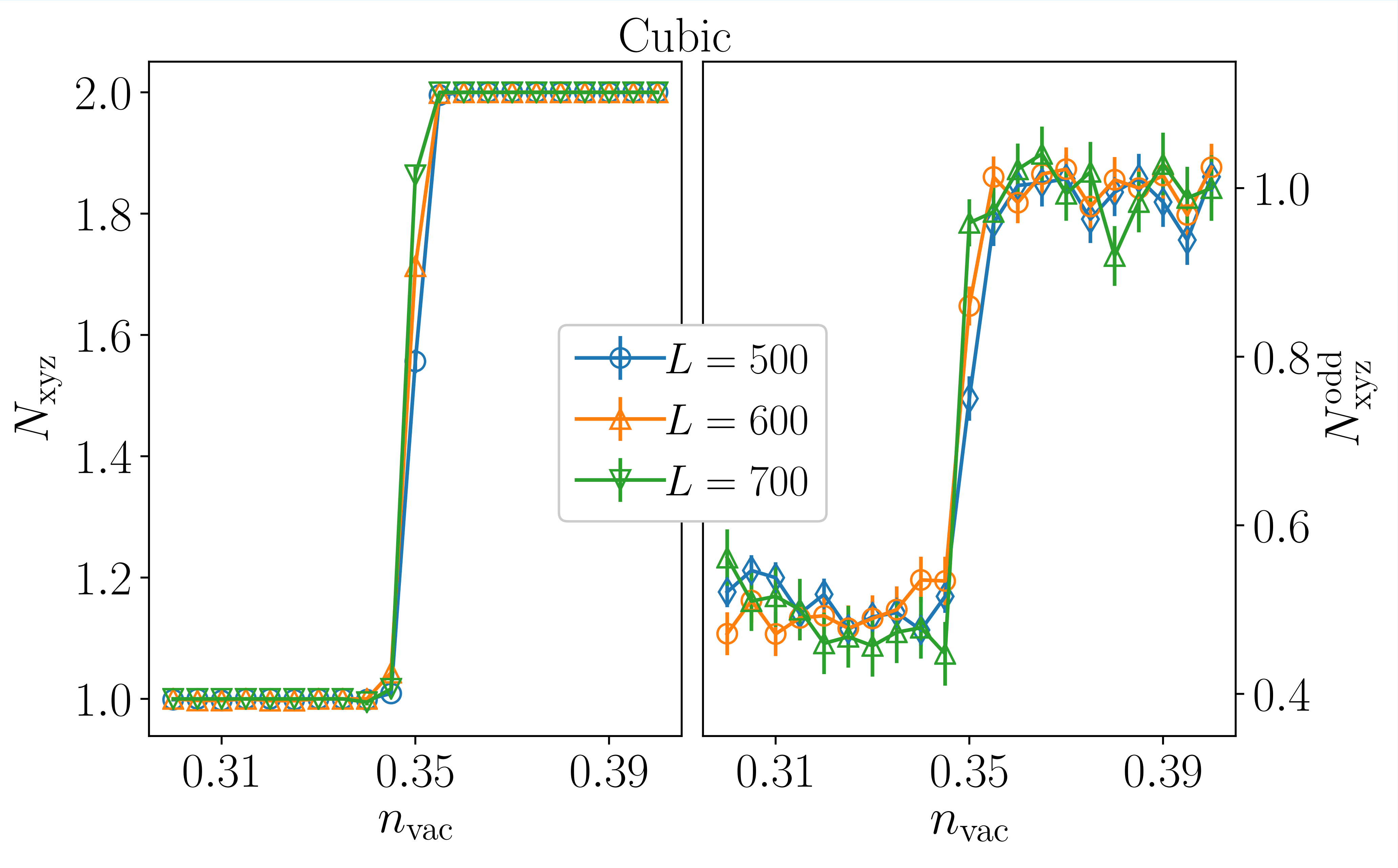


Anomalous exponent η indistinguishable from 0

A second transition?

