Percolation in maximum-density dimer packings: implications for transport & magnetism

Kedar Damle (TIFR-Mumbai) @ Harvard (January 2026)

unpublished: Bhola, KD, arXiv:2311.05634v2 (2025) Bhola, KD, arXiv:2512.23639 Ansari, Kundu, KD (in preparation)

recent: Ansari, KD, PRL 132 226504 (2024)
Bhola, Biswas, Islam, KD, PRX 12, 021058 (2022)
KD, PRB 105 235118 (2022)

background: Sanyal, KD, Motrunich, PRL 117 116806 (2016)

Disorder

Quenched disorder: Missing atoms, adatoms, lattice imperfections...

Static on the time-scale of experiment.

Quite common in condensed matter systems.

Strong disorder: new phases of matter (e.g. spin glasses, Anderson insulator, MBL...)

Weak disorder: probe correlations of underlying state (e.g. spin textures in frustrated magnets)

“Central dogma”

In large-size limit -

Strong version:

Self-averaging of properties: Sample-to-sample fluctuations small (average = typical)

Violations exist – e.g. Disordered quantum spin chains (infinite-disorder fixed points)

Weak version:

At a minimum, two samples prepared using some protocol must be in same phase.

Violations? May exist in infinite-range spin glass models (?)

Percolation

Sharp threshold behavior (in the limit of $L \to \infty$) of the end-to-end connectivity of a random medium as a function of the density n_v of blocked pathways or vacancies.

Simplest and entirely *geometric* example of a phase transition

Mathematical model: Regular lattice with random site or bond dilution n_v ("Bernoulli percolation")
(Broadbent and Hammersley Proc. Cam. Phil. Soc. 53, 629, 1957)

Boolean variable: Answer to YES/NO connectivity question.

$P(L,n_v)$: the probability of answering YES

Universal scaling at Bernoulli percolation transition

Scaling in the vicinity of the threshold: $P(L, n_v) = f((n_v - n_v^{crit})L^{1/\nu})$

$f(x) \to 1$ as $x \to -\infty$ $\quad$ $f(x) \to 0$ as $x \to +\infty$

Critical point exhibits scale invariant behavior

$n_v < n_v^{crit}$: diluted lattice has single infinite cluster with probability 1 (in limit $L \to \infty$)

In $L \to \infty$ limit: $P(L,n_v)$ can only be 0 or 1 at any dilution.

There is only one percolated phase and one unpercolated phase
(no reentrant transitions)

Our basic message

Maximum-density dimer packings of weakly-diluted lattices have unusual percolation transitions

Some of the percolated phases show violations of (even the weak form of the) “central dogma”

Root cause: Kinematic constraints that induce long-range correlations

Consequences for transport (and magnetism):

In pinned vortex lattice state of p+ip superconductors & short-range RVB spin liquids on triangular lattice

(vacancy disorder = missing vortices in former, nonmagnetic impurities in latter)

At a minimum: *Strong violations of thermodynamic self-averaging in the thermal conductivity/susceptibility*

Likely for p+ip SC:

"R-type samples" have high thermal conductance *but not "P-type" samples*

Likely for RVB liquid:

"R-type samples" have vacancy-induced spin-glass order but not "P-type" samples

In both cases: Chaotic (deterministic but unpredictable) response to changes in disorder configuration.

Background: vacancy disorder in particle-hole symmetric systems (with random hopping)

Original motivation - A kind of side story (going back to ‘91) in the localization saga:

about form of density $\rho(\epsilon)$ of eigenstates as $\epsilon \to 0$ in bipartite random hopping problems in two dimensions

Without vacancies: Renormalization group predicts a certain limiting singular behavior
(Gade ‘91, Motrunich, KD, Huse ‘02, Mudry Ryu Furusaki ‘03)

But: some claims that $n_v > 0$ changes asymptotics
(Ostrovsky et. al. & Hafner et al ’14)

Intriguing, since symmetries are unchanged...

Example: Tight-binding model for electrons in undoped graphene with vacancy defects

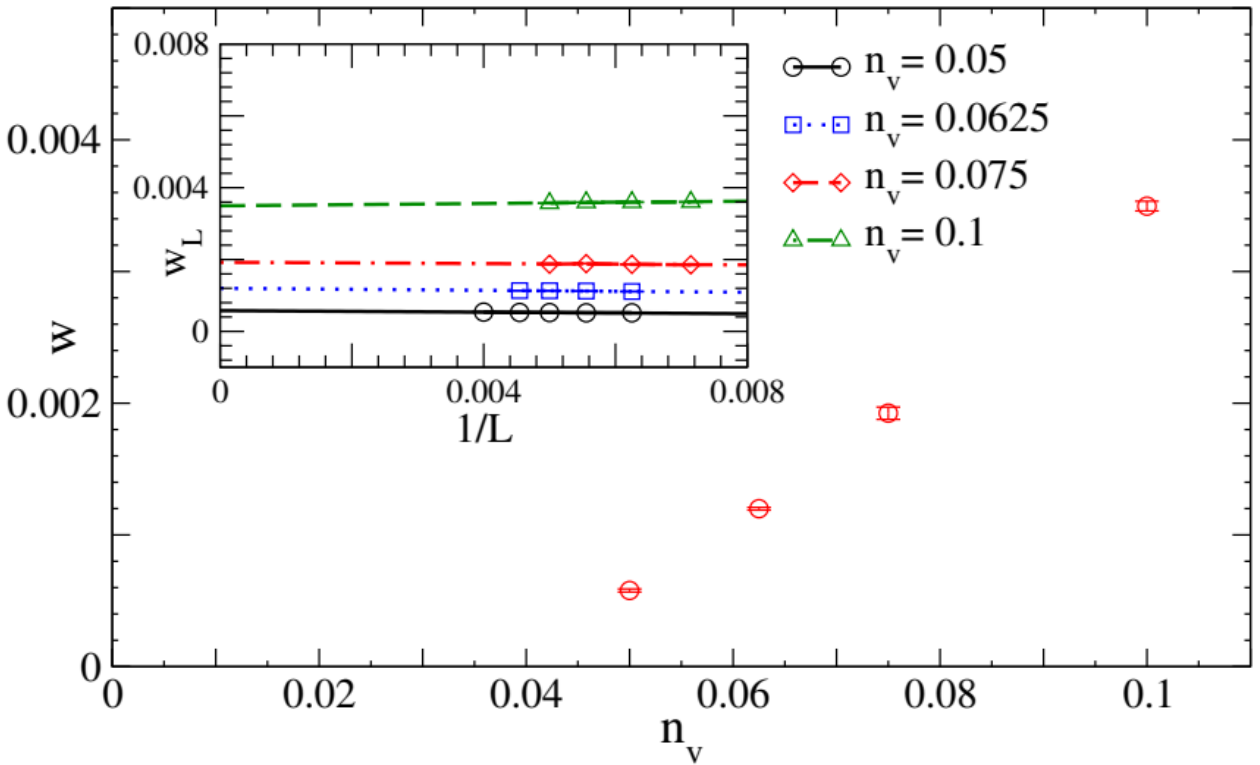
$$H = -\sum_{\langle rr' \rangle} \left(t_{rr'} |r\rangle\langle r'| + t^*_{rr'} |r'\rangle\langle r| \right)$$

r, r' Denotes sites of *diluted* honeycomb lattice (after random deletion of fraction n_v of vertices)

$\langle rr' \rangle$ Denotes *surviving* nearest-neighbor links (edges of diluted graph)

$t_{\langle rr' \rangle}$ Hopping amplitudes can vary from link to link

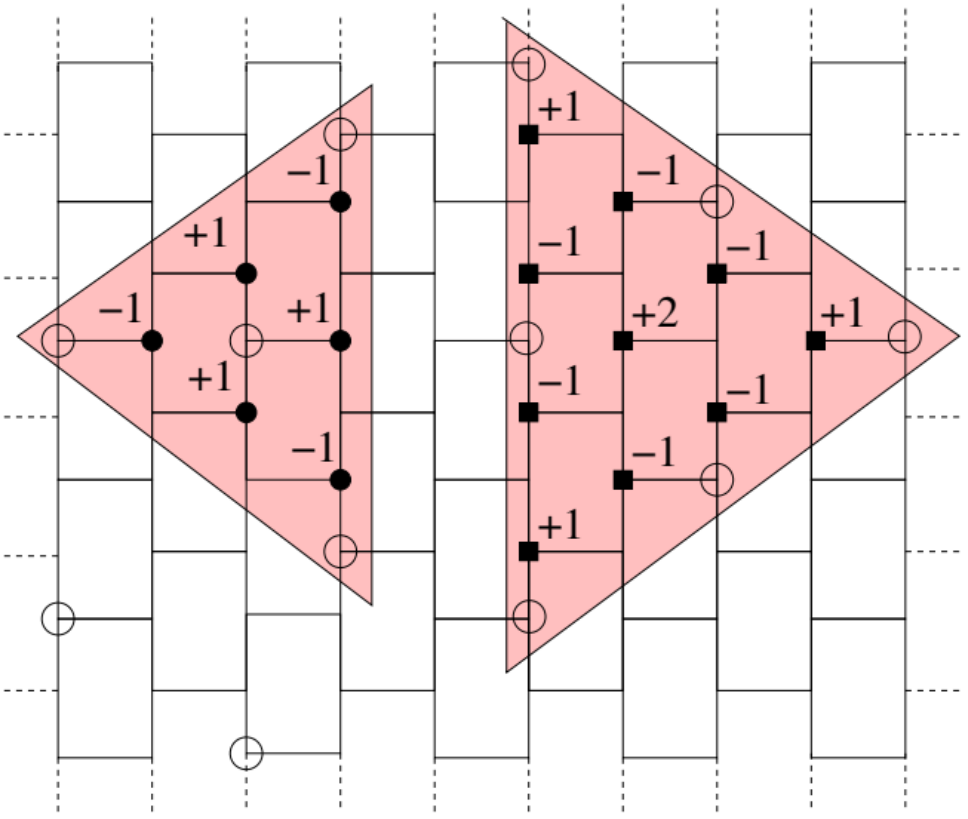
“Surprising” nonzero *density* w of ϵ = 0 states on diluted honeycomb lattice



Aside: Turns out crossover in density of states controlled by density of these zero modes...

Sanyal, KD, Motrunich PRL 2016

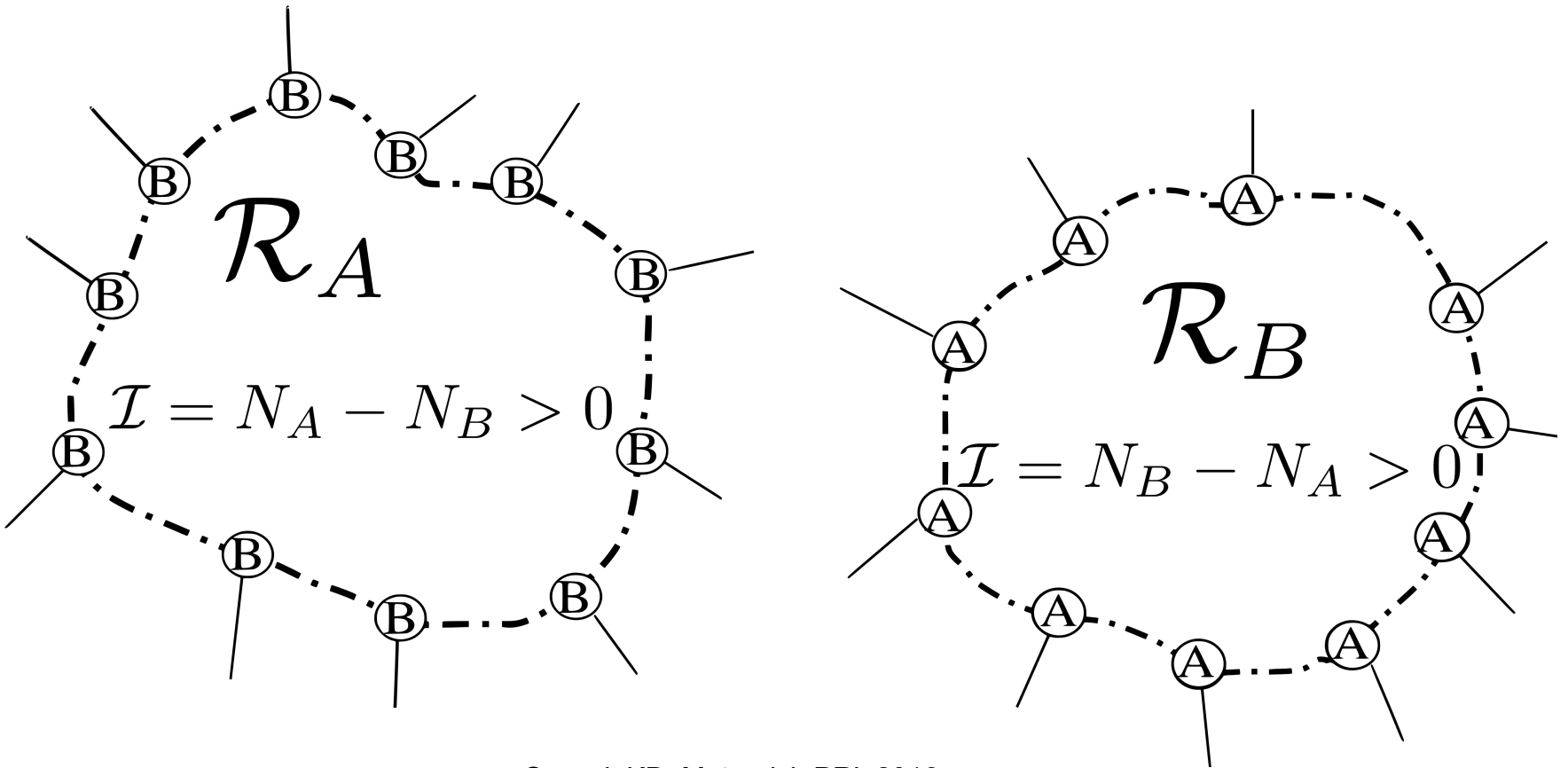
Origins: naive guesses and an important distinction



