Percolation in maximum-density dimer packings

Kedar Damle (TIFR-Mumbai) @ Geometric Methods in Percolation ICTS (March 2026)

unpublished: Bhola, KD, arXiv:2311.05634v2 (2025) Bhola, KD, arXiv:2512.23639 Ansari, Kundu, KD arXiv:2602.24203

recent: Ansari, KD, PRL 132 226504 (2024)
Bhola, Biswas, Islam, KD, PRX 12, 021058 (2022)
KD, PRB 105 235118 (2022)

background: Sanyal, KD, Motrunich, PRL 117 116806 (2016)

(Physics of) Electron waves and particles

Electrons waves in a crystalline metal:

Schrodinger: Frequencies (energies) of allowed eigenmodes in crystal potential

Bloch: Described by band theory for perfectly periodic crystals.

Pauli: Each wave mode can be occupied by two electrons

Fermi: Low energy physics controlled by states near ε_F .

Effect of Coulomb interactions:

Landau: Described by Fermi liquid (as opposed to gas) theory.

Basic idea: replace electrons by “dressed” quasiparticles

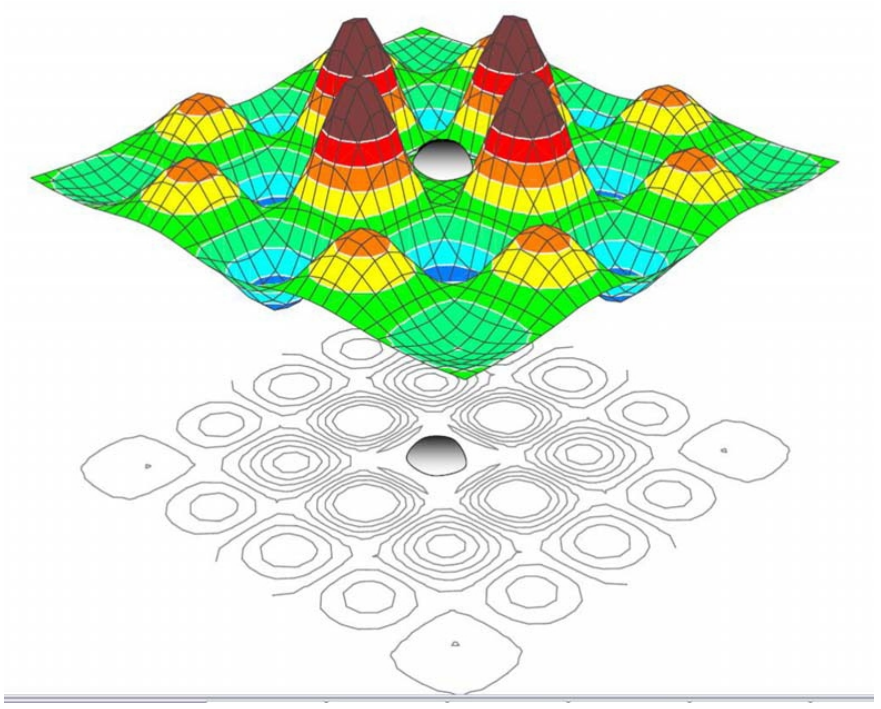
Disorder (in physics)

Quenched disorder: Missing atoms, adatoms, lattice imperfections...

Static on the time-scale of experiment.

Quite common in condensed matter systems.

Weak disorder (in a physical system)



Alloul et. al. Reviews of Modern Physics 81, 45 (2009)

Can serve as a probe correlations of underlying state (e.g. spin textures in frustrated magnets)

(Physics of) Strong disorder

Can lead to entirely new phases of matter:

Less well understood: spin glasses, many-body localization...

Much better understood: Anderson insulators

“Central dogma” (among physicists)

In large-size limit -

Strong version:

Self-averaging of properties: Sample-to-sample fluctuations small (average = typical)

Violations exist – e.g. Disordered quantum spin chains (infinite-disorder fixed points)

Weak version:

At a minimum, two samples prepared using some protocol must be in same phase.

Violations? May exist in infinite-range spin glass models (?)

Classical fluid in a porous medium

Simplest setting: Transport of a classical fluid in a porous medium

Random geometry of medium determines fluid transport

Paradigm of percolation

(more generally: diffusion in a random medium)

Percolation

Study end-to-end connectivity of a porous medium: Can you go from one end to the other?

Simplest model: randomly diluted regular lattice

Consider square grid or honeycomb net or cubic lattice of linear size L

Randomly delete a fraction of sites n_v

$P(L,n_v)$: the probability of answering YES to the question:
Can we go from one end to the other along existing sites and bonds?

How does this behave as a function of n_v , L ?

(Broadbent and Hammersley Proc. Cam. Phil. Soc. 53, 629, 1957)

Threshold behavior

Sharp threshold behavior (in the limit of $L \to \infty$) of $P(L,n_v)$

$$\lim_{L\to\infty} P(L, n_v) \to 1 \text{ for } n_v < n_v^{\text{crit}}$$

$$\lim_{L\to\infty} P(L, n_v) \to 0 \text{ for } n_v > n_v^{\text{crit}}$$

Physicist's view: Simplest example of universal critical behavior

Large-size limit in vicinity of threshold is interesting:

$$P(L, n_v) = f((n_v - n_v^{crit})L^{1/\nu})$$

$$f(x) \to 0 \text{ as } x \to +\infty$$
$$f(x) \to 1 \text{ as } x \to -\infty$$

Scaling function and exponent is universal, depends only on dimensionality

Critical point exhibits scale invariant behavior.

Random geometry of largest cluster doesn't look different if we rescale!
(within limits...)

Amazingly simple example of a scale-invariant critical point (!)

Peeking ahead:

Maximum-density dimer packings of weakly-diluted lattices have unusual percolation transitions

Some of the percolated phases show violations of (even the weak form of the) “central dogma”

Root cause: Kinematic constraints that induce long-range correlations

(Physics of) Matter waves in disordered medium: Anderson localization

Electron waves can be localized even when classical fluid percolates

Effect of destructive quantum interference on large-scale transmission coefficient

Even weak disorder localizes electron waves in two dimensions

In three dimensions: Localization above threshold of disorder strength

Many variants well-understood

e.g. quasiparticle localization in dirty superconductors

Key determinant: Global symmetries (particle-hole, time-reversal, spin rotation...)

Anderson (1958)
Abrahams, Anderson, Licciardello, Ramakrishnan (1979)

(Physics of) Quantum percolation: Anderson localization meets Bernoulli percolation

Motivated by simple models of binary AB alloys (1970's story...)

Electrons cannot hop to B sites (ion at B site lacks relevant orbital at fermi energy)

Modeled by randomly diluting crystalline lattice
(vacancies correspond to locations of B atoms)

Special case of Anderson localization:
No random potential (to first approximation)
Only forbidden hops/sites of lattice.

Basic question:

Does an eletron wave remain delocalized whenever classical particles percolate?

Or: is there a "quantum percolation" transition inside the classical percolated phase?

“Tight-binding model” for electrons in undoped graphene with vacancy defects

$$H = -\sum_{\langle rr' \rangle} \left(t_{rr'} |r\rangle\langle r'| + t^*_{rr'} |r'\rangle\langle r| \right)$$

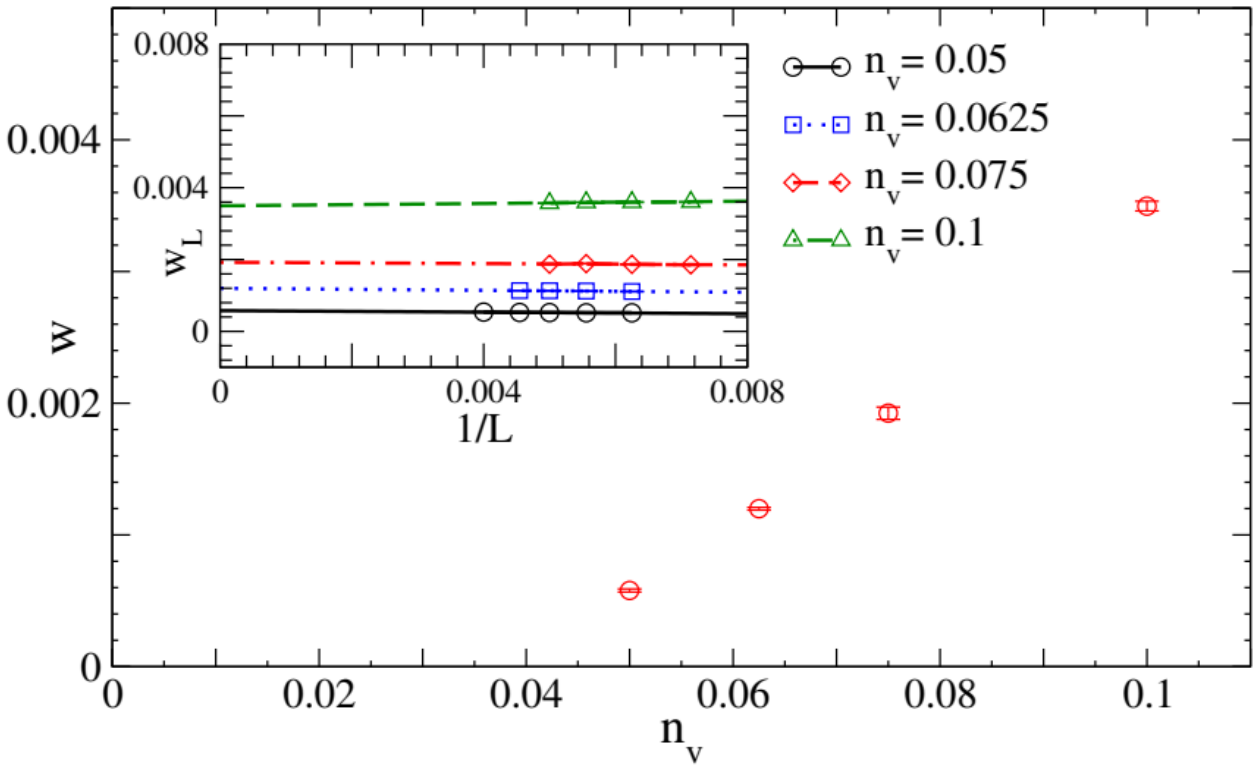
r, r' Denotes sites of *diluted* honeycomb lattice (after random deletion of fraction n_v of vertices)

$\langle rr' \rangle$ Denotes *surviving* nearest-neighbor links (edges of diluted graph)

$t_{\langle rr' \rangle}$ Hopping amplitudes can vary from link to link

Math problem: Spectrum of Hermitean (weighted) adjacency matrix of this graph in large-size limit