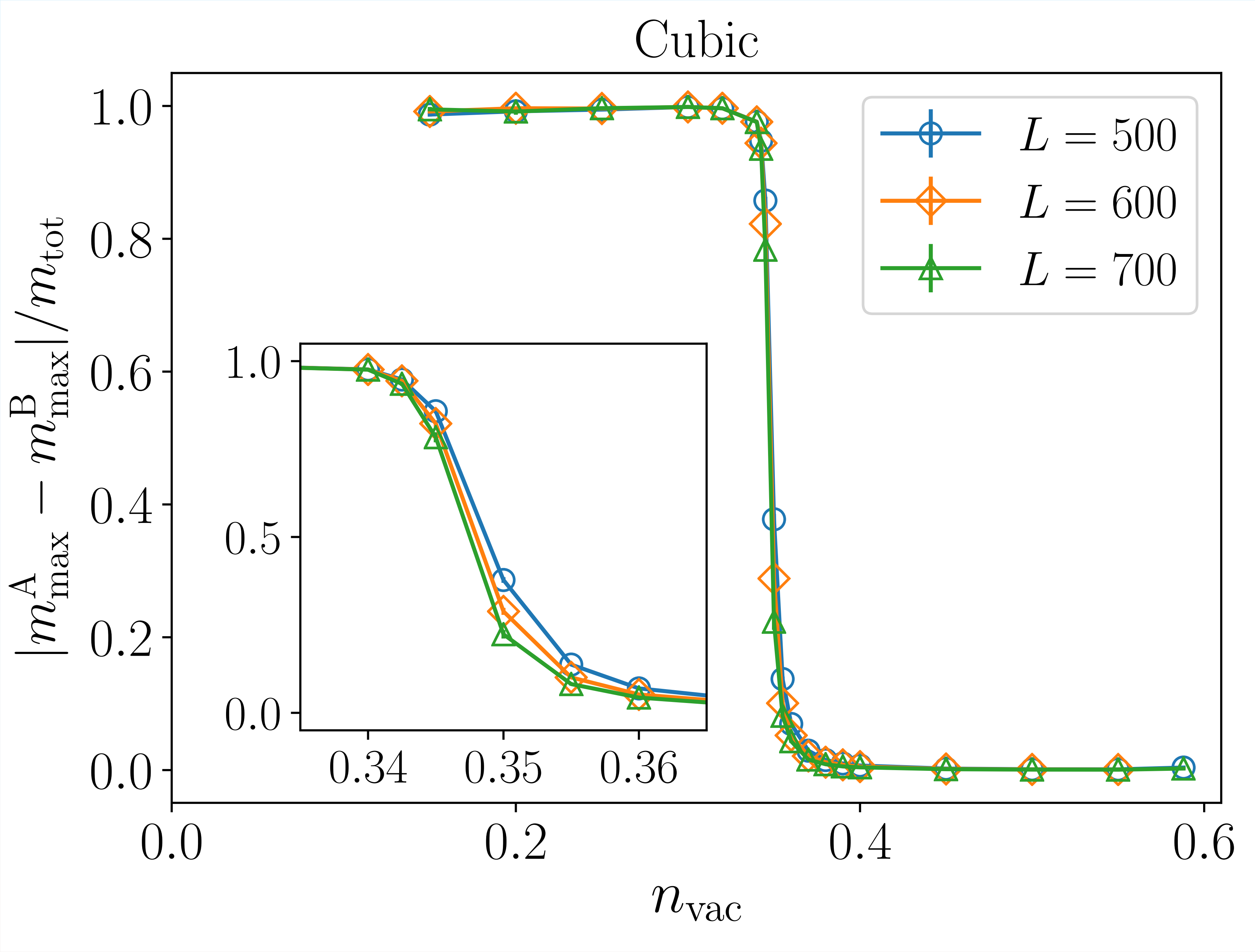


A second transition (!)