Sanyal, KD, Motrunich PRL 2016

Key distinction: On the right - *topologically protected zero mode wavefunction.*
On the left - only yields zero mode for uniform t

Conjecture: *Rare* “R-type regions”:



Sanyal, KD, Motrunich PRL 2016

Key unanswered question

Hand-drawn examples provide lower bounds on DOS at E=0.

But computed value much greater.

What actually determines density of zero modes??

Note: zero modes are not a floating point precision artifact...

Also confirmed by Weik et. al. (2016)

Local sublattice imbalance and dimers: A first clue

Such zero modes only depend on connectivity, not hopping strengths.

R-type regions rely on local imbalance between A and B type site densities.

Suggests thinking in terms of *matchings* a.k.a *dimer covers*

Regions of lattice that cannot be covered perfectly by dimers host wavefunctions

Bhola, Biswas, Islam, KD, PRX 12, 021058 (2022)

Language primer: Fully-packed dimers (perfect matchings)

Fully-packed hard-core dimer models in stat-mech: Match **each** site to an adjacent site monogamously

In graph theory/computer science: The perfect matching problem

Easy to see (for regular lattices like square, triangular, honeycomb, kagome...):

Extensive entropy of fully-packed dimer covers (perfect matchings)

(exact computation of entropy on planar graphs: Classic papers by Kasteleyn & Fisher)

Maximum matchings of disordered lattices

Basic question: Can a diluted lattice with even number of vertices be perfectly matched?

If bipartite, need |A| = |B|

But: generally not possible *even with* |A|=|B|

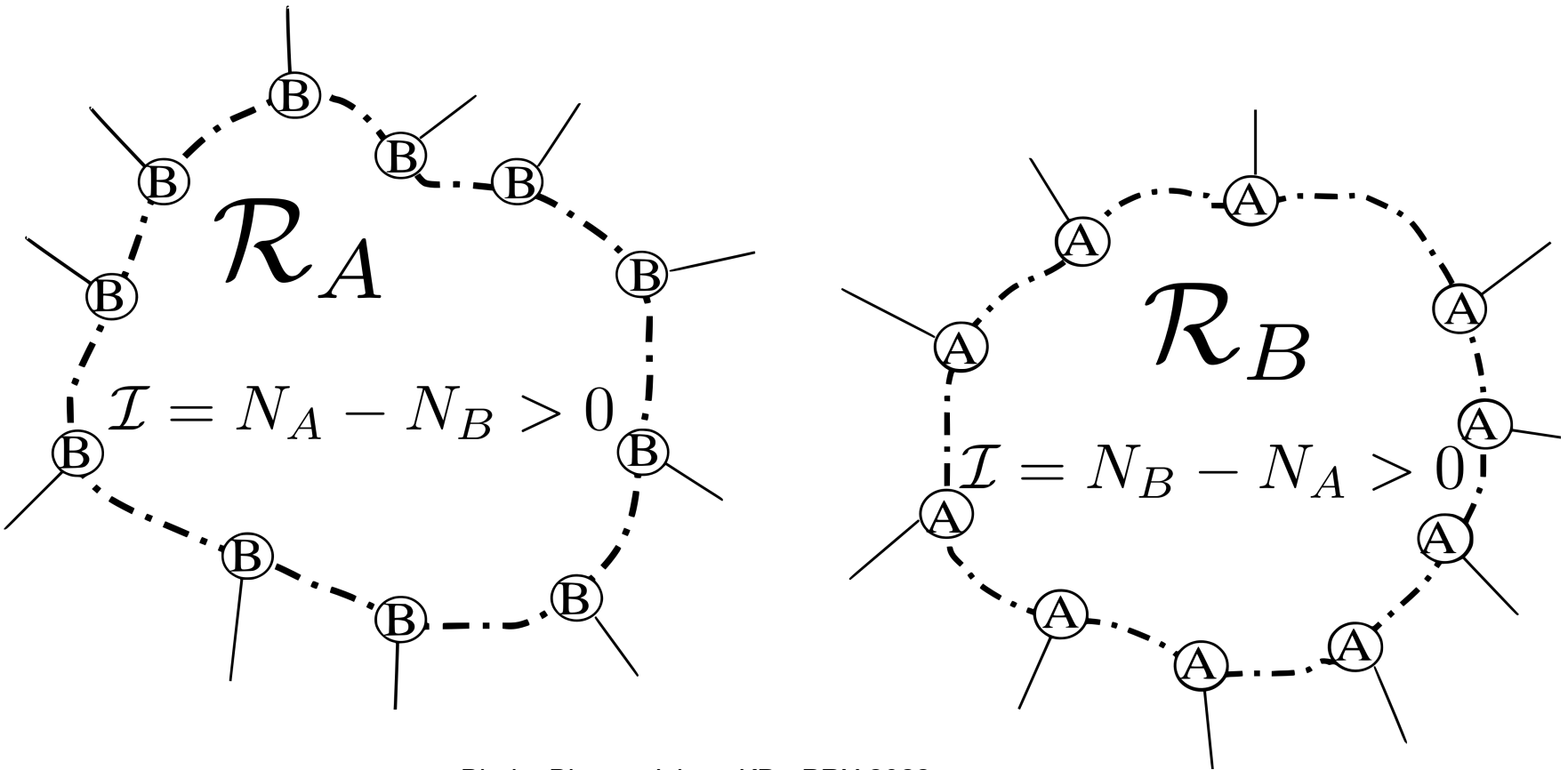
Then have *maximum matchings* but not *perfect matchings*

Maximum matchings have unmatched sites that host *monomers*

Generically: nonzero density of vacancies implies nonzero density of monomers

Bhola, Biswas, Islam, KD, PRX 12, 021058 (2022)

Key observation: “R-type” regions trap monomers



Bhola, Biswas, Islam, KD, PRX 2022

Gels with: Longuet-Higgins on zero modes

AIP | The Journal of Chemical Physics

Some Studies in Molecular Orbital Theory I. Resonance Structures and Molecular Orbitals in Unsaturated Hydrocarbons

1950

H. C. Longuet-Higgins

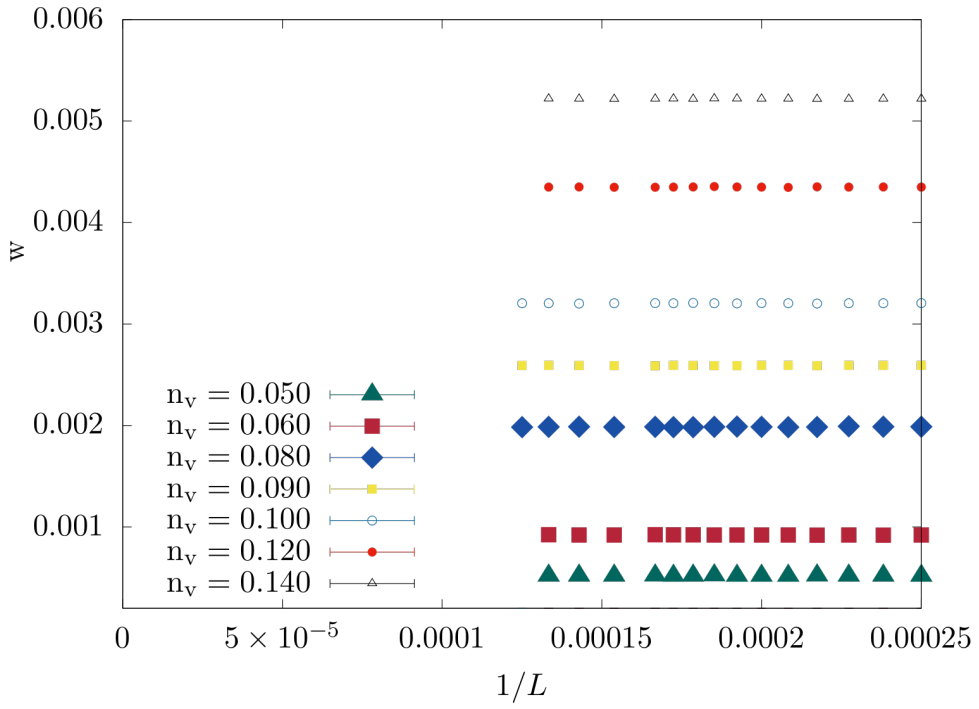
E=0 molecular orbitals correspond to magnetic moments in MO theory of benzenoid molecules

Effectively studying a tight-binding model and asking about E=0 states.

Result: (transcribed to our language)

Total number of monomers in maximum matching = Total number of topolgically protected zero modes (global statement)

A first step: Check convergence to thermodynamic limit



But: we want local information (for transport etc)...

So “where” do the modes “live”?

How does one find a complete set of R-type regions?

What does this question even mean in algebraic terms??

One answer:

Identify a “maximally-localized” basis for the topologically-protected part of zero-energy subspace

Digression:
Topologically protected collective Majorana modes of Majorana tight-binding models?

$$\mathcal{H}_{\text{network}} = \frac{i}{4} \sum_{rr'} \mathcal{A}_{rr'} \eta_r \eta_{r'}$$

$\mathcal{A}_{rr'}$ Antisymmetric matrix of quantum mechanical mixing amplitudes

η_r Majorana operators corresponding to localized Majorana modes

Zeroth order descriptions of:

Mixing of Majorana modes in vortex lattice state of topological p+ip superconductors

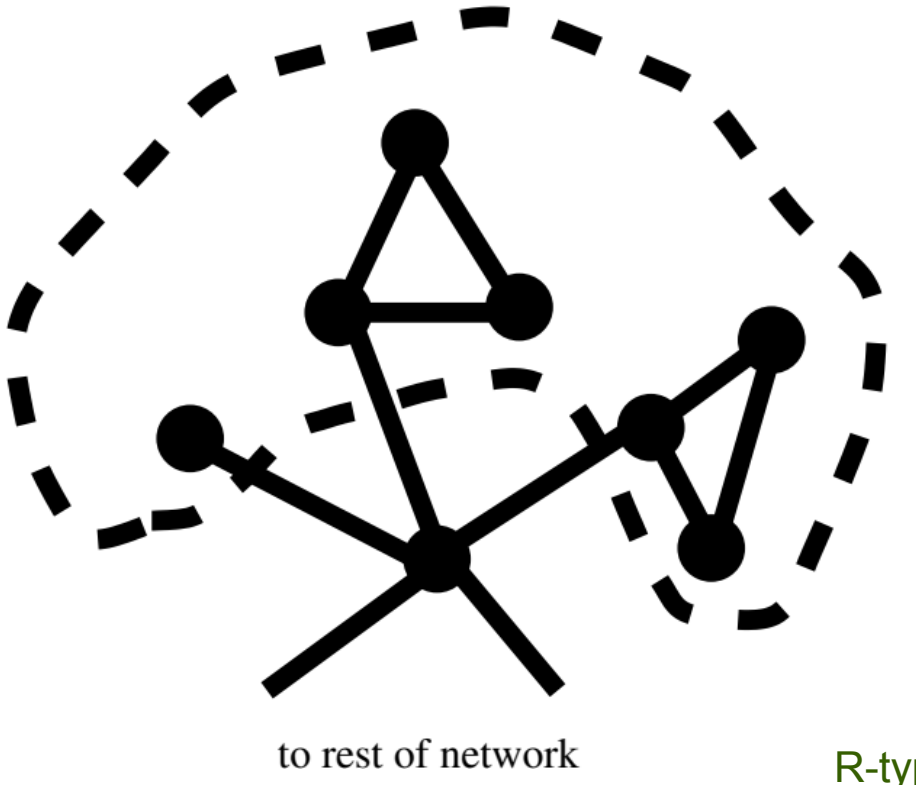
Majorana excitations of Majorana spin liquid states

Can such networks have topologically protected collective Majorana modes?