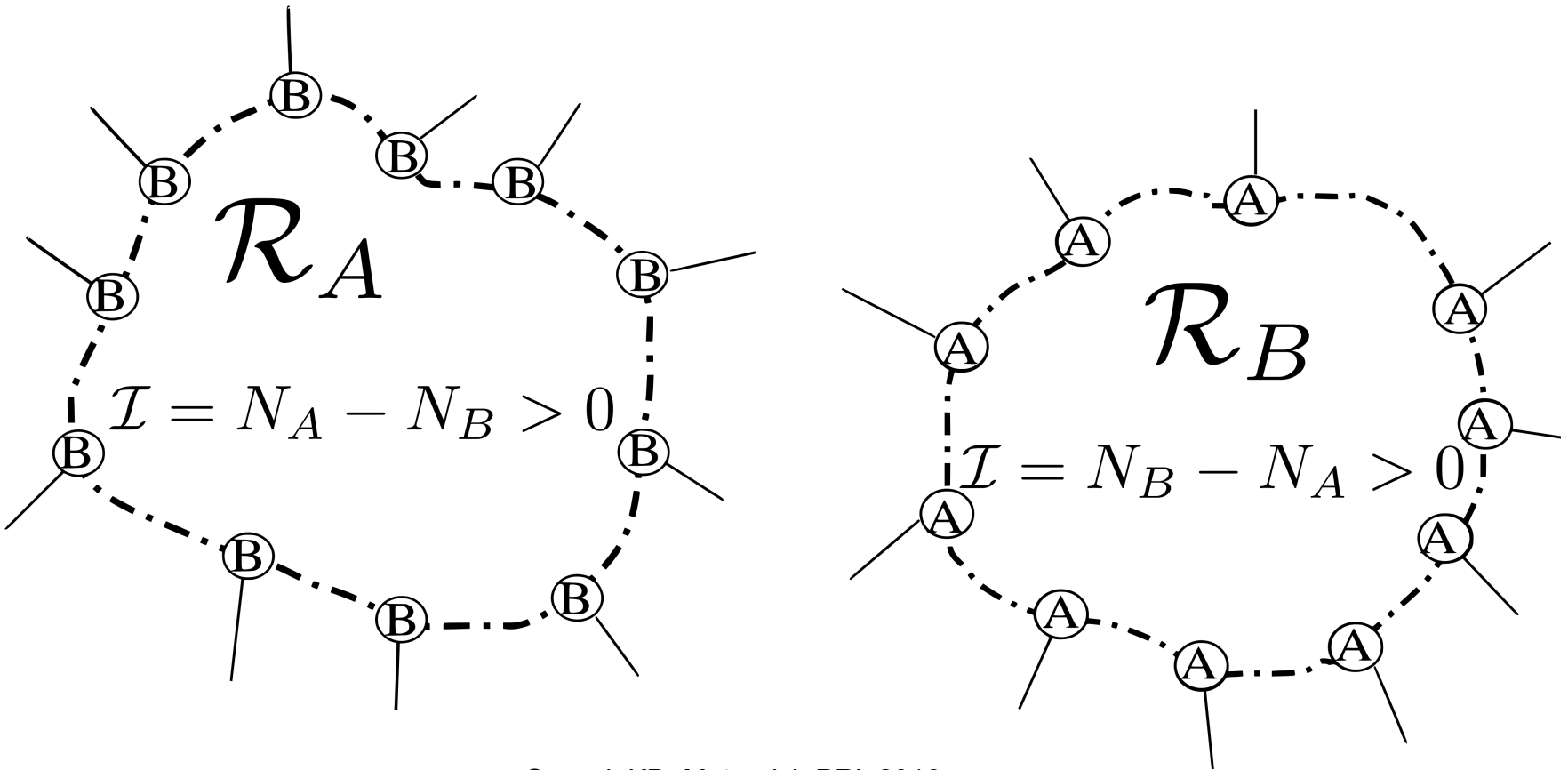
“Surprising” nonzero *density* w of ϵ = 0 states on diluted honeycomb lattice



Aside: Turns out crossover in density of states controlled by density of these zero modes...

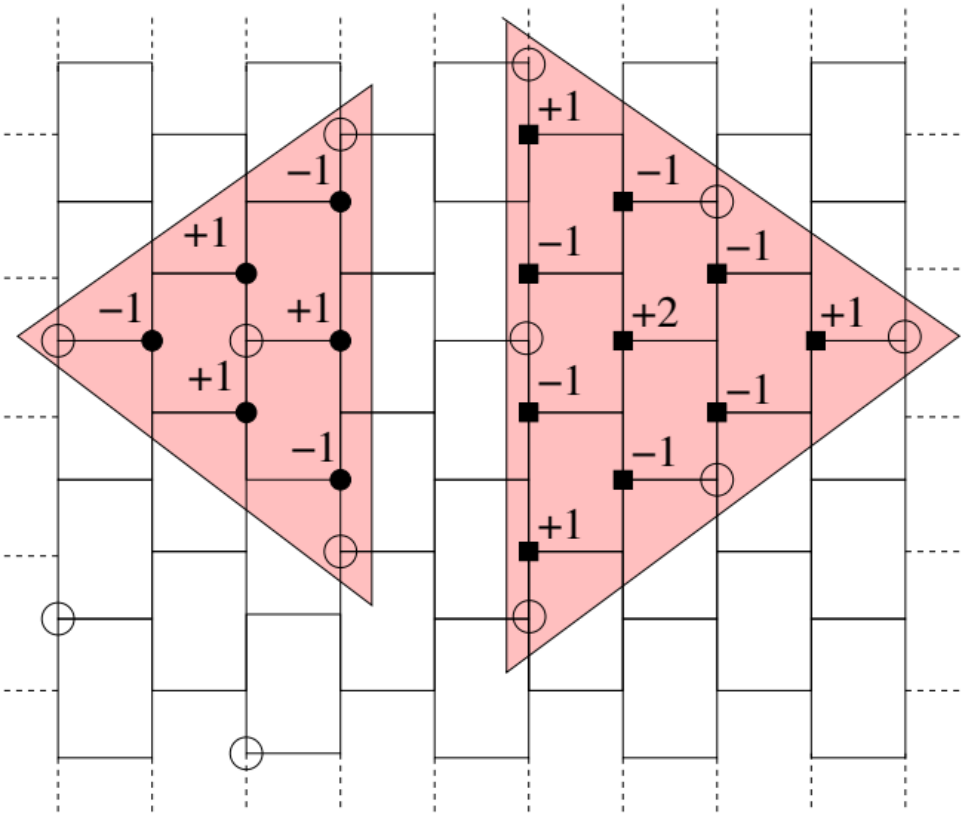
Sanyal, KD, Motrunich PRL 2016

Conjecture for mechanism: *Rare* “R-type regions”:



Sanyal, KD, Motrunich PRL 2016

An example and an important distinction



Sanyal, KD, Motrunich PRL 2016

Key distinction: On the right - *topologically protected zero mode wavefunction.*
On the left - only yields zero mode for uniform t

Key unanswered question

Hand-drawn examples provide lower bounds on DOS at E=0.

But computed value much greater.

What actually determines density of zero modes??

Note: zero modes are not a floating point precision artifact...

Also confirmed by Weik et. al. (2016)

Local sublattice imbalance and dimers: A first clue

Such zero modes only depend on connectivity, not hopping strengths.

R-type regions rely on local imbalance between A and B type site densities.

Suggests thinking in terms of *matchings* a.k.a *dimer covers*

Regions of lattice that cannot be covered perfectly by dimers host wavefunctions

Bhola, Biswas, Islam, KD, PRX 12, 021058 (2022)

Language primer: Fully-packed dimers (perfect matchings)

Fully-packed hard-core dimer models in stat-mech: Match **each** site to an adjacent site monogamously

In graph theory/computer science: The perfect matching problem

Easy to see (for regular lattices like square, triangular, honeycomb, kagome...):

Extensive entropy of fully-packed dimer covers (perfect matchings)

(exact computation of entropy on planar graphs: Classic papers by Kasteleyn & Fisher)

Maximum matchings of disordered lattices

Basic question: Can a diluted lattice with even number of vertices be perfectly matched?

If bipartite, need |A| = |B|

But: generally not possible *even with* |A|=|B|

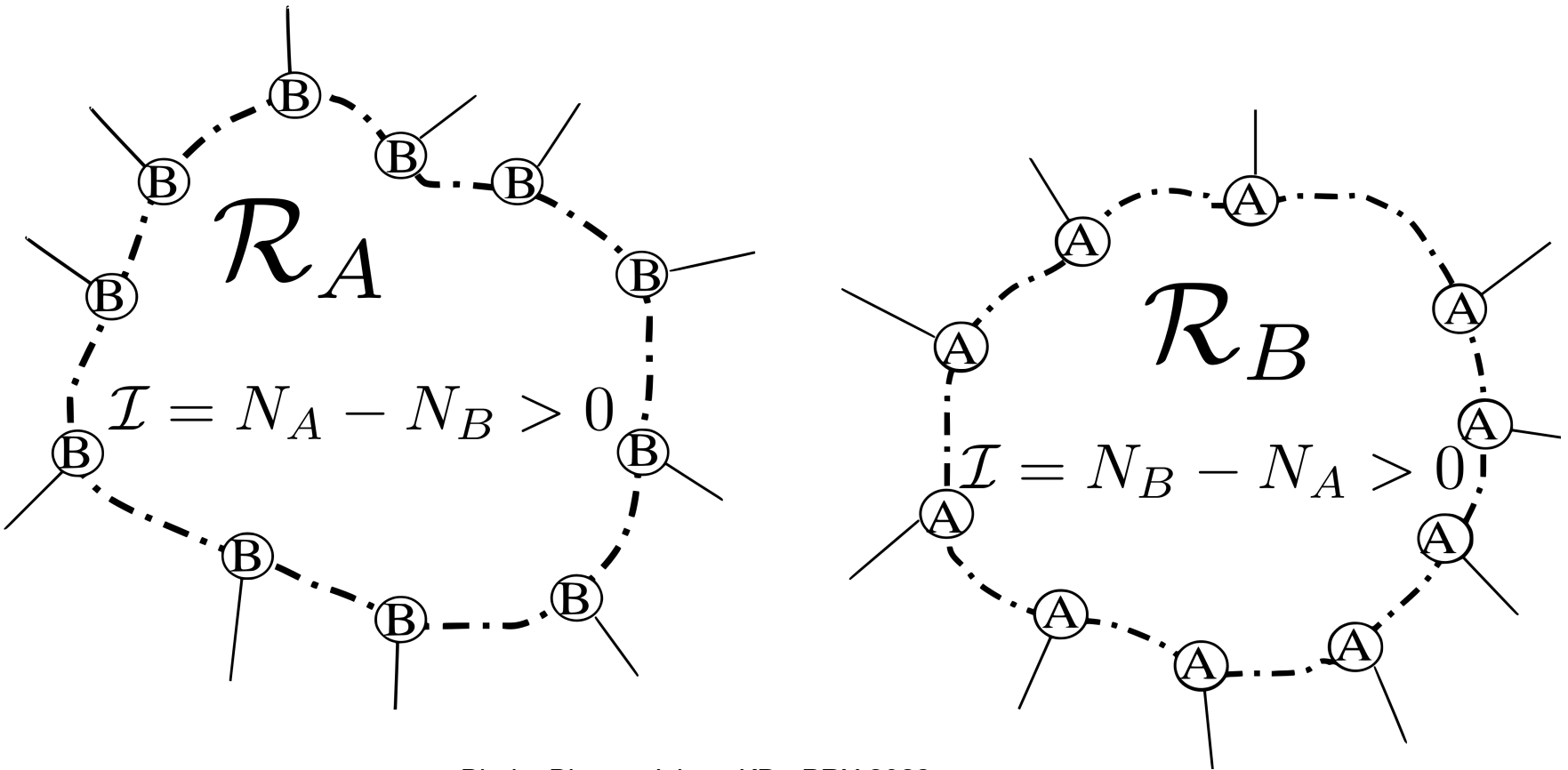
Then have *maximum matchings* but not *perfect matchings*

Maximum matchings have unmatched sites that host *monomers of the maximum-density dimer packing*

Generically: nonzero density of vacancies implies nonzero density of monomers

Bhola, Biswas, Islam, KD, PRX 12, 021058 (2022)

Key observation: “R-type” regions trap monomers



Bhola, Biswas, Islam, KD, PRX 2022

Gels with: Longuet-Higgins on zero modes

AIP | The Journal of Chemical Physics

Some Studies in Molecular Orbital Theory I. Resonance Structures and Molecular Orbitals in Unsaturated Hydrocarbons