Number of percolating clusters goes from 2 to 1

Sublattice symmetry breaking inside percolated phase



Spontaneous sublattice-symmetry breaking transition within percolated phase

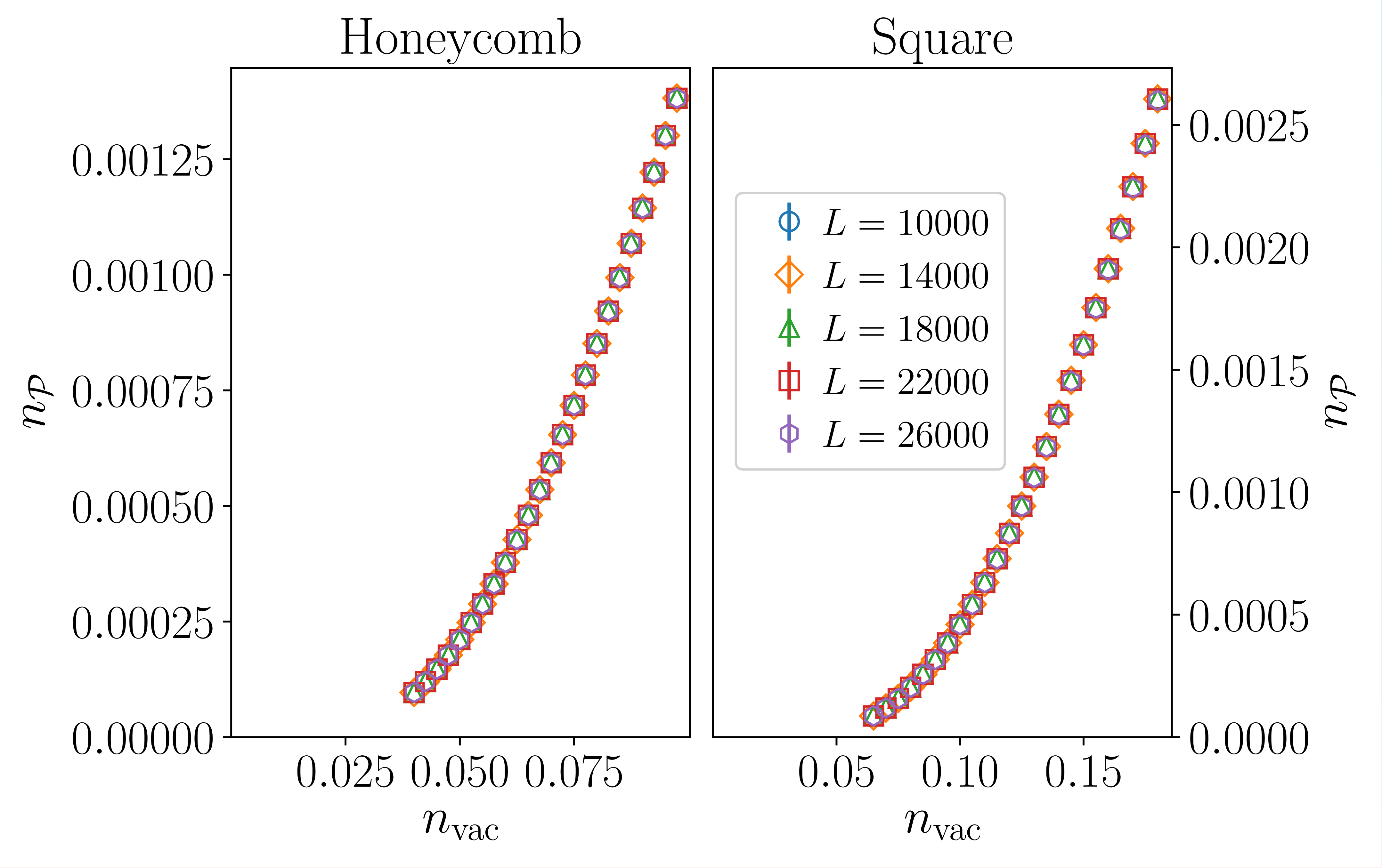
Consequences

- Infinitesimal dilution localises monomers of the maximally-packed dimer model in two dimensions
- There is no bipartite quantum percolation transition in two dimensions (long story, starting ‘70s)
- Precise determination of quantum percolation threshold in three dimensions.
- Infinitesimal dilution causes sublattice symmetry breaking in the monomer gas in three dimensions. Consequences for electronic system(?)
- In corresponding Majorana network: Majorana zero modes hosted by R-type regions with odd imbalance undergo a percolation transition
- Perhaps most directly interesting: Low energy triplet excitations in diluted quantum antiferromagnets in the extremely low-dilution regime.