Linear algebra question : Classify/construct topologically protected null vectors of $iA_{rr'}$

Note: Bipartite random hopping special case of this

R-type regions hosting collective Majorana modes



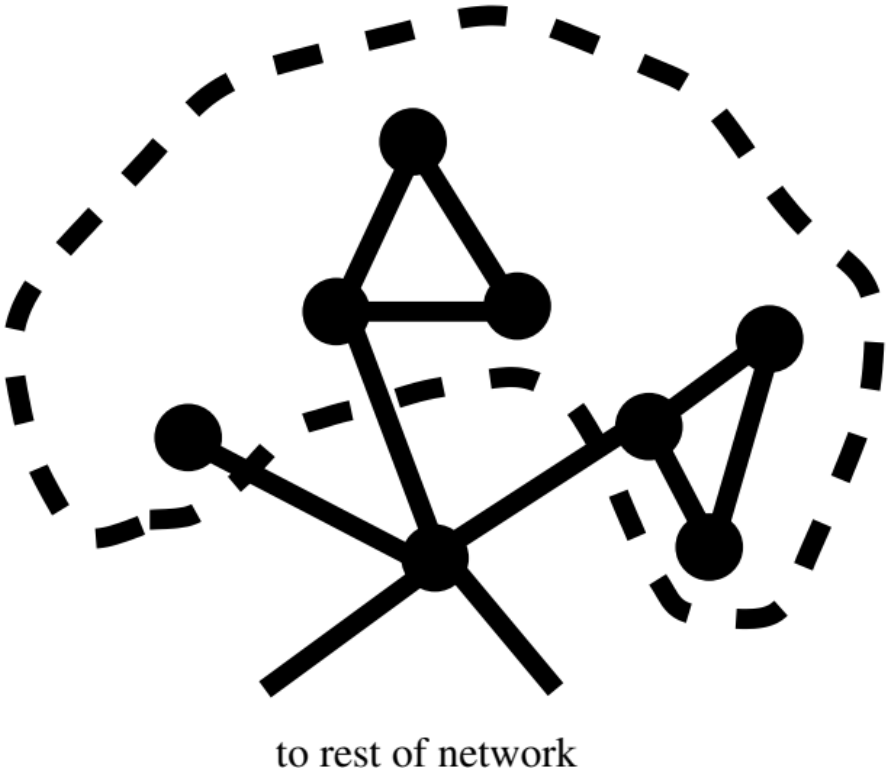
Odd cyles in isolation have a zero mode.

Generically: such modes mix and are destroyed

R-type regions host linear combinations that survive mixing

KD, PRB 2022

Key observation: R-type regions trap monomers



The same R-type region also traps two monomers

KD, PRB 2022

Gels with theorem of Lovasz

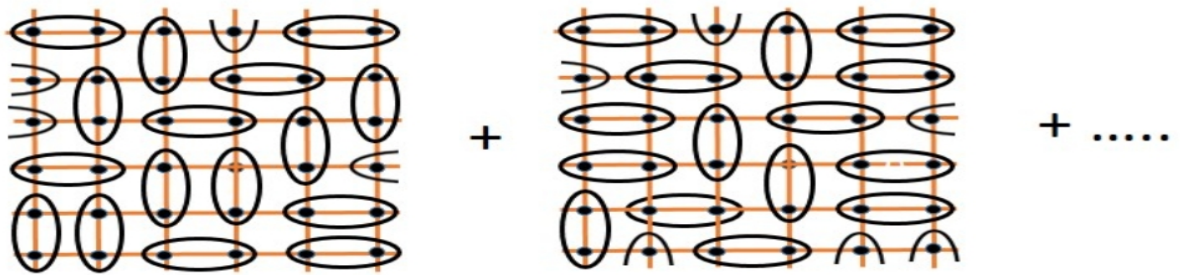
ON DETERMINANTS, MATCHINGS, AND RANDOM ALGORITHMS

by L. Lovász*

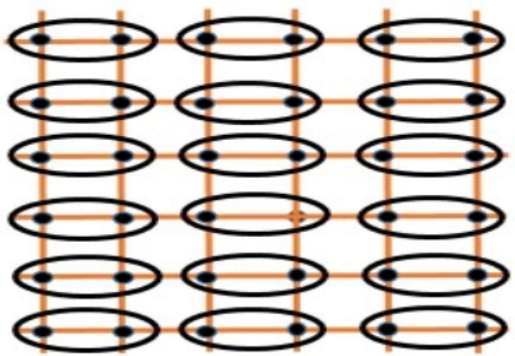
Fund. Comp. Th. 1979

Total monomer number = Total number of topologically protected zero modes of $i\mathcal{A}_{rr'}$

Digression: non-magnetic impurities in VBS and sRVB states



Short-range resonating valence bond (sRVB) spin liquid (gapped)



Valence bond solid (VBS)

(with spontaneous lattice symmetry breaking)

Quantum dimer model framework for RVB/VBS states

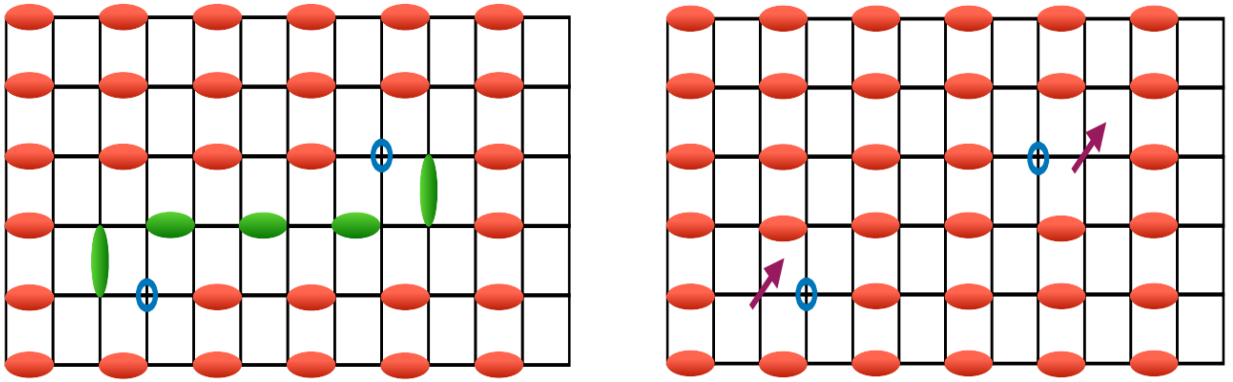
Rokhsar and Kivelson: Effective Hamiltonian living in subspace of singlets spanned by nn VB

$$H_{QDM} = -t(\,|\,=\rangle\langle\,=\,|\,+\,|\,=\,\rangle\langle\,|\,=\,|\,) + \ldots$$

More generally: Ring-exchange kinetic terms on “flippable” plaquettes, and local interactions

effect of matrix elements to further-neighbor singlet states captured in additional terms (“overlap expansion”)

Single-impurity effect in VBS state



Each vacancy, even if isolated from other vacancies, seeds a local moment in a VBS state

Signature: Large intermediate temperature range with Curie tail in susceptibility

Quenched below scale set by residual interactions

$$\chi_{\rm imp} \sim \frac{\mathcal{C}}{T} \text{ for } J_{\rm eff} \ll T \ll J$$
$$\mathcal{C} \propto n_v$$

Ansari, KD, PRL 132 226504 (2024)

Vacancy disorder in sRVB spin liquid states: QDM framework

If disordered lattice has maximum matchings but no perfect matchings (fully-packed dimer covers):

Nonzero monomer density corresponds to density of vacancy-induced local moments

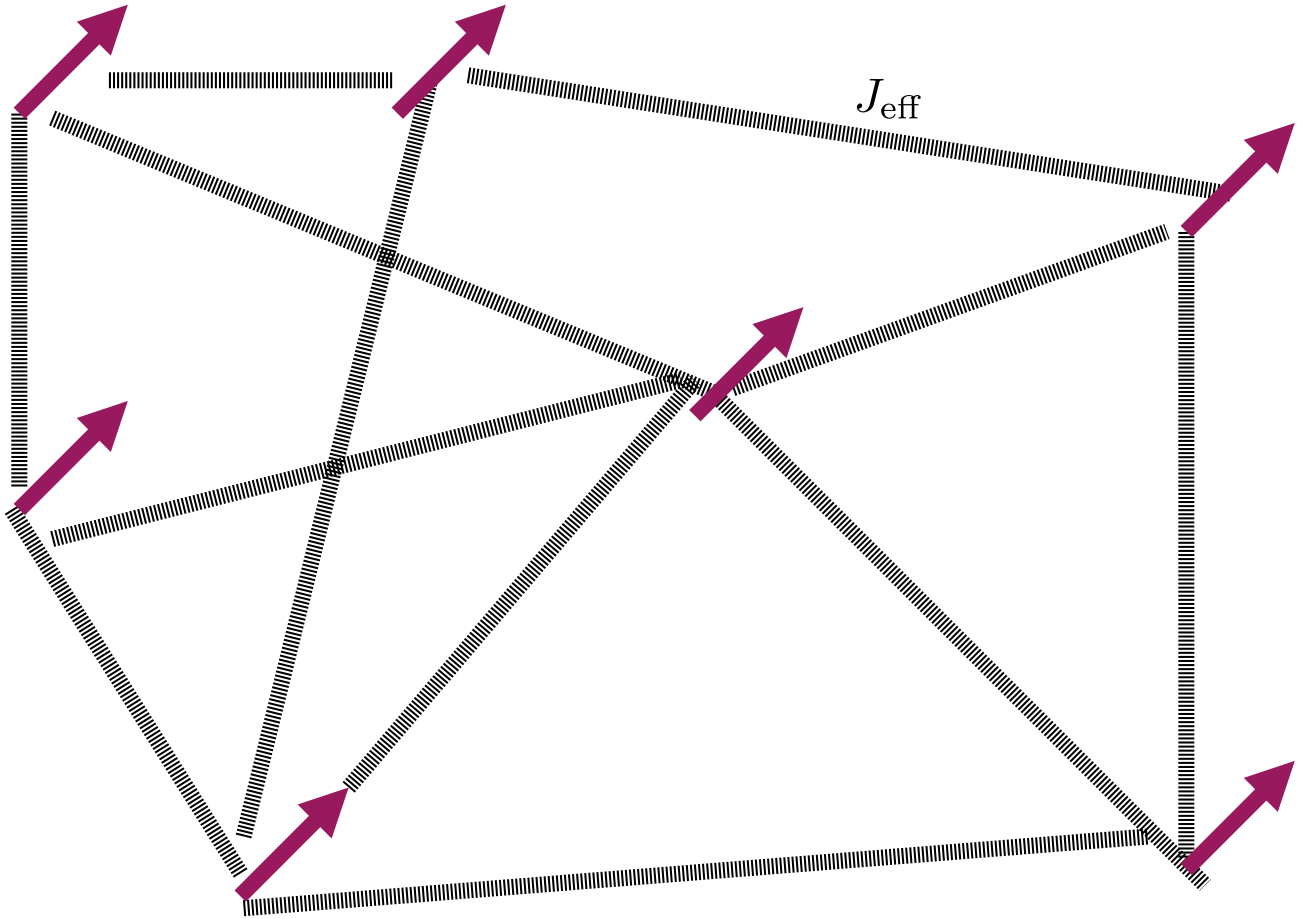
("emergent", at intermediate scale...)