1950

H. C. Longuet-Higgins

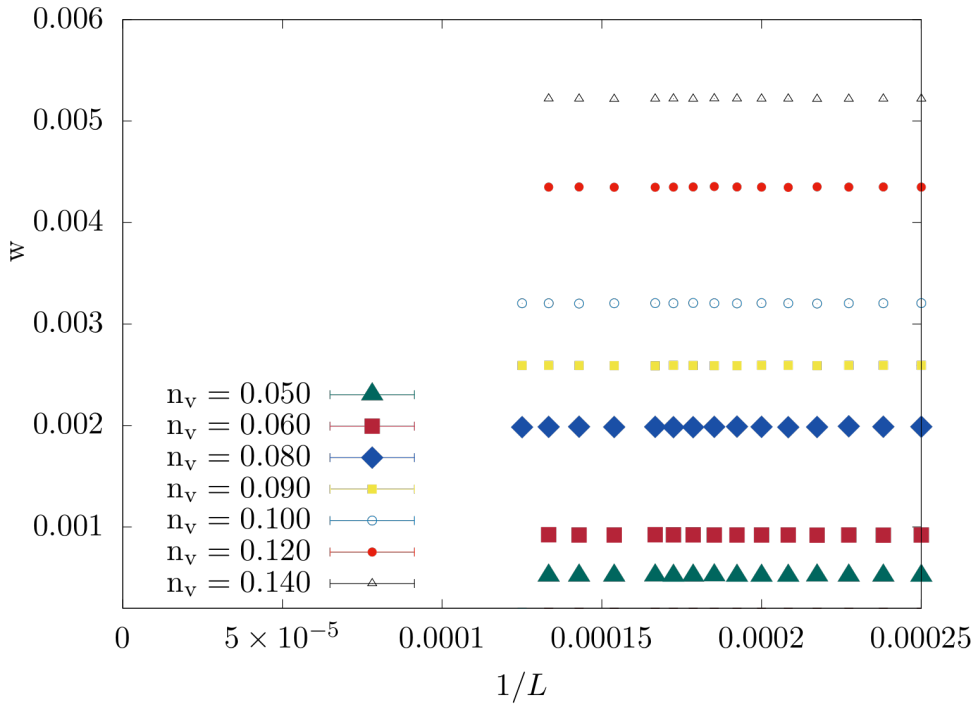
E=0 molecular orbitals correspond to magnetic moments in MO theory of benzenoid molecules

Effectively studying a tight-binding model and asking about E=0 states.

Result: (transcribed to our language)

Total number of monomers in maximum matching = Total number of topolgically protected zero modes (global statement)

A first step: Check convergence to thermodynamic limit



But: we want local information (for transport etc)...

So “where” do the modes “live”?

How does one find a complete set of R-type regions?

What does this question even mean in algebraic terms??

One answer:

Identify a “maximally-localized” basis for the topologically-protected part of zero-energy subspace

Digression:
(Physics of) Topologically protected collective Majorana modes of Majorana tight-binding models?

$$\mathcal{H}_{\text{network}} = \frac{i}{4} \sum_{rr'} \mathcal{A}_{rr'} \eta_r \eta_{r'}$$

$\mathcal{A}_{rr'}$ Antisymmetric matrix of quantum mechanical mixing amplitudes

η_r Majorana operators corresponding to localized Majorana modes

Zeroth order descriptions of:

Mixing of Majorana modes in vortex lattice state of topological p+ip superconductors

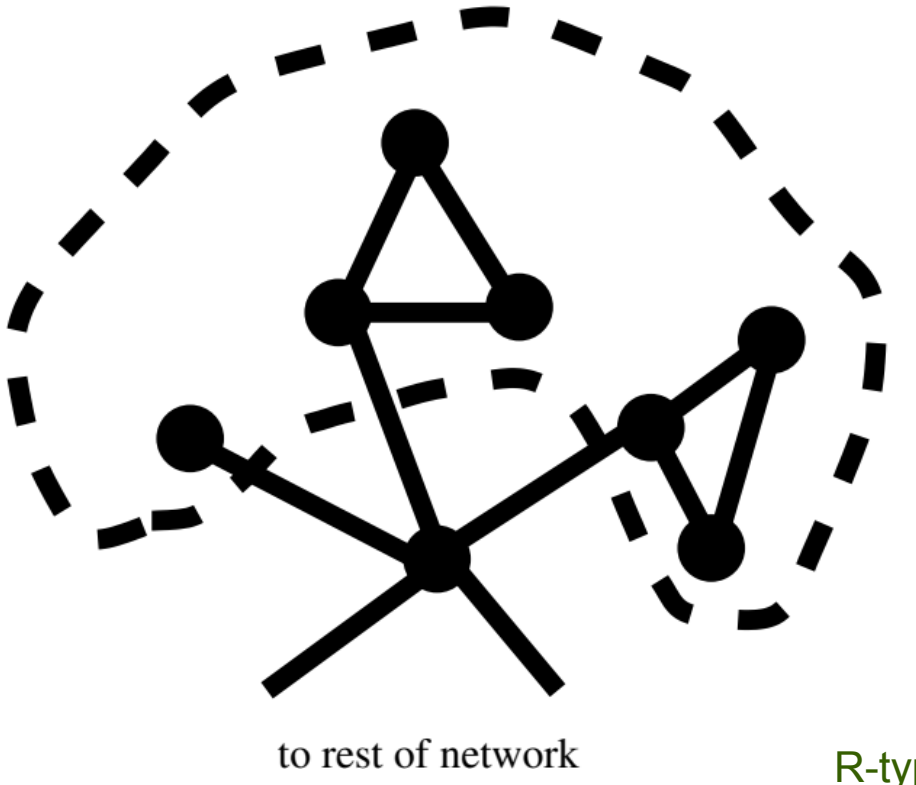
Majorana excitations of Majorana spin liquid states

Can such networks have topologically protected collective Majorana modes?

Linear algebra question : Classify/construct topologically protected null vectors of $iA_{rr'}$

Note: Bipartite random hopping special case of this

R-type regions hosting such zero modes



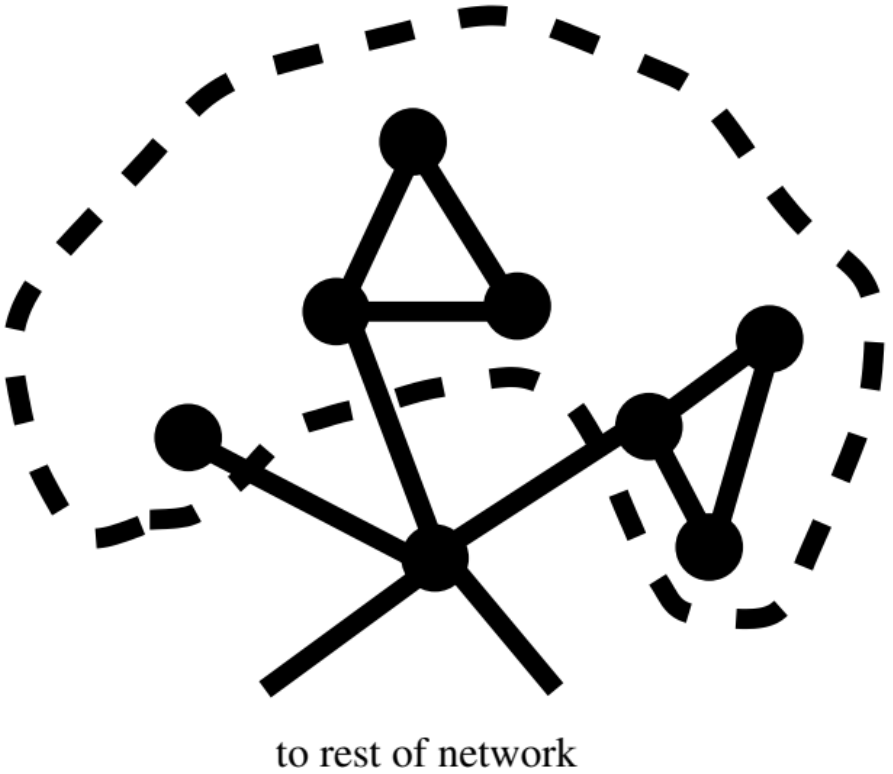
Odd cyles in isolation have a zero mode.

Generically: such modes mix and are destroyed

R-type regions host linear combinations that survive mixing

KD, PRB 2022

Key observation: R-type regions trap monomers



The same R-type region also traps two monomers

KD, PRB 2022

Gels with theorem of Lovasz

ON DETERMINANTS, MATCHINGS, AND RANDOM ALGORITHMS

by L. Lovász*

Fund. Comp. Th. 1979

Total monomer number = Total number of topologically protected zero modes of $i\mathcal{A}_{rr'}$

Focus on random geometry of maximum-density dimer packings

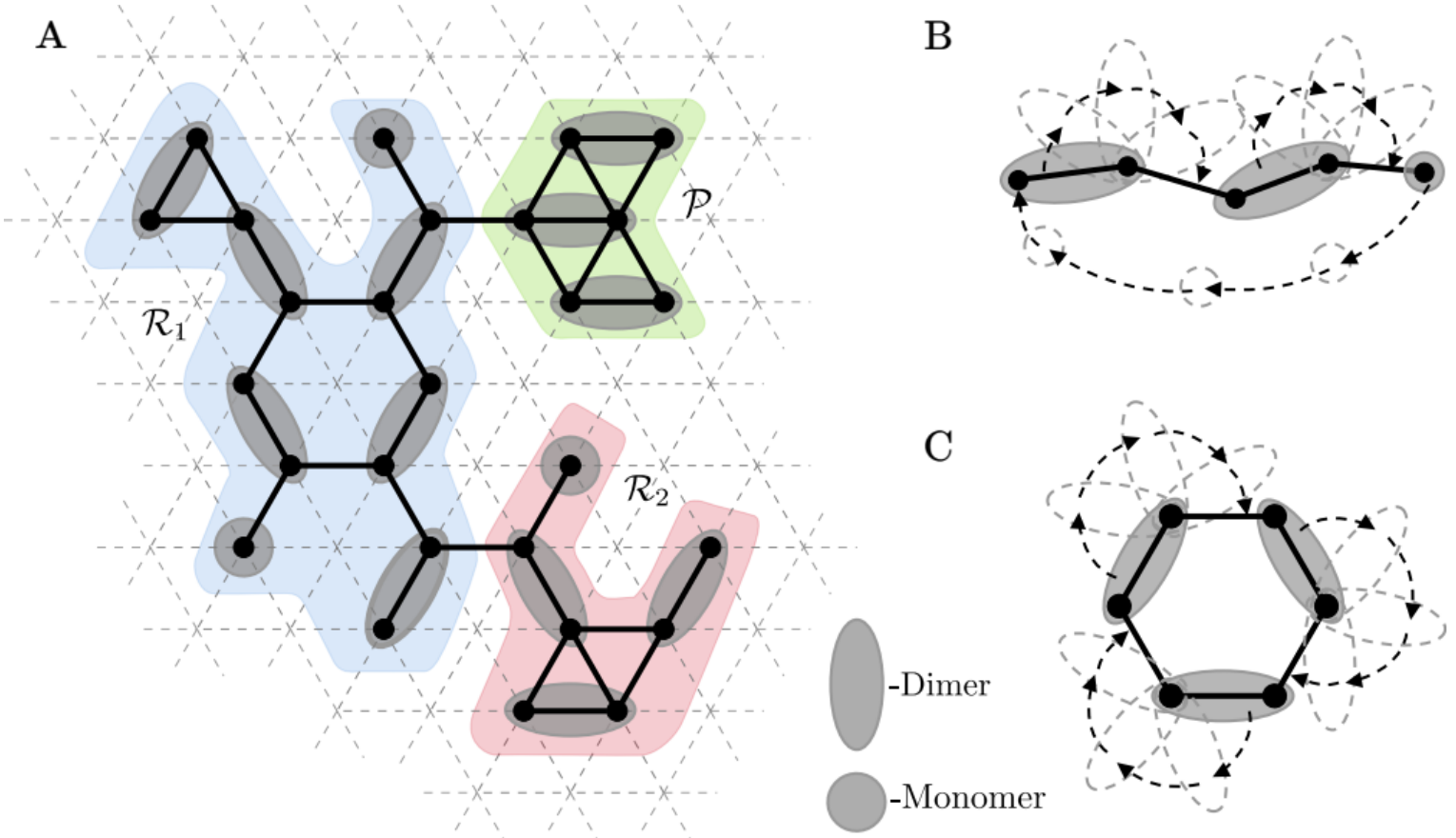
We ask: Where on the lattice do the monomers live? (in the ensemble of maximum matchings)

The answer should give us

partial information on zero-modes/collective Majorana modes

How does one implement this?