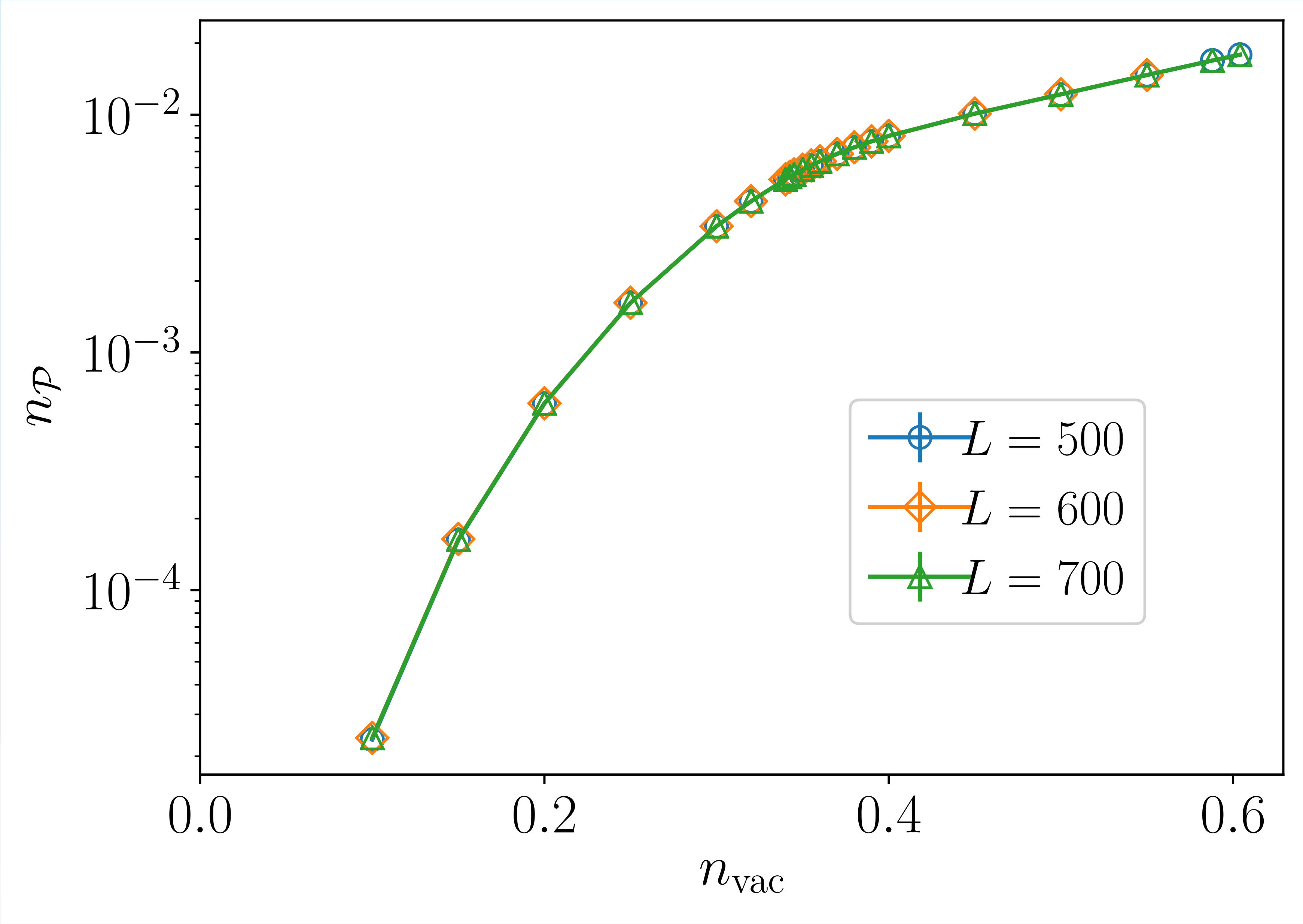
Acknowledgements

- T. Kavitha and A. Mondal for introduction to graph decompositions
- Computational resources of DTP-TIFR
- Discussions with D. Sen, D. Dhar, Mahan Mj, J. Radhakrishnan and many others...
- Research at TIFR supported by DAE, and in part by Infosys-Chandrasekharan Random Geometry Centre @ TIFR and SERB (JC Bose Fellowship)

Thermodynamic densities of number of P-type regions



Cubic lattice: Very similar basic picture...



Summary a la Wodehouse

- Patient perseverance produces percolative paradigms!