Generally, nonzero vacancy density implies nonzero density of monomers

(effect of irregularity and local structure, true on lattices like triangular, square, honeycomb...)

$$\chi_{\rm imp} \sim \frac{\mathcal{C}}{T} \text{ for } J_{\rm eff} \ll T \ll J$$
$$\mathcal{C} \propto n_{\rm monomer}$$

Summary: Distinct vacancy-induced local moment instabilities of RVB and VBS states



In RVB case, only if

$$n_{\text{monomer}} \neq 0$$

$$(\mathcal{C} \propto n_{\text{monomer}})$$

In VBS case, even when

$$n_{\text{monomer}} = 0$$

$$(\mathcal{C} \propto n_v)$$

Ansari, KD, PRL 132 226504 (2024)

Second thoughts?

This conclusion seems to rely too much on having only nearest-neighbor singlets?

Does it really hold for generic short-range RVB liquid states?

To answer: large-N route to quantum dimer model

$$
\begin{aligned}
H &= J \sum_{\langle rr' \rangle} \vec{S}_r \cdot \vec{S}_{r'} + \dots \\
&= -J \sum_{\langle rr' \rangle} \left(\mathcal{P}_{rr'} - \frac{1}{4} \right) + \dots
\end{aligned}
$$

Enlarge symmetry group:

$$
H = -\frac{J_m}{N} \sum_{\langle r_1 r_2 \rangle} \sum_{\alpha,\beta=1}^{N} |\alpha\rangle_{r_1} |\alpha\rangle_{r_2} \langle\beta|_{r_1} \langle\beta|_{r_2} + \cdots ,
$$

Affleck, Read, Sachdev, Auerbach, Penc, Mila, Coleman, Sandvik, Alet, Kawashima, Beach, Kaul...(1988 - now)

What's the enlarged symmetry?

$$
\begin{aligned}
\mathcal{A}_{\alpha\beta}(r) &= -i(|\alpha\rangle_r\langle\beta|_r - |\beta\rangle_r\langle\alpha|_r) \;\; \forall \;\; \text{pairs} \;\; \alpha < \beta \\
\mathcal{S}_{\alpha\beta}(r) &= (|\alpha\rangle_r\langle\beta|_r + |\beta\rangle_r\langle\alpha|_r) \;\; \forall \;\; \text{pairs} \;\; \alpha < \beta \\
\mathcal{Q}_{\alpha\alpha}(r) &= (|\alpha\rangle_r\langle\alpha|_r - 1/N) \;\; \forall \;\; \alpha = 1 \ldots N-1
\end{aligned}
$$

$$\mathcal{A}^{\text{tot}}_{\alpha\beta} = \sum_r \mathcal{A}_{\alpha\beta}(r)$$

SO(N) symmetry on any arbitrary lattice

$$\mathcal{S}^{\text{tot}}_{\alpha\beta} = \sum_r (-1)^r \mathcal{S}_{\alpha\beta}(r)$$

$$\mathcal{Q}^{\text{tot}}_{\alpha\beta} = \sum_r (-1)^r \mathcal{Q}_{\alpha\beta}(r)$$

Bipartite case: Enhanced "staggered" SU(N) symmetry

Large N limit in pure case

Any perfect (fully packed) dimer cover is a ground state (each dimer interpreted as singlet state)

Leading 1/N corrections: Captured precisely by QDM Hamiltonian with ring-exchange

Higher orders in 1/N: Additional local terms in QDM Hamiltonian

(Affleck, Read, Sachdev, Kaul...)

Recover the same QDM framework---without nearest-neighbor singlet assumption.

Disordered case: Large N limit

Any maximum matching now gives a large-N ground state.

Monomers correspond to free moments (additional degeneracy)

Leading 1/N corrections: QDM Hamiltonian with ring-exchange + monomer kinetic energy terms

Higher orders in 1/N: Additional local terms in QDM Hamiltonian

Correspond to residual interactions between local moments...(?)

These control fate of system at lowest energies

So: Large N also gives maximally-packed QDM description of disorder effects in short-range RVB liquid

Focus on random geometry of maximum-density dimer packings

We ask: Where on the lattice do the monomers live? (in the ensemble of maximum matchings)

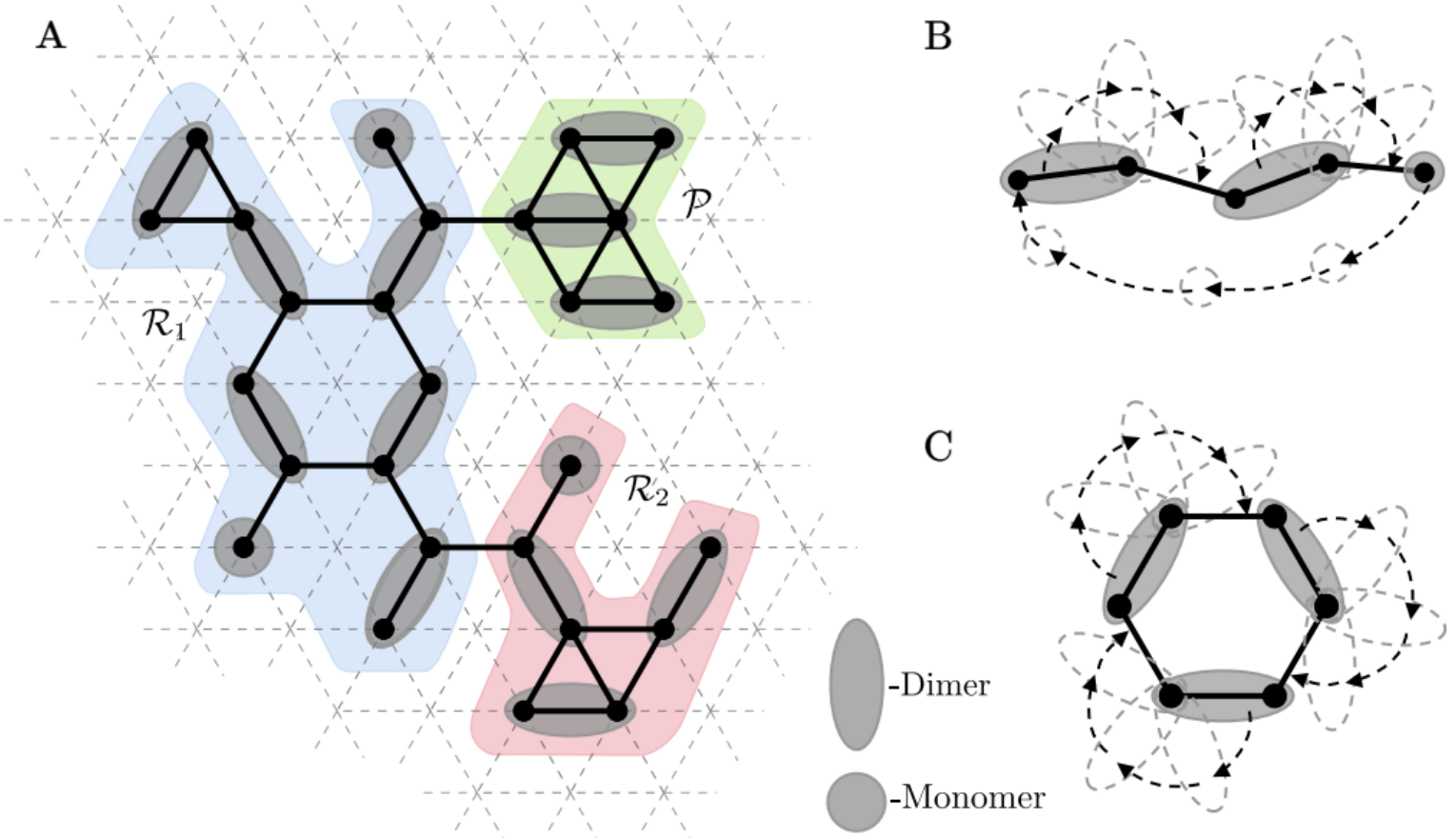
The answer should give us

partial information on zero-modes/collective Majorana modes

Vacancy-induced local moment instabilities

How does one implement this?

The setting: Maximum-density dimer packings of diluted lattices



Conclusions (from pictures):

Consequences of hard-core and maximum-density constraints:

Constrained kinematics: ring-exchange or monomer-hopping

Constraint on links of ring-exchange and monomer-hopping process paths:

Each such link must be occupied by a dimer in at least one such dimer packing

Constraint on monomer and dimer motion:

Monomers confined to well-defined regions of disordered lattice. Other regions fully-packed.

Identifying monomer-carrying and perfectly-matched regions:

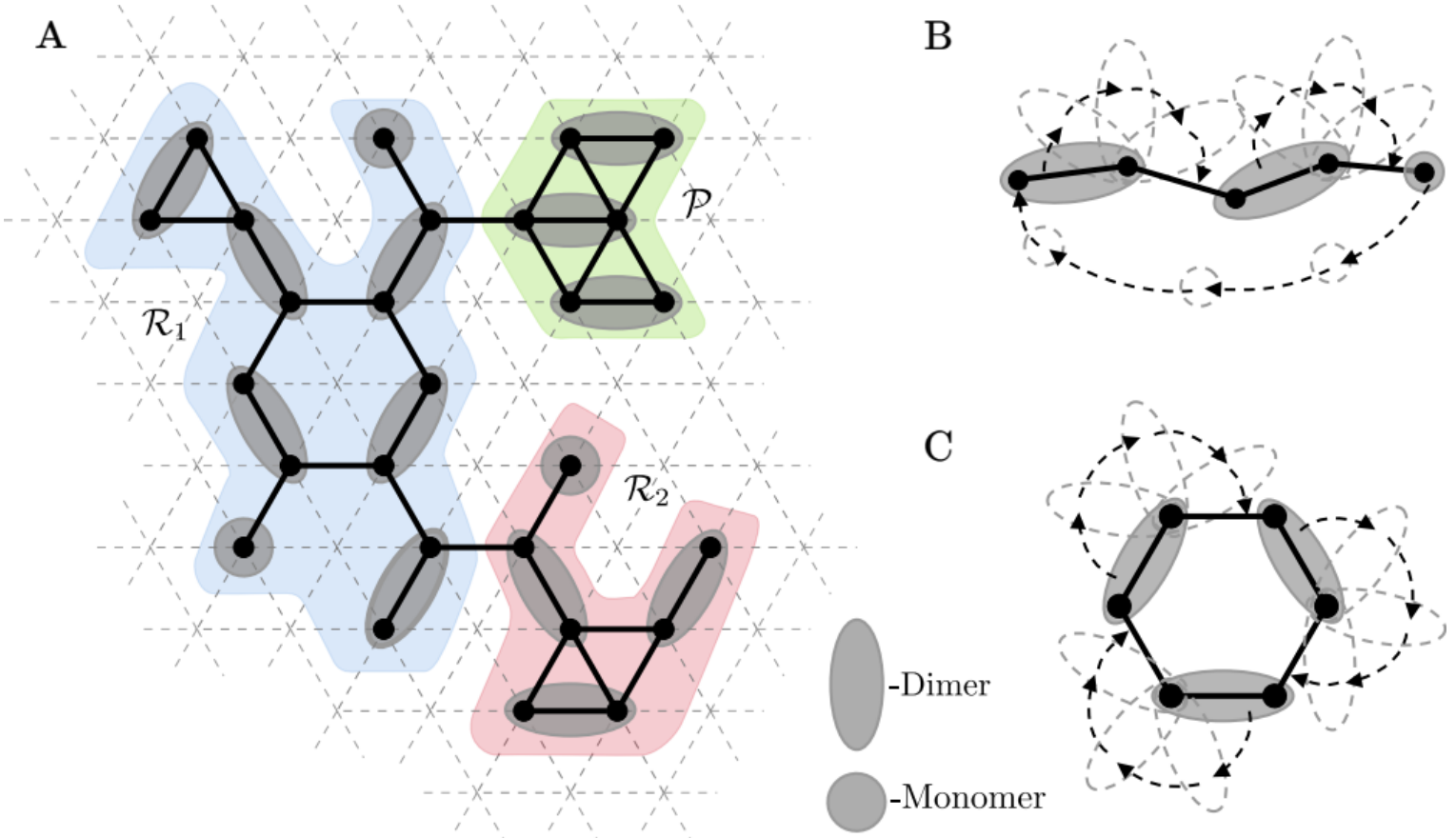
Boundaries of monomer-carrying $\mathcal{R}$-type, fully-packed $\mathcal{P}$-type regions:

Some "forbidden" links of disordered lattice can never be occupied by a dimer in any such packing

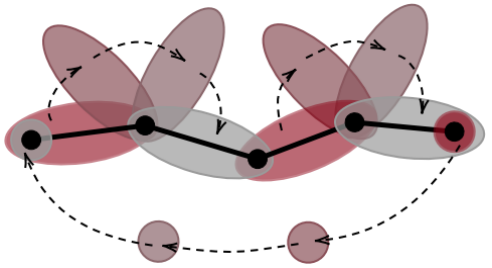
Boundaries of these regions demarcated by the "forbidden" links

Question becomes: Can we identify the forbidden links in a systematic way given a disordered sample?