The setting: Maximum-density dimer packings of diluted lattices



Conclusions (from pictures):

Consequences of hard-core and maximum-density constraints:

Constrained kinematics: ring-exchange or monomer-hopping

Constraint on links of ring-exchange and monomer-hopping process paths:

Each such link must be occupied by a dimer in at least one such dimer packing

Constraint on monomer and dimer motion:

Monomers confined to well-defined regions of disordered lattice. Other regions fully-packed.

Identifying monomer-carrying and perfectly-matched regions:

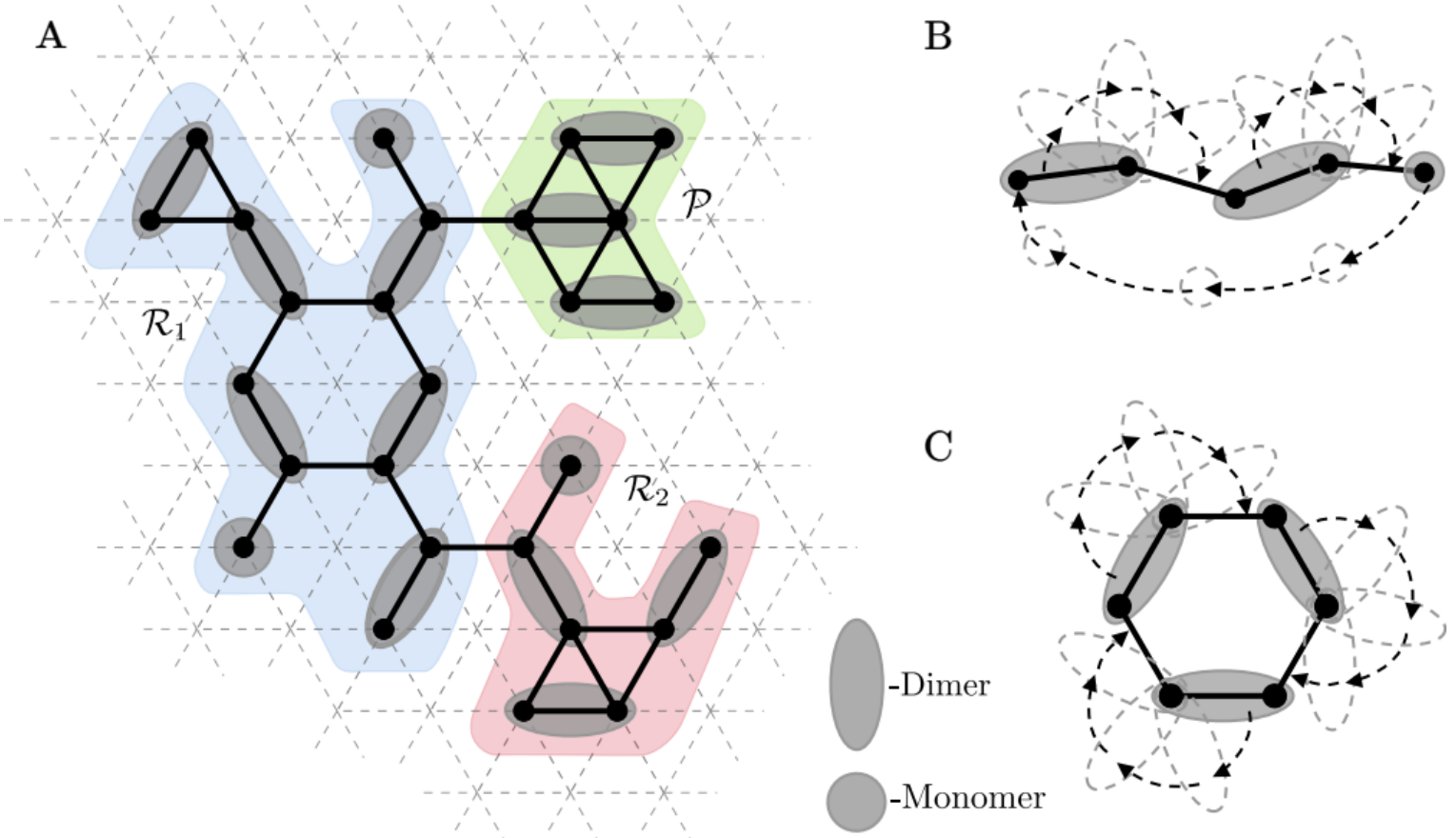
Boundaries of monomer-carrying $\mathcal{R}$-type, fully-packed $\mathcal{P}$-type regions:

Some "forbidden" links of disordered lattice can never be occupied by a dimer in any such packing

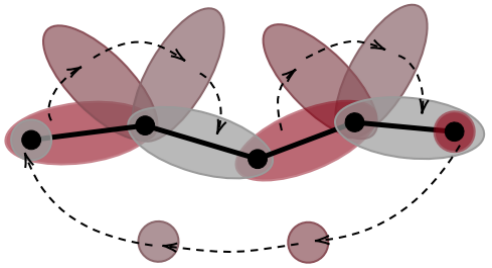
Boundaries of these regions demarcated by the "forbidden" links

Question becomes: Can we identify the forbidden links in a systematic way given a disordered sample?

Geometry of monomer-carrying and fully-packed regions



Math to the rescue: Gallai-Edmonds & Dulmage-Mendelsohn theory



Definitions:

Pick favorite maximum-density dimer packing

Explore forest of alternating paths starting from all monomers

Label vertices e (even) if they can be reached along an even-length path of this forest

Label vertices u (unreachable) if they cannot be reached along any paths of this forest

Label vertices o (odd) otherwise (i.e. can be reached by odd-length path but not even-length path)

Theorem of Gallai-Edmonds (general case) & Dulmage-Mendelsohn (bipartite case):

Labeling is property of disordered lattice, not of your favorite maximum-density dimer packing.

COVERINGS OF BIPARTITE GRAPHS

A. L. DULMAGE AND N. S. MENDELSOHN

Dulmage & Mendelsohn, Canadian J. Math 1958

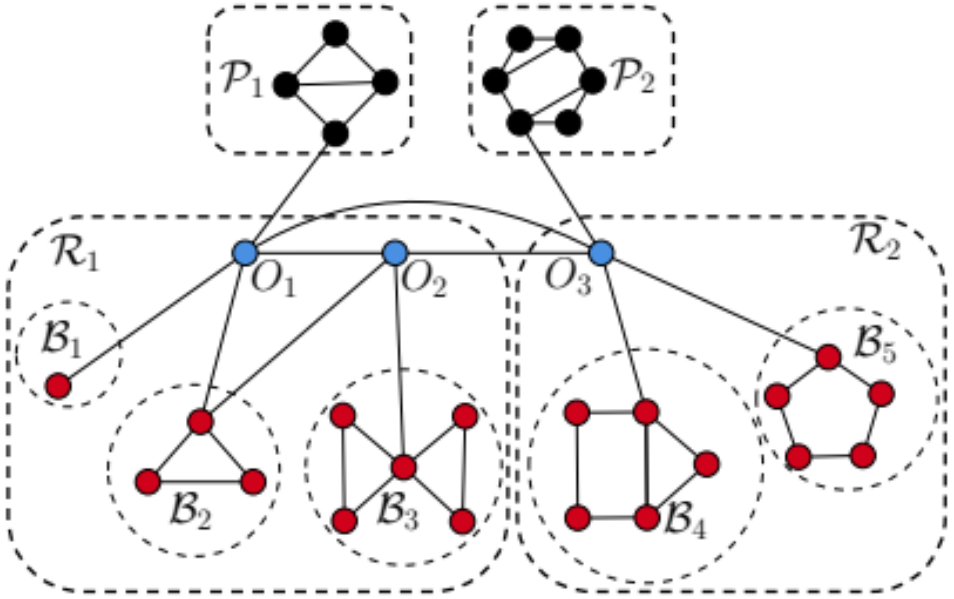
T. Gallai 1963,'64

PATHS, TREES, AND FLOWERS

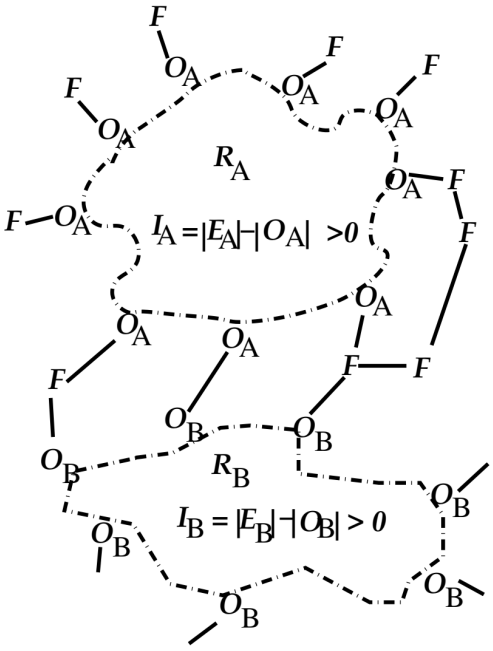
JACK EDMONDS

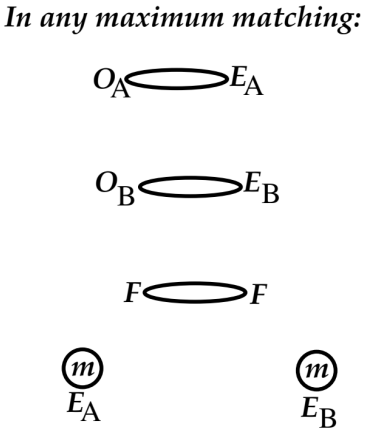
J. Edmonds, 1965

Our answer



KD, PRB 105 235118 (2022)





Bhola, Biswas, Islam, KD, PRX 2022

Key observation: o – o and o – u links are the "forbidden" links. Delete!

Gives us an alternate local proof of Longuet-Higgins & Lovasz results.

Number of monomers in an R-type region = Number of zero modes localized in same region.
(adding contributions from all regions gives older global statements of Longuet-Higgins and Lovasz)

Striking theorem on kagome lattice

Any odd parity connected cluster of the site-diluted kagome lattice is a single R-type region with exactly one monomer living in it, and any even parity connected cluster hosts a perfect matching.

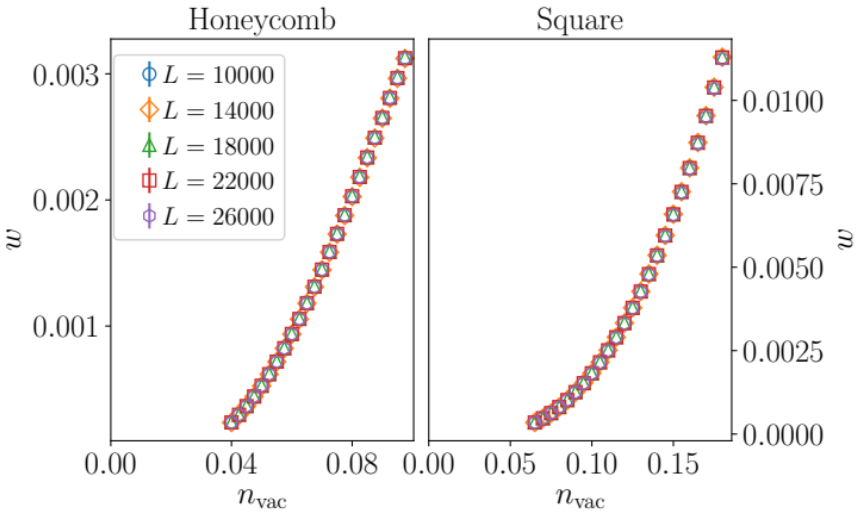
For the infinite connected cluster of the kagome lattice below its percolation transition, this implies:

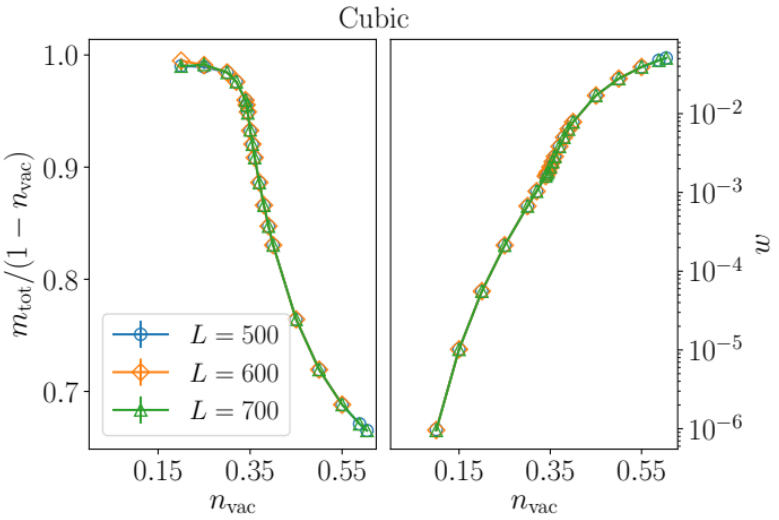
$$n_{\mathrm{monomer}} = w = 0$$

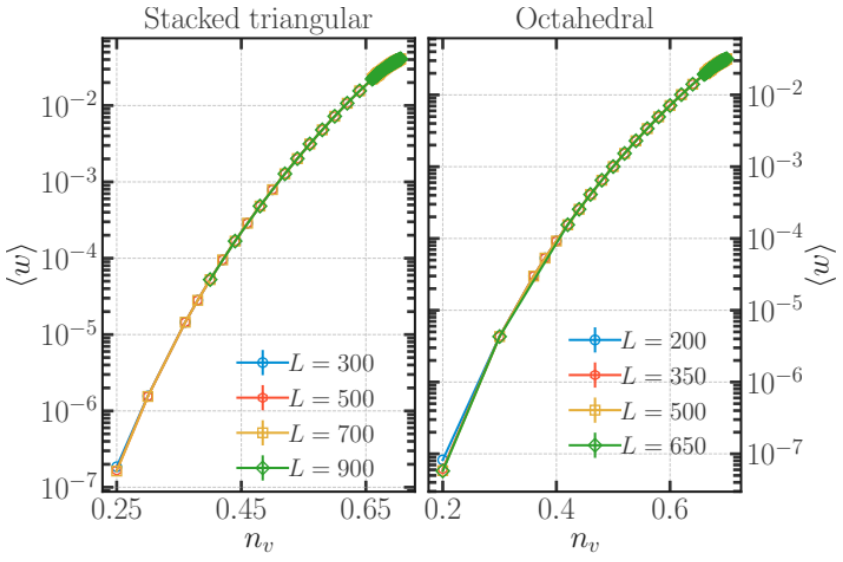
Result works on any "claw-free lattice" (pyrochlore lattice, star lattice etc)

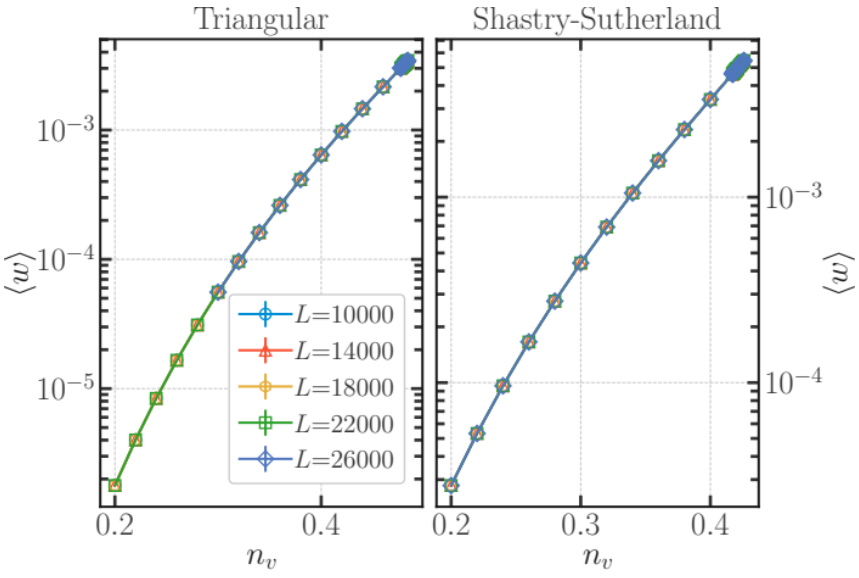
Sumner, Proc. AMS 42, 8 (1974)
Ansari, KD, PRL 132 226504 (2024)
Bhola, KD, arXiv:2512.23639 (2025)

All other diluted graphs generically have nonzero and smoothly varying w









Generic case amenable to tractable computations(!)

Can obtain complete set of R-type and P-type regions from one maximum matching of diluted lattice

via BFS for augmenting alternating paths (Edmonds' Blossoms in non-bipartite case)

Opens door to detailed computational study of random geometry of $\mathcal{R}$-type and $\mathcal{P}$-type regions

(..and thence, (hopefully) to deductions about the physics...)

Bhola, KD, arXiv:2311.05634v2 (2025)

KD, PRB 105 235118 (2022)

Bhola, Biswas, Islam, KD, PRX 2022

Enter: Percolation...

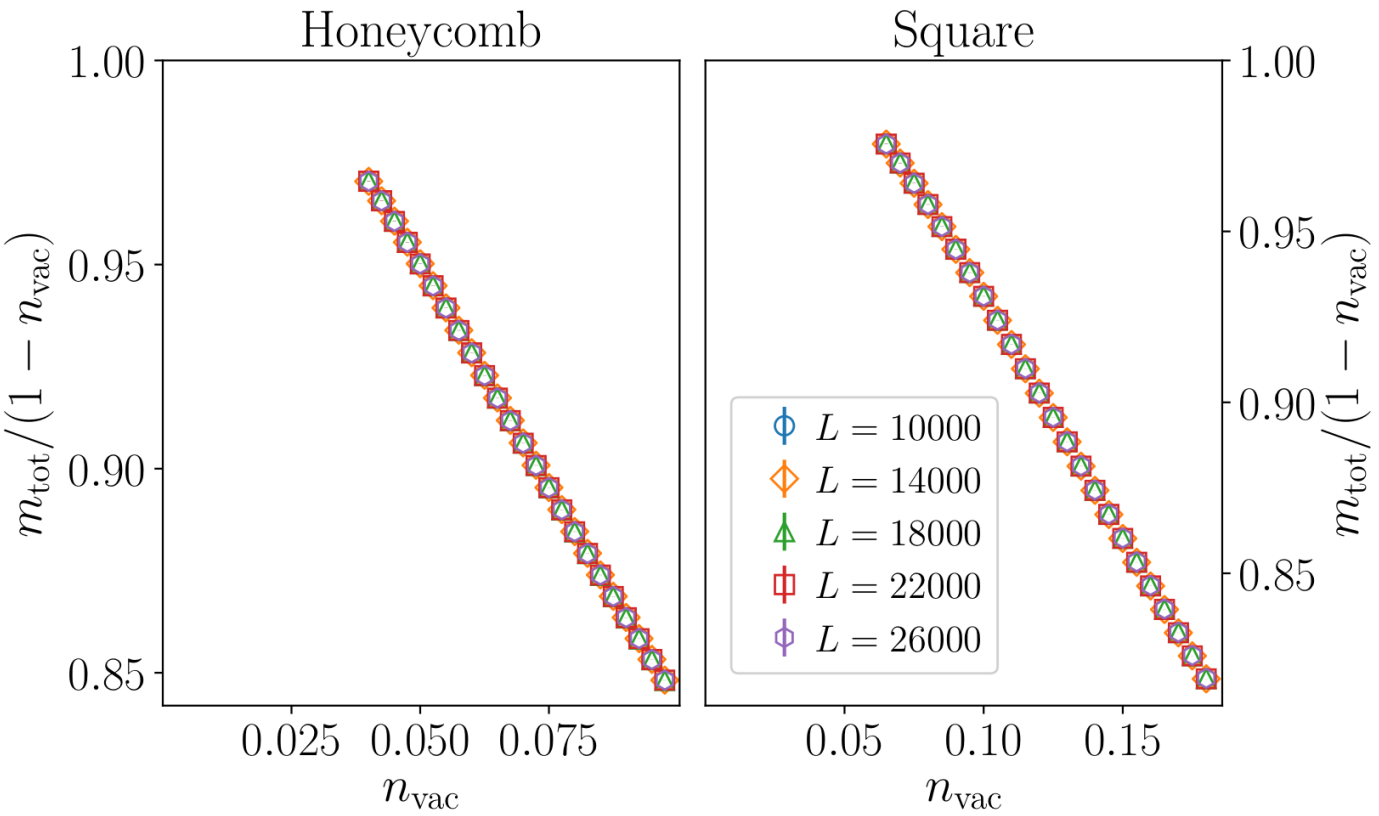
Typical regions are large at low dilution: Think in terms of percolation

The “right” yes/no question to ask: Can one walk from one end of a sample, staying within a single region?