Geometry of monomer-carrying and fully-packed regions



Math to the rescue: Gallai-Edmonds & Dulmage-Mendelsohn theory



Definitions:

Pick favorite maximum-density dimer packing

Explore forest of alternating paths starting from all monomers

Label vertices e (even) if they can be reached along an even-length path of this forest

Label vertices u (unreachable) if they cannot be reached along any paths of this forest

Label vertices o (odd) otherwise (i.e. can be reached by odd-length path but not even-length path)

Theorem of Gallai-Edmonds (general case) & Dulmage-Mendelsohn (bipartite case):

Labeling is property of disordered lattice, not of your favorite maximum-density dimer packing.

COVERINGS OF BIPARTITE GRAPHS

A. L. DULMAGE AND N. S. MENDELSOHN

Dulmage & Mendelsohn, Canadian J. Math 1958

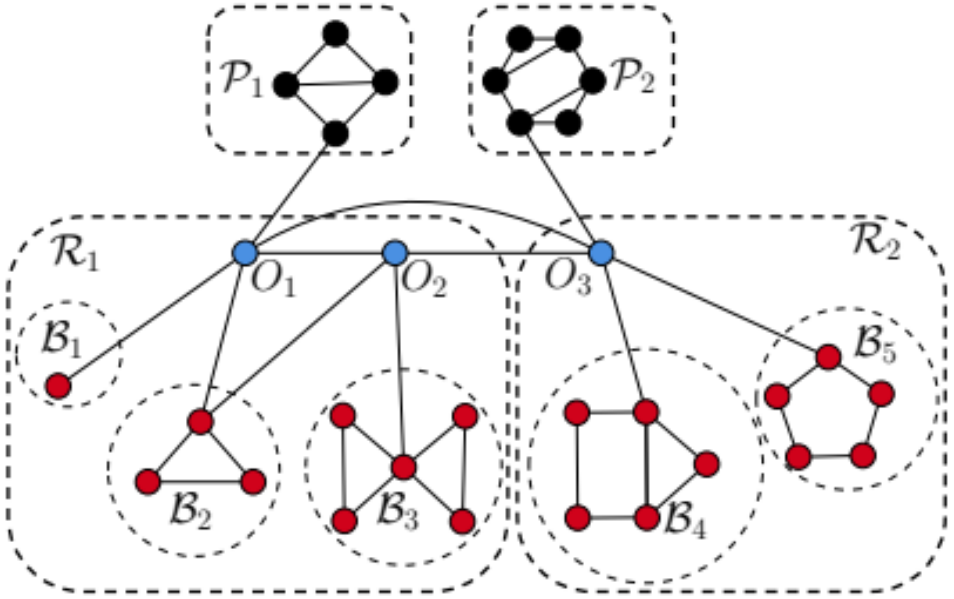
T. Gallai 1963,'64

PATHS, TREES, AND FLOWERS

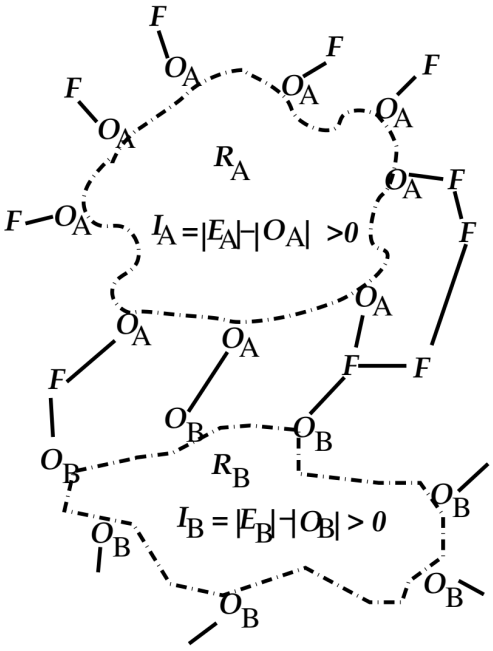
JACK EDMONDS

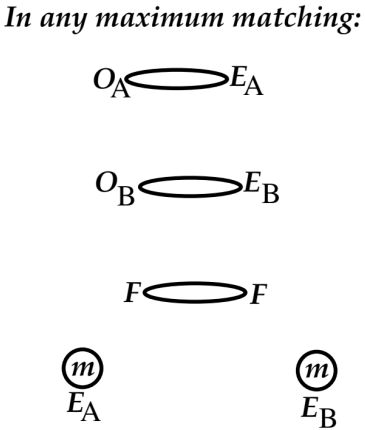
J. Edmonds, 1965

Our answer



KD, PRB 105 235118 (2022)





Bhola, Biswas, Islam, KD, PRX 2022

Key observation: o – o and o – u links are the "forbidden" links. Delete!

Gives us an alternate local proof of Longuet-Higgins & Lovasz results.

Number of monomers in an R-type region = Number of zero modes localized in same region.
(adding contributions from all regions gives older global statements of Longuet-Higgins and Lovasz)

Striking theorem on kagome lattice

$$n_{\mathrm{monomer}} = 0$$

in (infinite connected cluster of) the diluted kagome lattice with nonzero vacancy density

Short-range RVB state has no vacancy-induced local moment instability on kagome lattice (!)

Theorem generally true on all "claw-free lattices" (pyrochlore lattice, star lattice etc)

Ansari, KD, PRL 132 226504 (2024)
Bhola, KD, arXiv:2512.23639 (2025)

More generally: tractable computations(!)

Can obtain complete set of R-type and P-type regions from one maximum matching of diluted lattice

via BFS for augmenting alternating paths (Blossoms in non-bipartite case)

Opens door to detailed computational study of random geometry of $\mathcal{R}$-type and $\mathcal{P}$-type regions

(..and thence, (hopefully) to deductions about the physics...)

Bhola, KD, arXiv:2311.05634v2 (2025)

KD, PRB 105 235118 (2022)

Bhola, Biswas, Islam, KD, PRX 2022

Enter: Percolation...

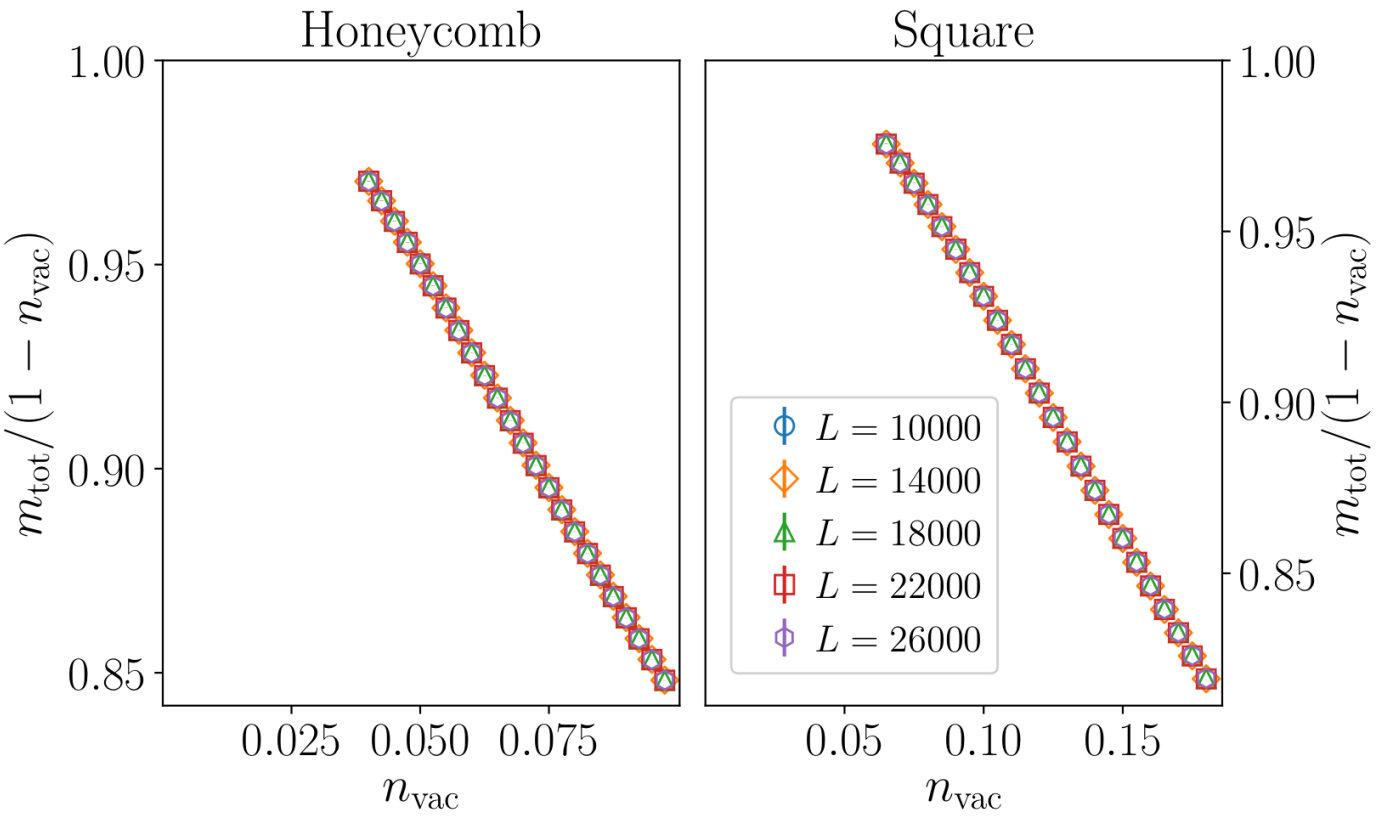
Typical regions are large at low dilution: Think in terms of percolation

The “right” yes/no question to ask: Can one walk from one end of a sample, staying within a single region?