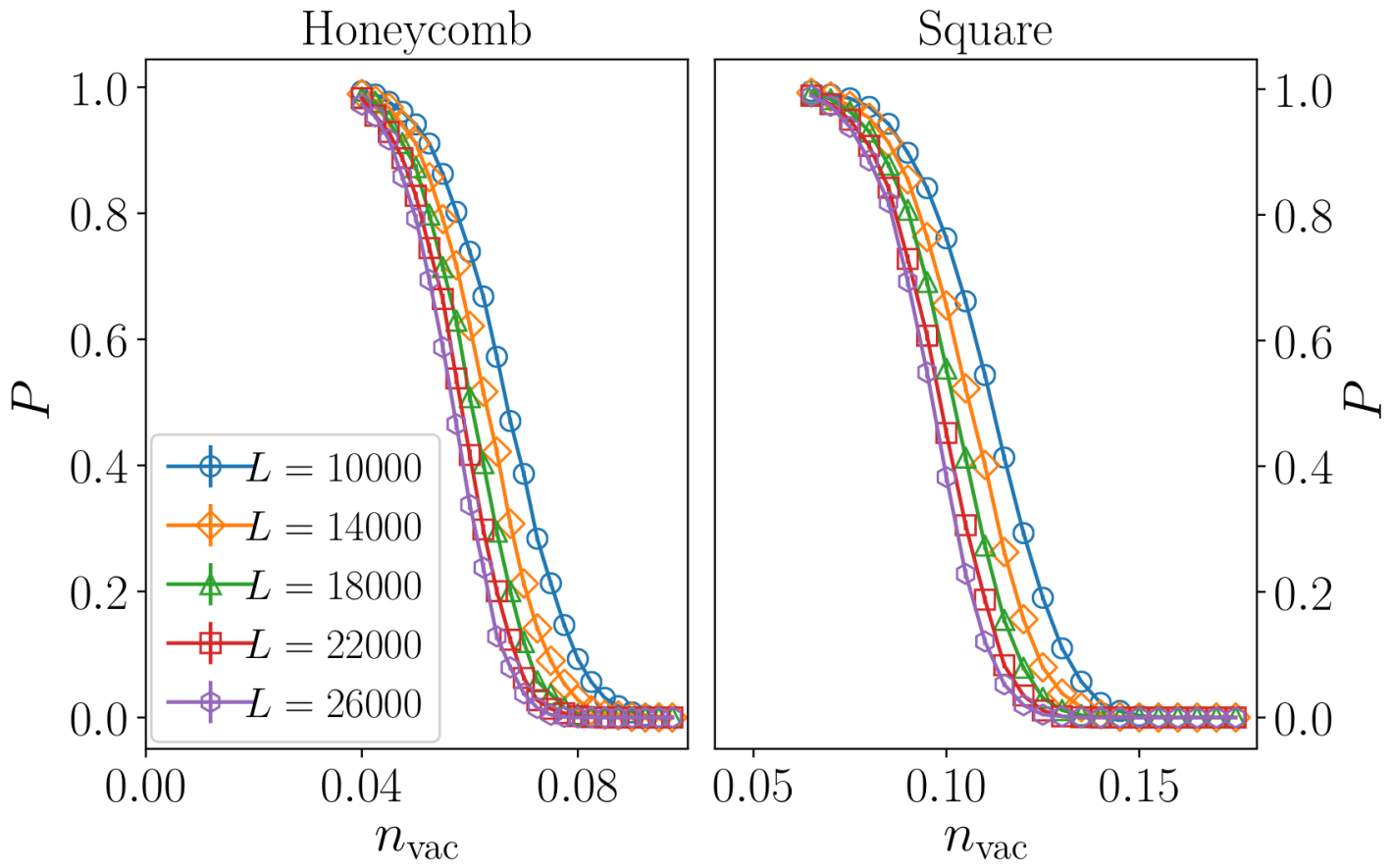
Bipartite case

Bhola, Biswas, Islam, KD, PRX 2022

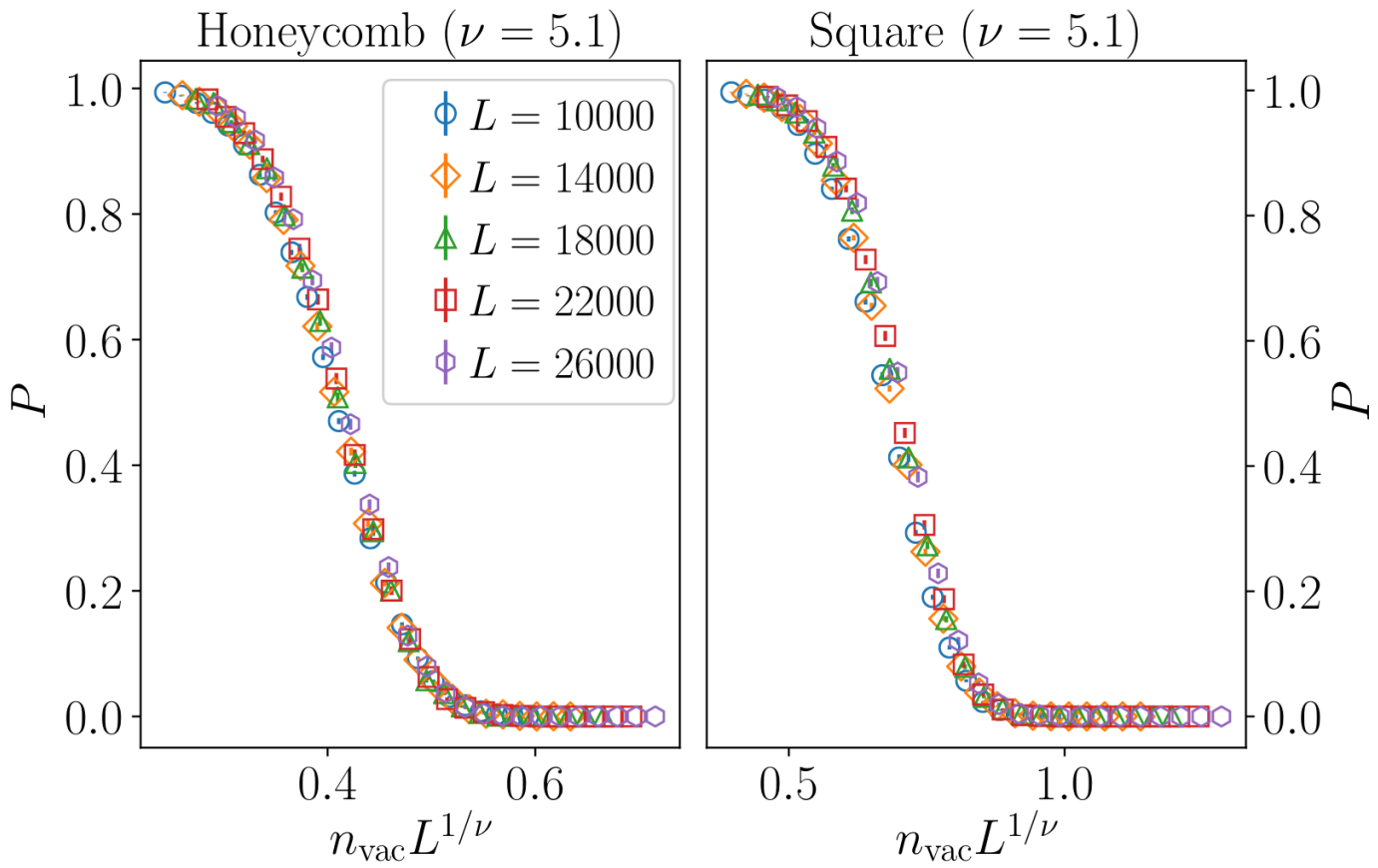
$\mathcal{R}$-type regions take over lattice



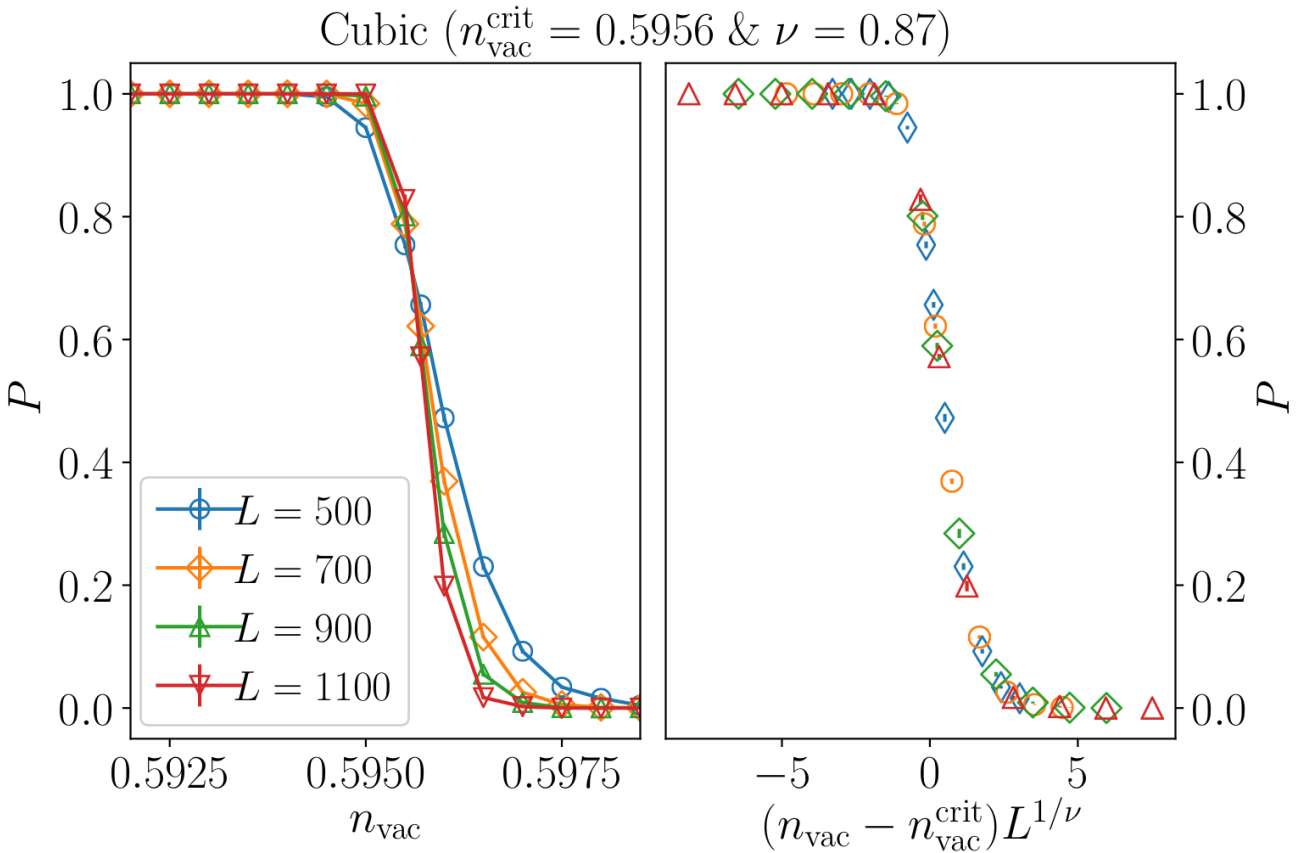
Incipient percolation at $n_v = 0$ (?)



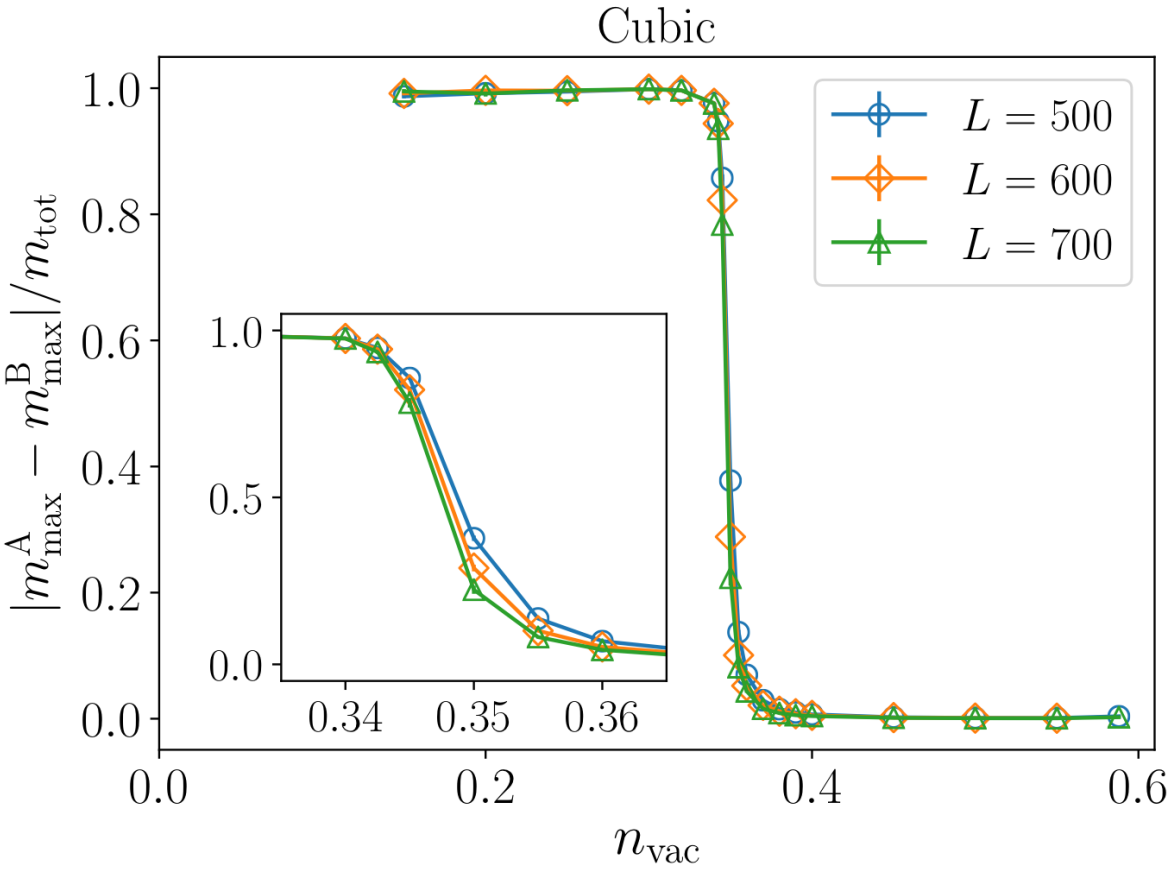
Universal scaling at $n_v = 0$ critical point



Percolation transition on cubic lattice



Spontaneous sublattice symmetry breaking deep inside percolated phase



Consequences for “quantum percolation” and Kitav systems...