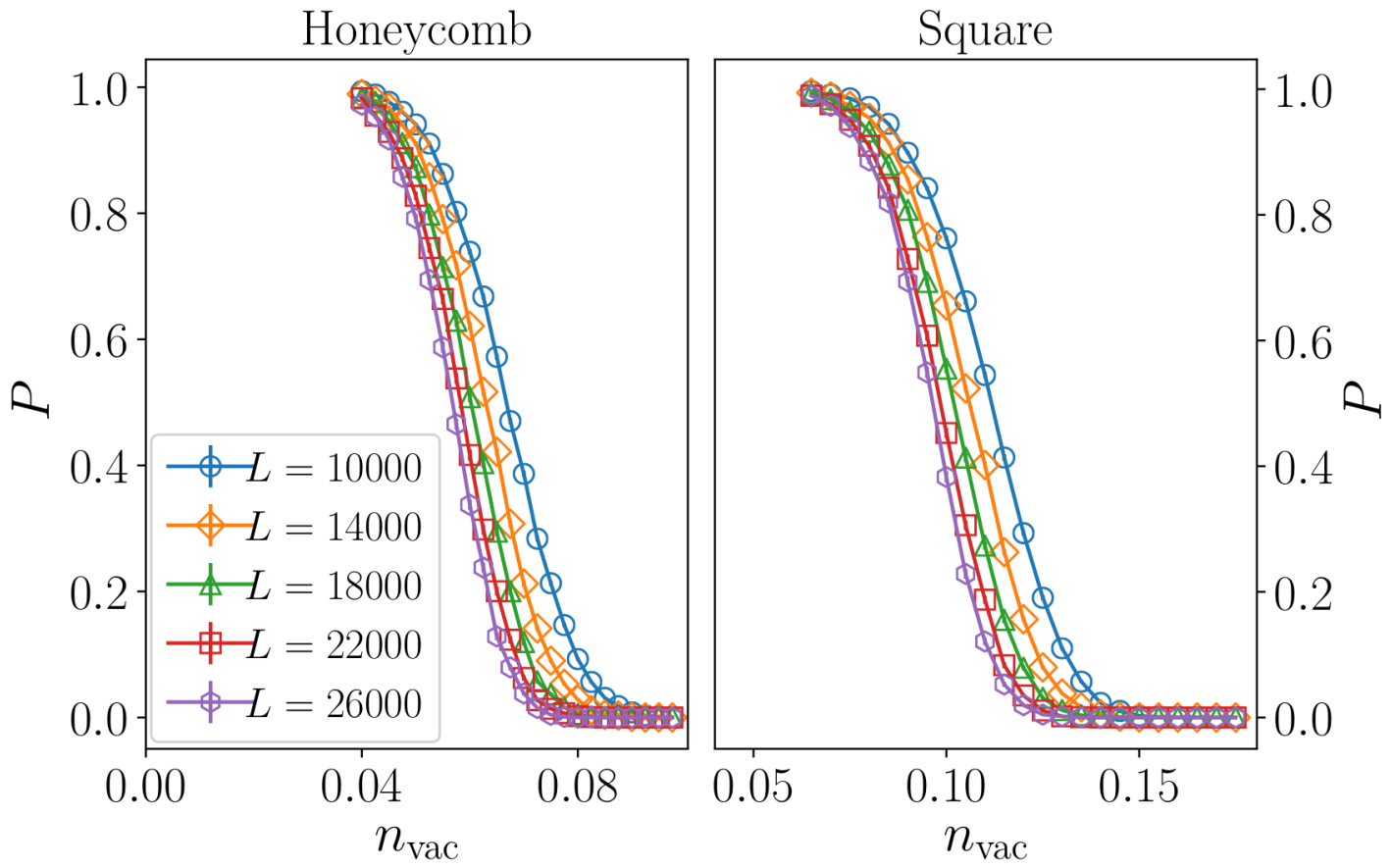
Bipartite case

Bhola, Biswas, Islam, KD, PRX 2022

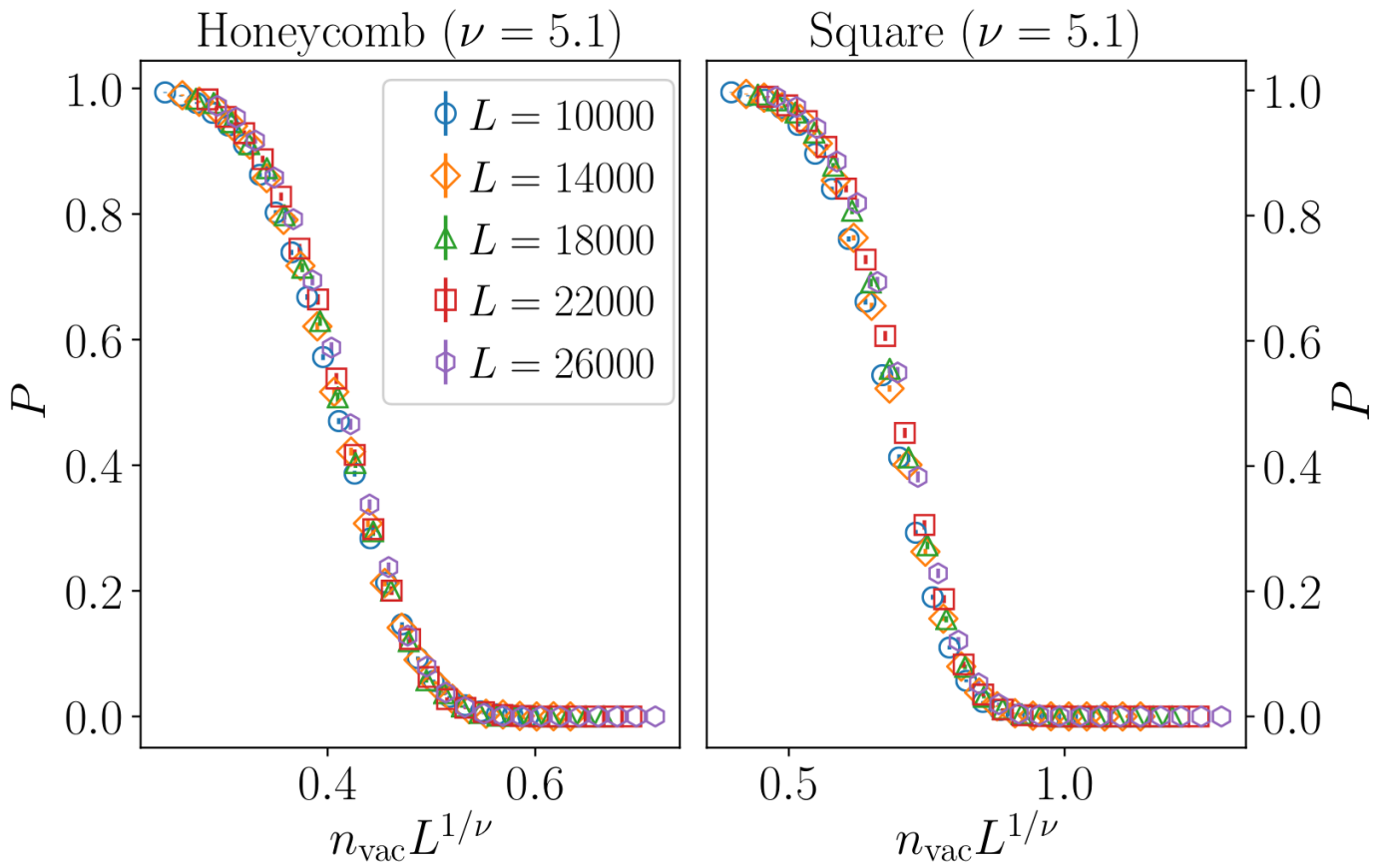
$\mathcal{R}$-type regions take over lattice



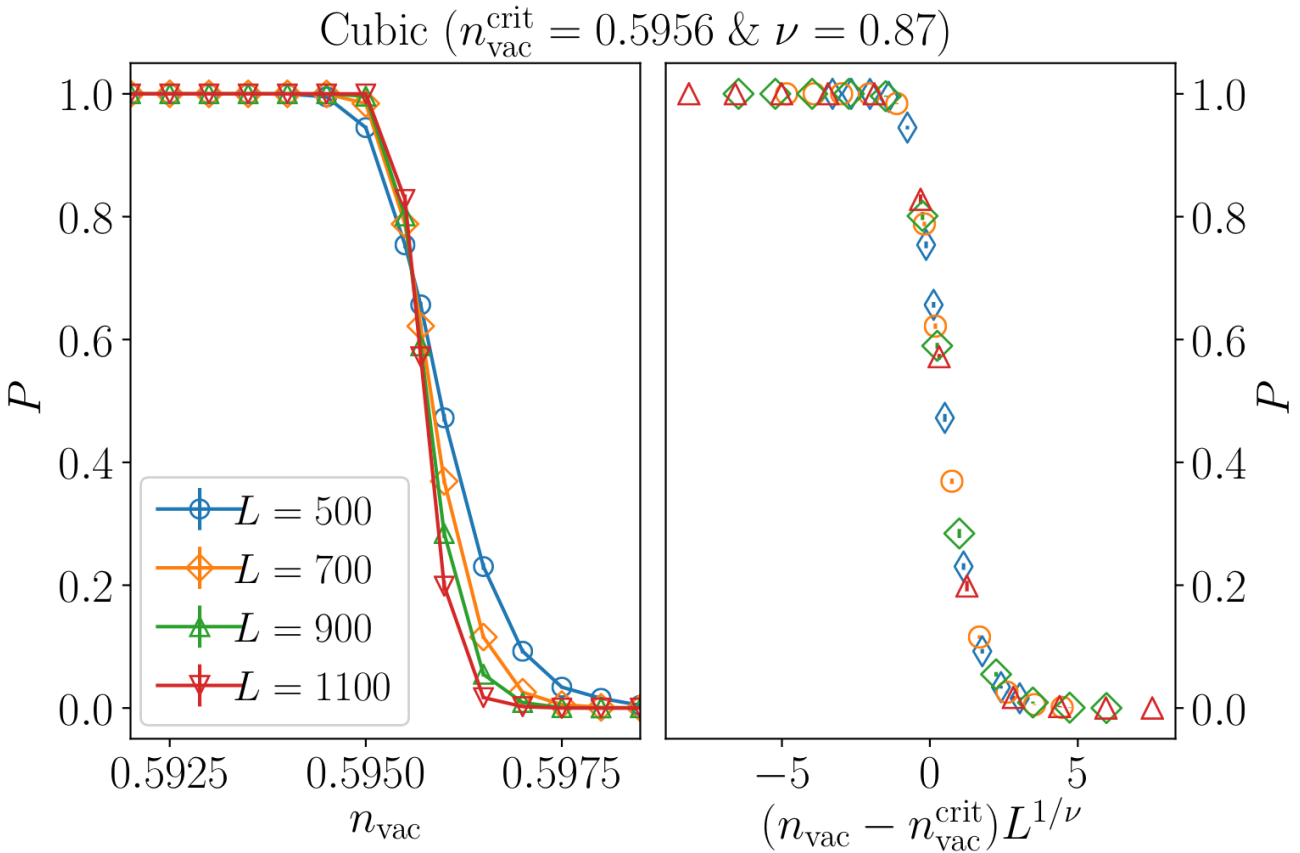
Incipient percolation at $n_v = 0$ (?)



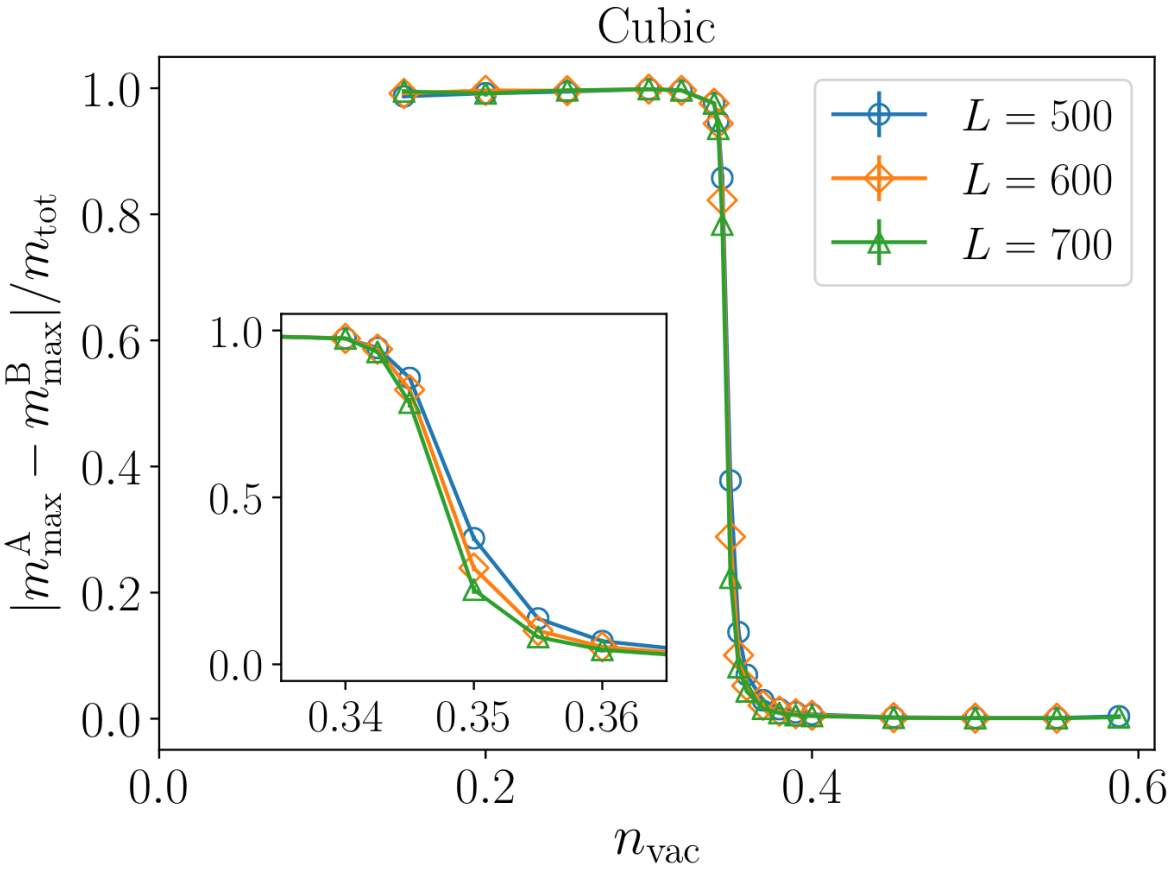
Universal scaling at $n_v = 0$ critical point



Percolation transition on cubic lattice



Spontaneous sublattice symmetry breaking deep inside percolated phase



Consequences for “quantum percolation” and Kitav systems...