No quantum percolation transitions on the square or honeycomb lattices at half-filling

(very large crossover length scale)

Quantum percolation transition on diluted cubic lattices understood in terms of monomer percolation

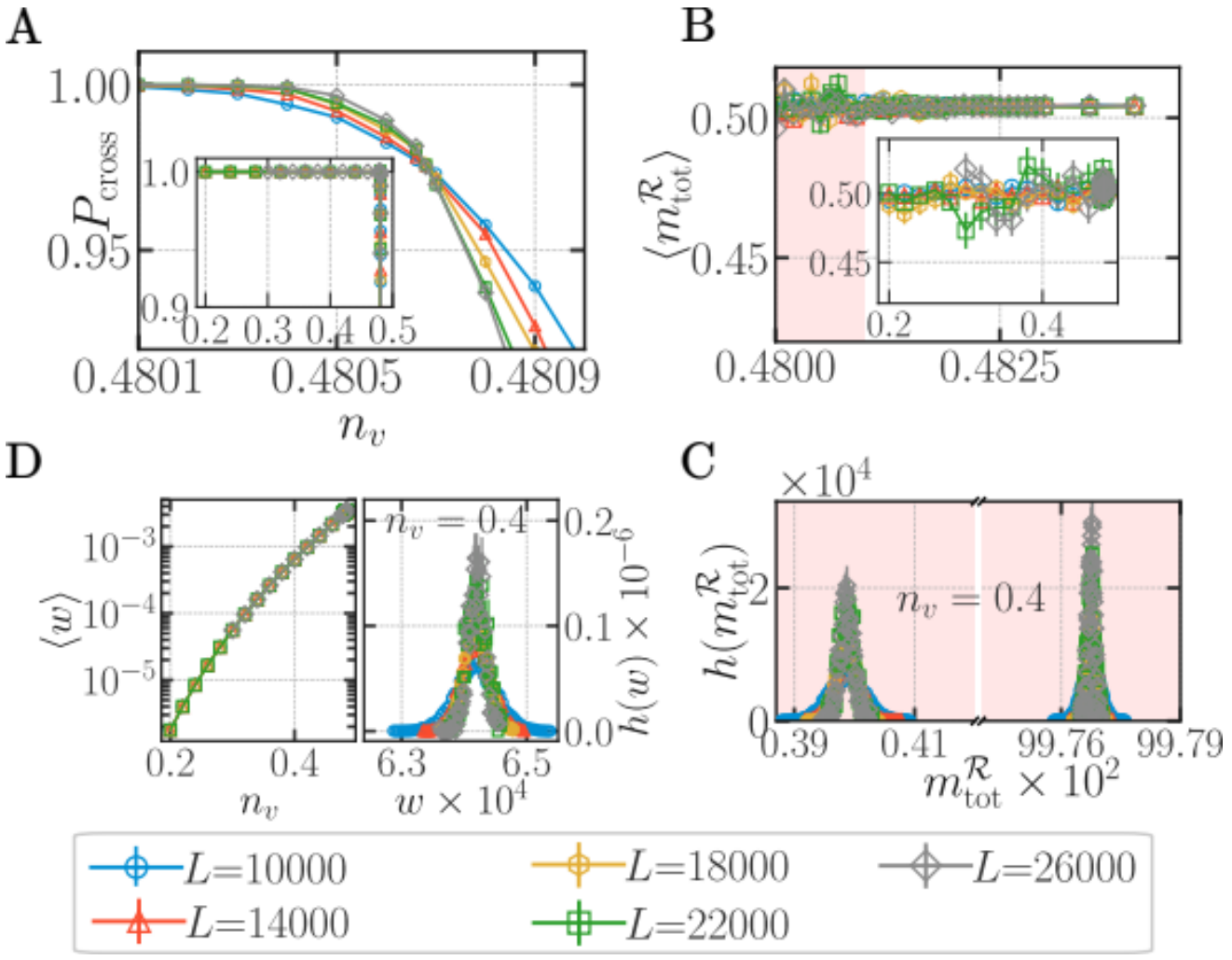
(With interactions: sublattice symmetry breaking transition could have implications for magnetic response)

Bhola, Biswas, Islam, KD, PRX 2022

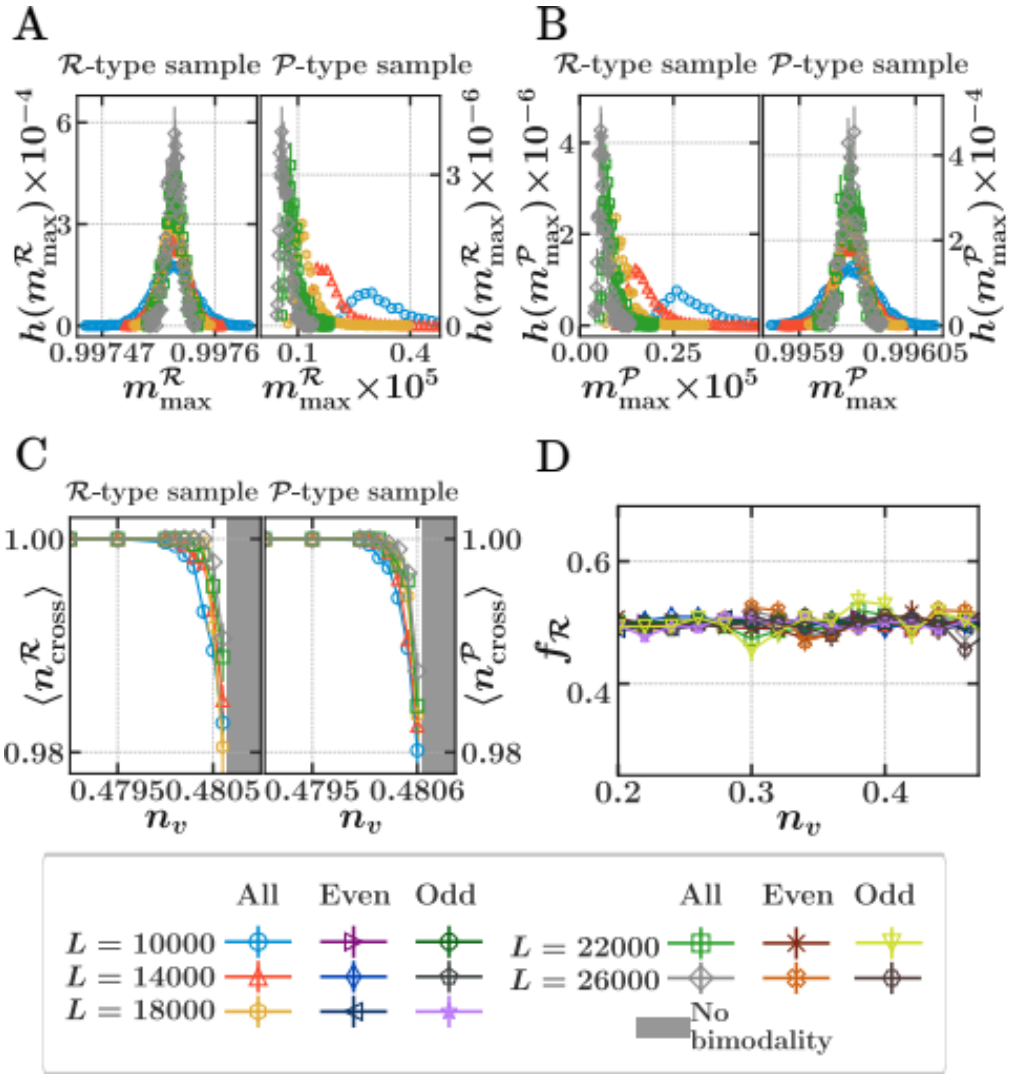
Nonbipartite case

Bhola, KD, arXiv:2311.05634v2 (2025)

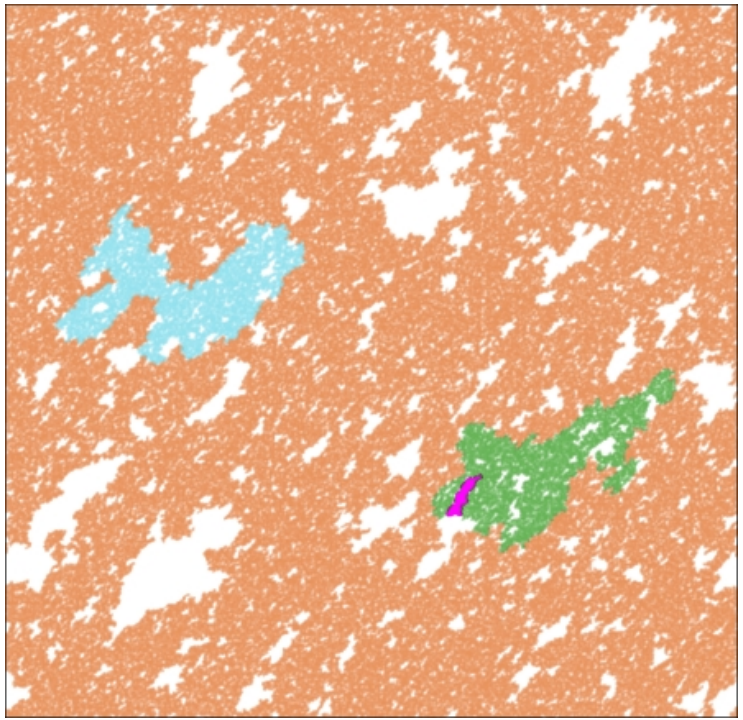
On the diluted triangular lattice



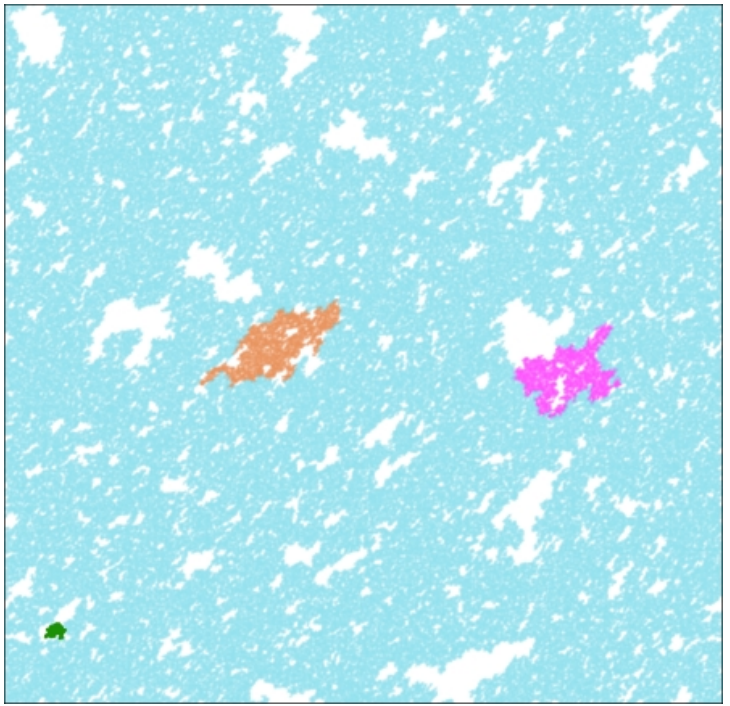
On the diluted triangular lattice



Pictorially on the diluted triangular lattice



R-type sample



P-type sample

On the diluted triangular lattice

$$f_{\mathcal{R}} = 1/2$$

Why???