No quantum percolation transitions on the square or honeycomb lattices at half-filling

(very large crossover length scale)

Quantum percolation transition on diluted cubic lattices understood in terms of monomer percolation

(With interactions: sublattice symmetry breaking transition could have implications for magnetic response)

Bhola, Biswas, Islam, KD, PRX 2022

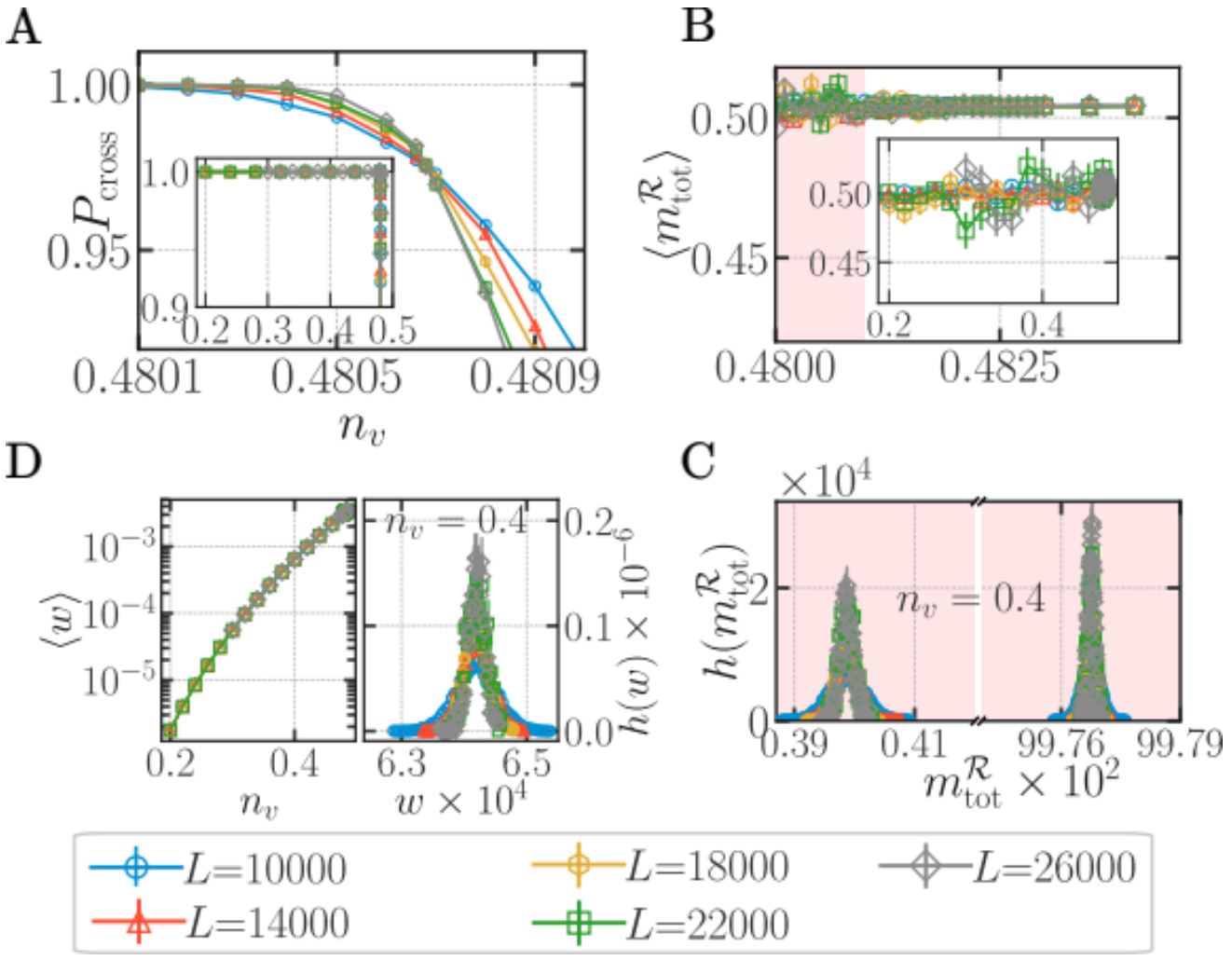
(obvious) spin-off: Kitaev magnets have vacancy-induced Curie tails with Curie constant $\mathcal{C} \propto w$

Sanyal, KD, Chalker, Moessner PRL 127 127201 (2021)

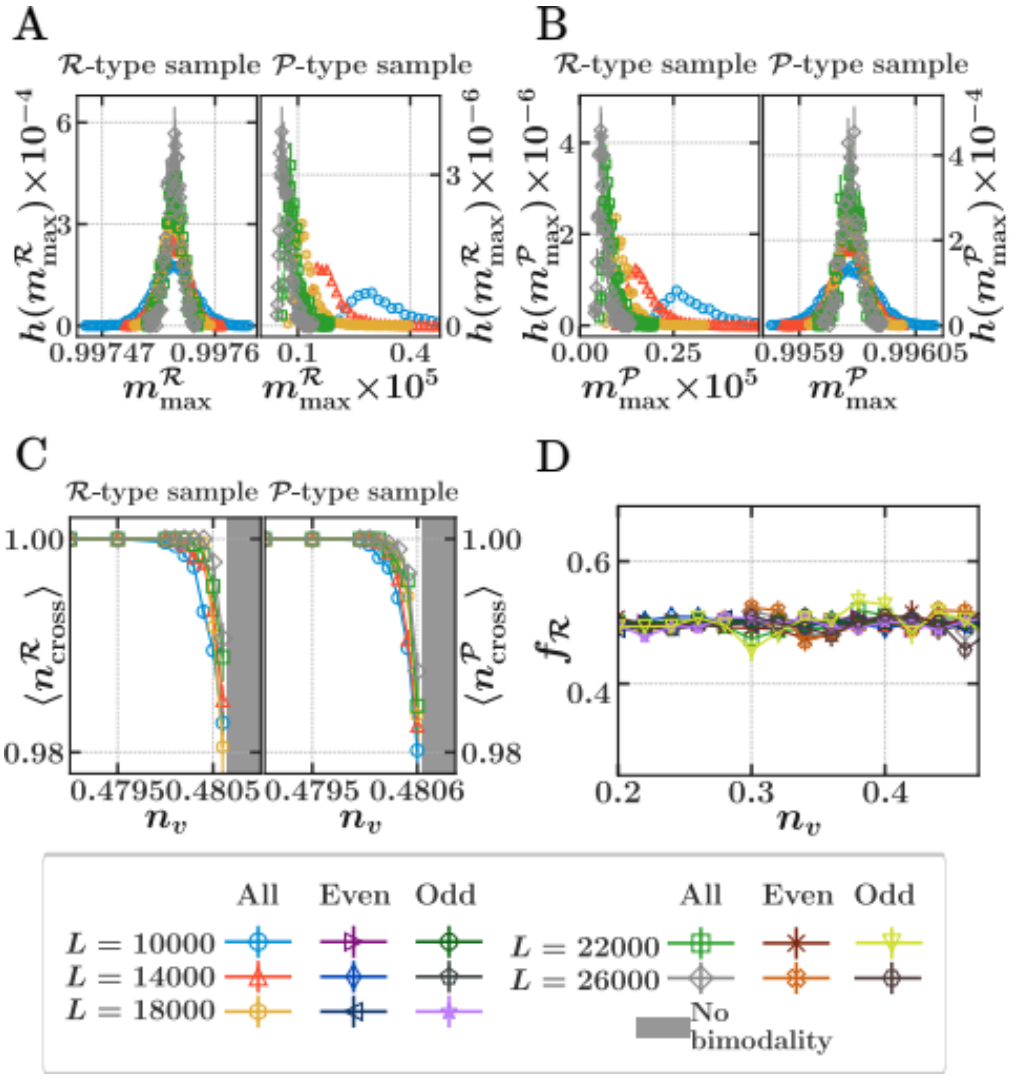
Nonbipartite case

Bhola, KD, arXiv:2311.05634v2 (2025)

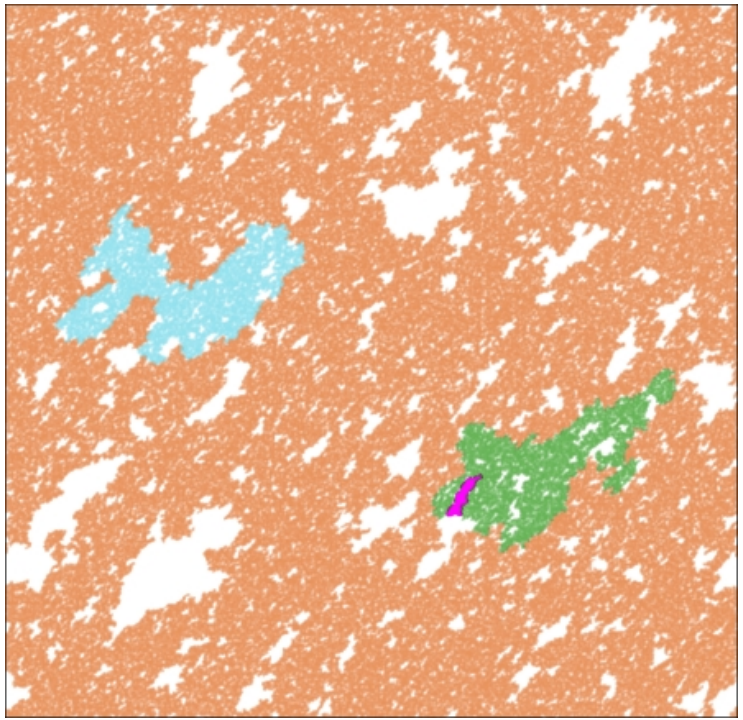
On the diluted triangular lattice



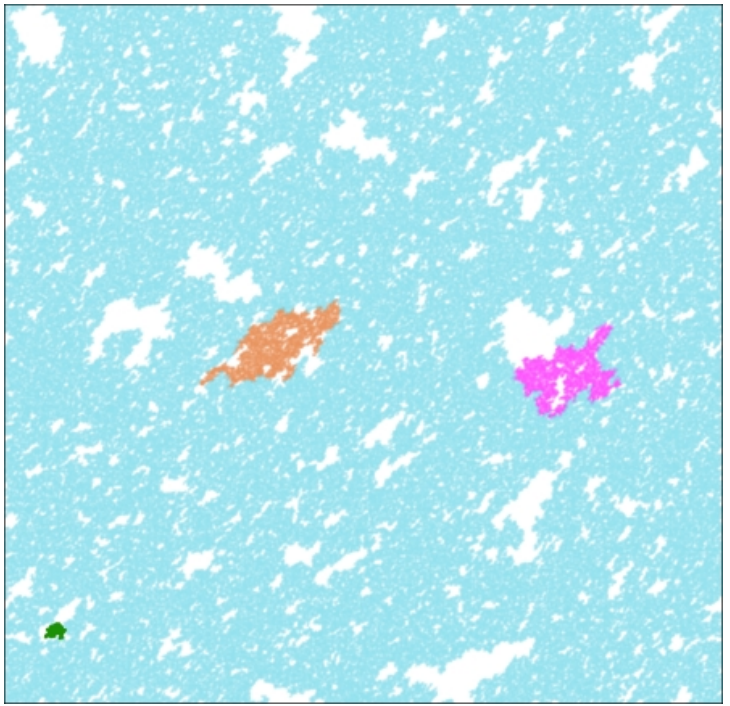
On the diluted triangular lattice



Pictorially on the diluted triangular lattice



R-type sample



P-type sample

On the diluted triangular lattice

$$f_{\mathcal{R}} = 1/2$$

Why???

Does this suggest some emergent symmetry between monomer-carrying and fully-packed regions

Again: Parity of largest geometric cluster plays no role!

On the diluted triangular lattice

Violation of even the weak form of "central dogma" at low vacancy concentration:

Monomers delocalized in half the samples, localized to O(1) regions in the other half!

All samples identically prepared, randomly diluted, with the exact same density of vacancies

On the diluted triangular lattice

Suggests extreme sensitivity of large-scale geometry to micro-scale details of disorder configuration

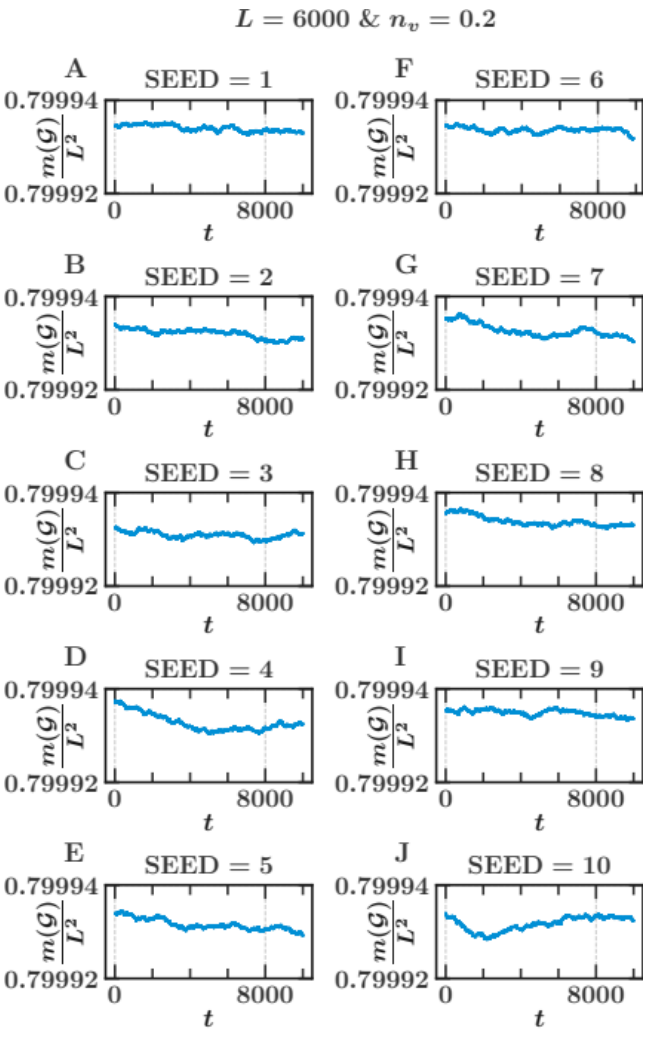
Can we quantify this?

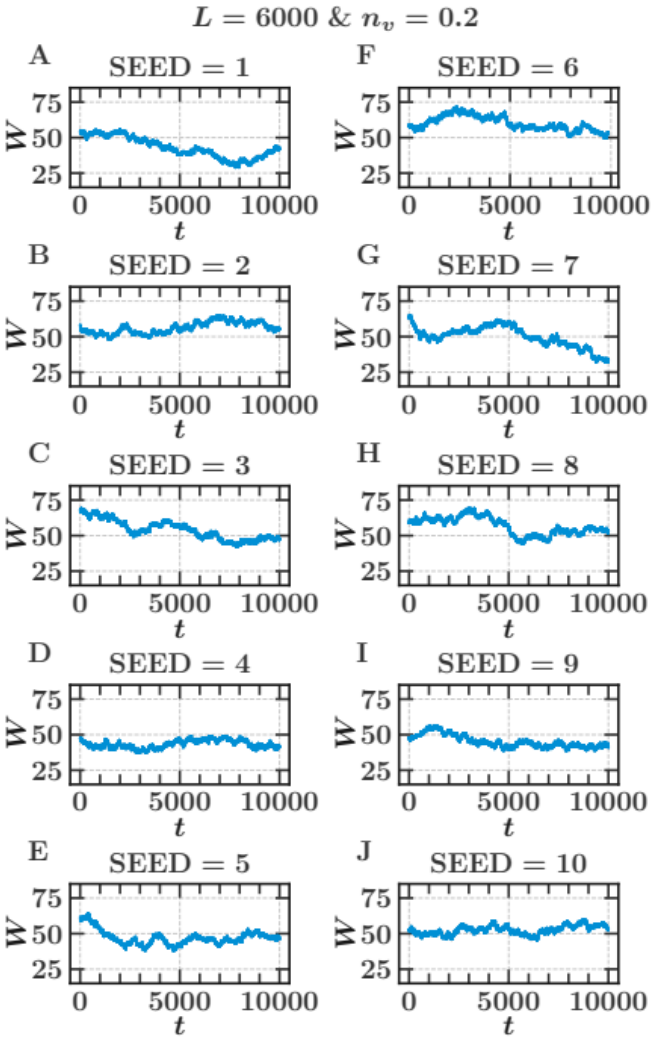
Model dynamics: Set vacancies in motion and watch what happens!

Small fraction of vacancies exchange position with neighboring surviving site at each time step

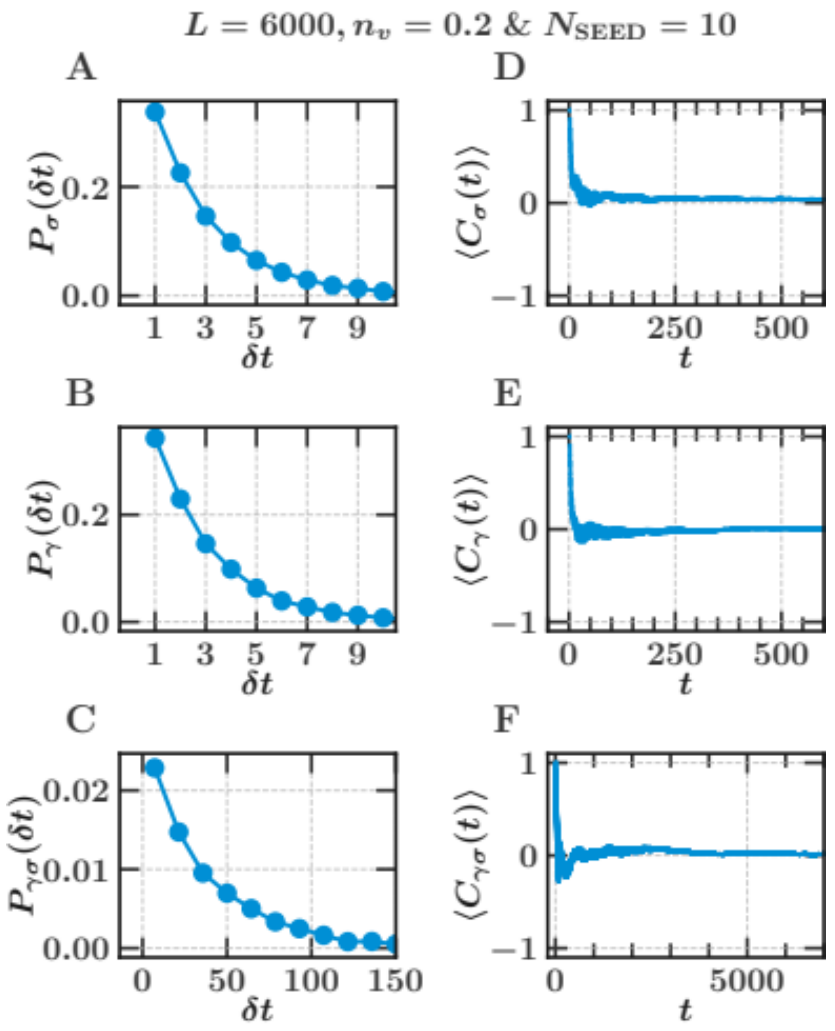
How does the large-scale geometry of these regions react?

Dynamics doesn't disturb underlying lattice much





Yet: Large-scale geometry of monomer-carrying/fully-packed regions responds chaotically



And thus: Consequences for transport

Effects of weak vacancy disorder (missing vortices) in pinned vortex lattice state of p+ip superconductors:

At a minimum: *Strong violations of thermodynamic self-averaging in the thermal conductivity*

Likely: *"R-type samples" have high thermal* conductance *but not "P-type" samples*

Also: Chaotic (deterministic but unpredictable) conductance response to changes in disorder configuration.

...and for magnetism

Effects of weak vacancy disorder (nonmagnetic impurities) in short-range RVB spin liquids on triangular lattice

At a minimum: *Strong violations of thermodynamic self-averaging in the susceptibility*

Likely: *“R-type samples” have spin-glass order but not “P-type” samples*

Also: Chaotic (deterministic but unpredictable) susceptibility response to changes in disorder configuration.

More results...

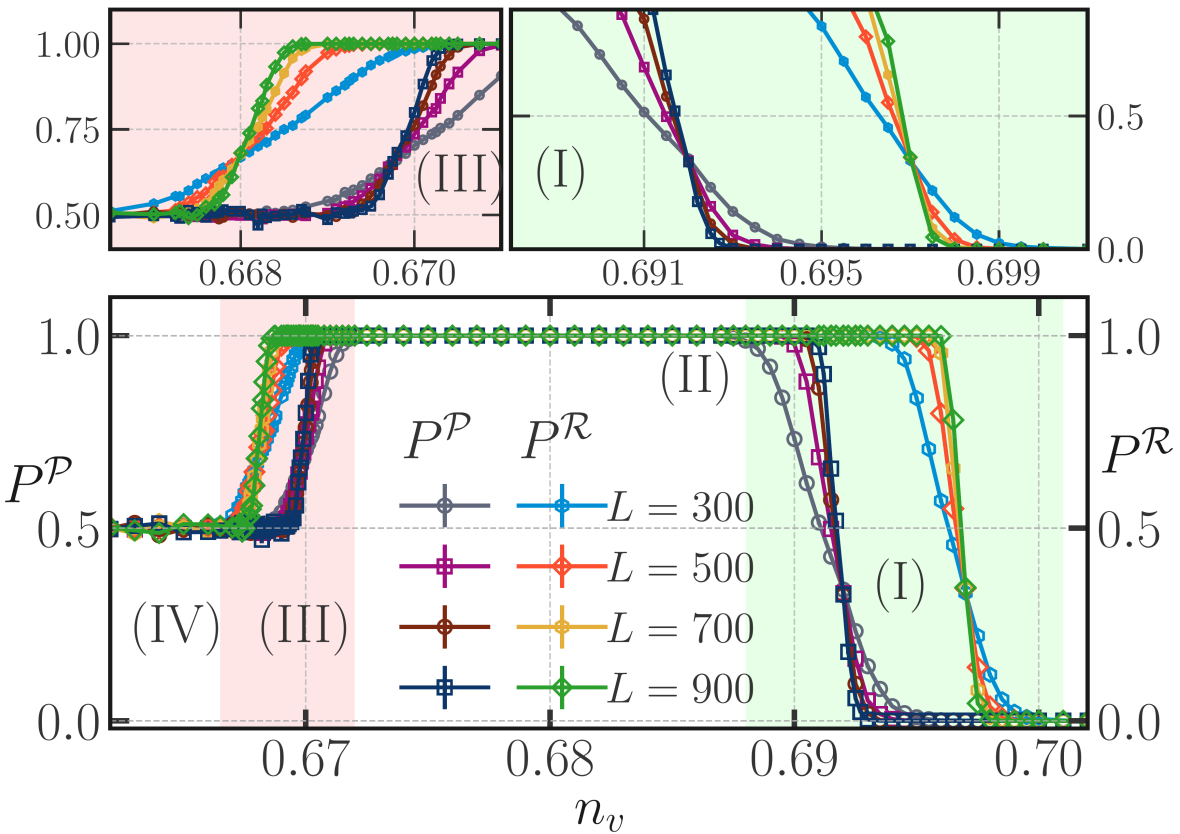
Similar phenomena on other non-bipartite lattices:

Checked: Percolation transition and low-dilution phase essentially the same on the Shastry-Sutherland lattice

A totally baroque phase diagram in three dimensions

Interesting from the vantage point of percolation theory

Phase diagram in 3d via wrapping probabilities



Bhola, KD, arXiv:2311.05634v2 (2025)

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