Does this suggest some emergent symmetry between monomer-carrying and fully-packed regions

Again: Parity of largest geometric cluster plays no role!

On the diluted triangular lattice

Violation of even the weak form of "central dogma" at low vacancy concentration:

Monomers delocalized in half the samples, localized to O(1) regions in the other half!

All samples identically prepared, randomly diluted, with the exact same density of vacancies

On the diluted triangular lattice

Suggests extreme sensitivity of large-scale geometry to micro-scale details of disorder configuration

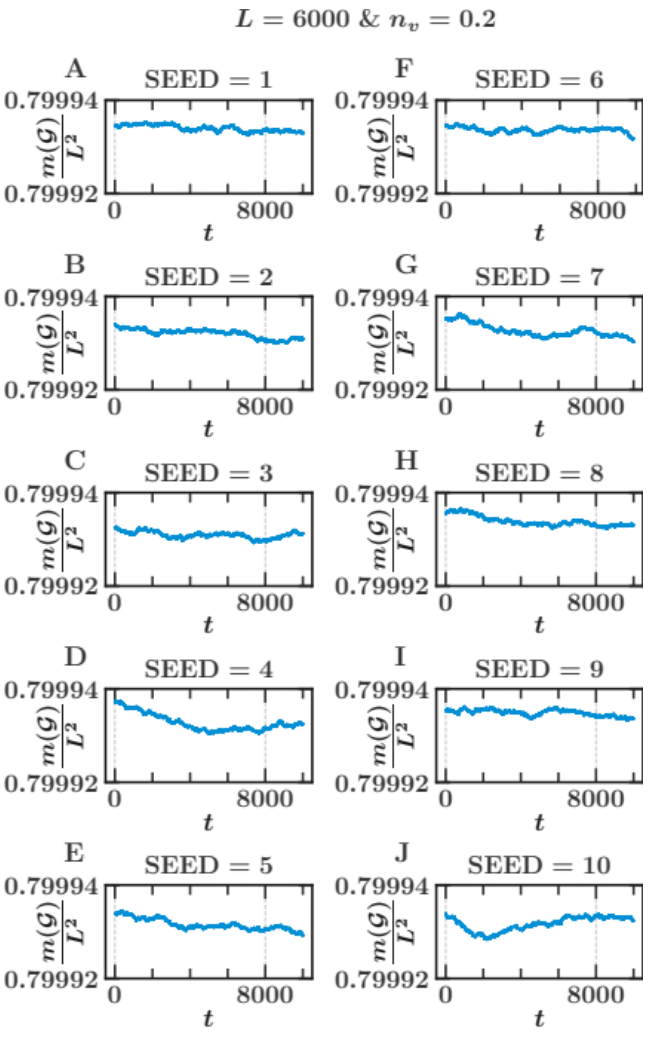
Can we quantify this?

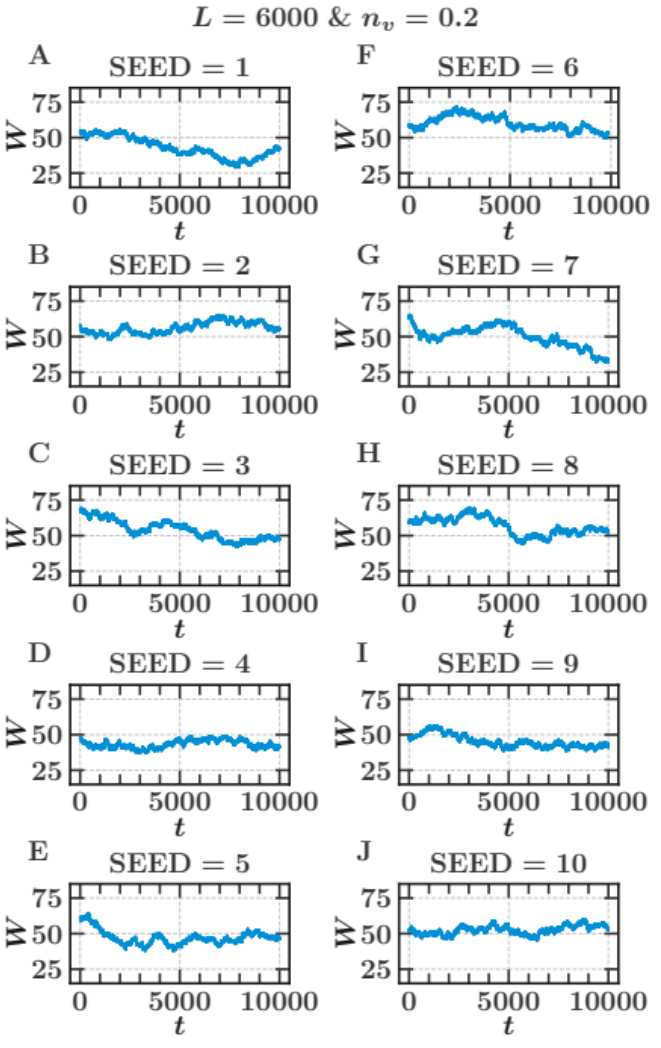
Model dynamics: Set vacancies in motion and watch what happens!

Small fraction of vacancies exchange position with neighboring surviving site at each time step

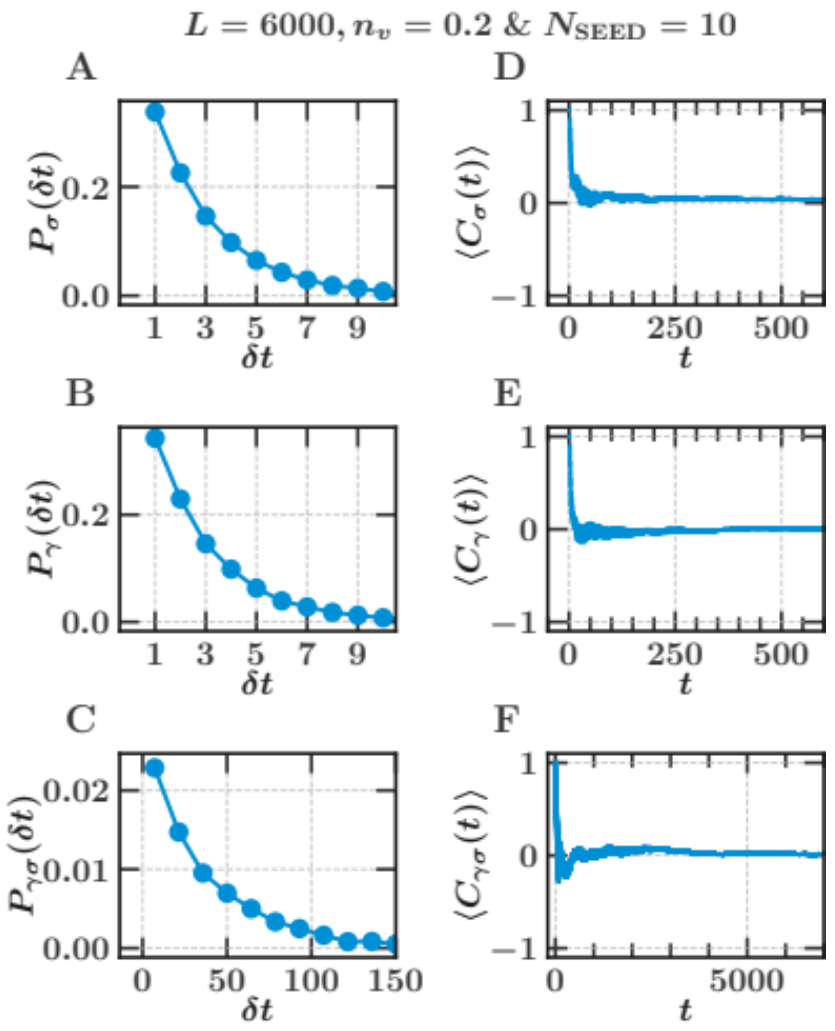
How does the large-scale geometry of these regions react?

Dynamics doesn't disturb underlying lattice much





Yet: Large-scale geometry of monomer-carrying/fully-packed regions responds chaotically



More results...

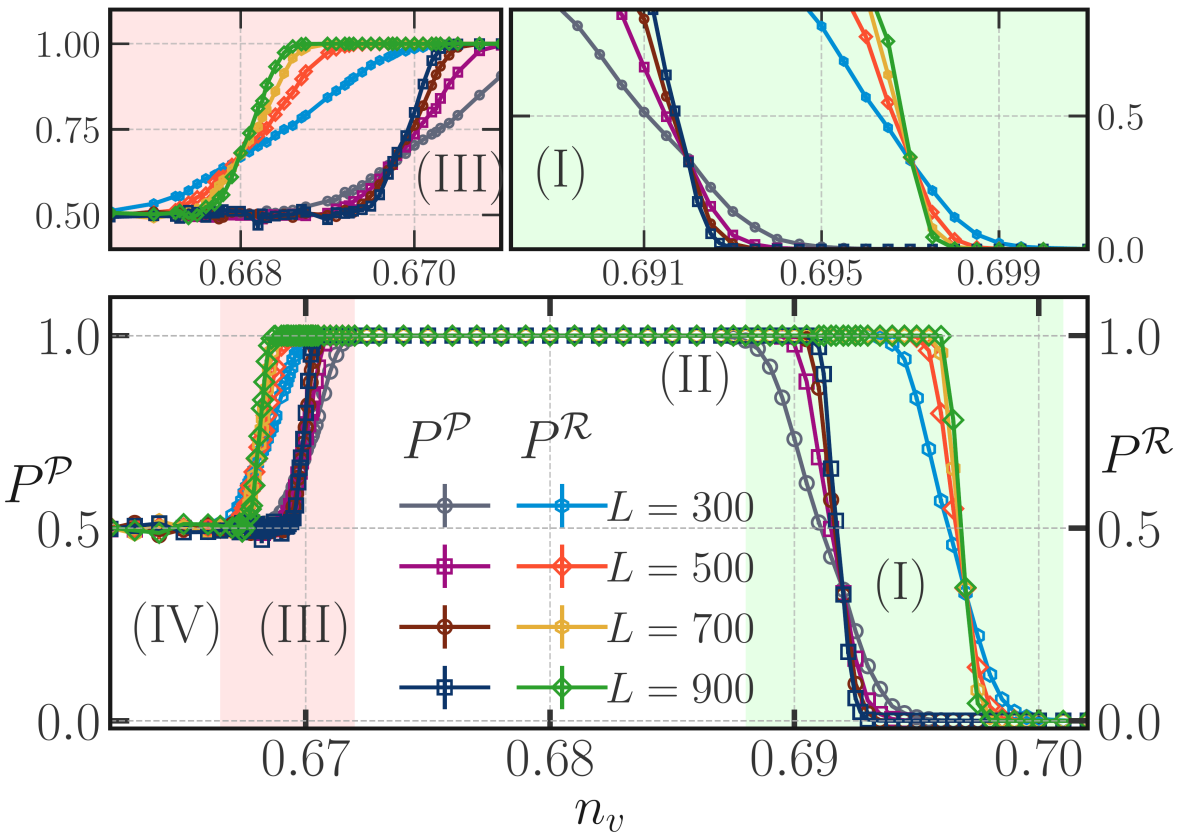
Similar phenomena on other non-bipartite lattices:

Checked: Percolation transition and low-dilution phase essentially the same on the Shastry-Sutherland lattice

A totally baroque phase diagram in three dimensions

Interesting from the vantage point of percolation theory

Phase diagram in 3d via wrapping probabilities



Bhola, KD, arXiv:2311.05634v2 (2025)

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