Chaotic percolation in maximum-density dimer packings: implications for spin liquids and topological superconductors

Kedar Damle (TIFR-Mumbai) @ ICTS-TIFR (29July2025)

Unpublished:

Bhola, KD, arXiv:2311.05634v2; Ansari, Kundu, KD (in preparation

recent:

Ansari, KD, PRL 132 226504 (2024)

Bhola, Biswas, Islam, KD, PRX 12, 021058 (2022)

KD, PRB 105 235118 (2022)

background:

Sanyal, KD, Chalker, Moessner PRL 127 127201 (2021)

Sanyal, KD, Motrunich, PRL 117 116806 (2016)






Disorder

Quenched disorder: Missing atoms, adatoms, lattice imperfections...

Quite common in condensed matter systems

Effects varied:

Weak disorder: Can be irrelevant for low energy properties (not always).

Strong disorder: new phases of matter (e.g. spin glasses)

Can probe correlations of underlying state (e.g. spin textures in frustrated magnets)

“Central dogma”

In large-size limit -

Strong version:

Self-averaging of properties: Sample-to-sample fluctuations small (average = typical)

Violations exist – e.g. Disordered quantum spin chains (infinite-disorder fixed points)

Weak version:

At a minimum, two samples prepared using some protocol must be in same phase.

Violations? May exist in infinite-range spin glass models (?)

Our basic message

Violations of “central dogma”

Weak disorder can lead to:

Violations of not just strong but also weak form of the “central dogma”

Root cause: Kinematic constraints induce long-range correlations

(caveat emptor: merely post-facto rationalization, no detailed understanding)

Where's the "Fractionalized Quantum Matter" in this story?

Weak vacancy disorder predicted to lead to similar effects in:

Triangular lattice short-range RVB spin liquids

Pinned triangular vortex-lattice state of topological p+ip superconductors

Triangular lattice Majorana spin liquids

Some predictions for observable effects

Consequences:

Weak vacancy disorder leads to similar effects in short-range RVB spin liquids on the triangular lattice

At a minimum: Strong violations of thermodynamic self-averaging in low-temperature susceptbtibility

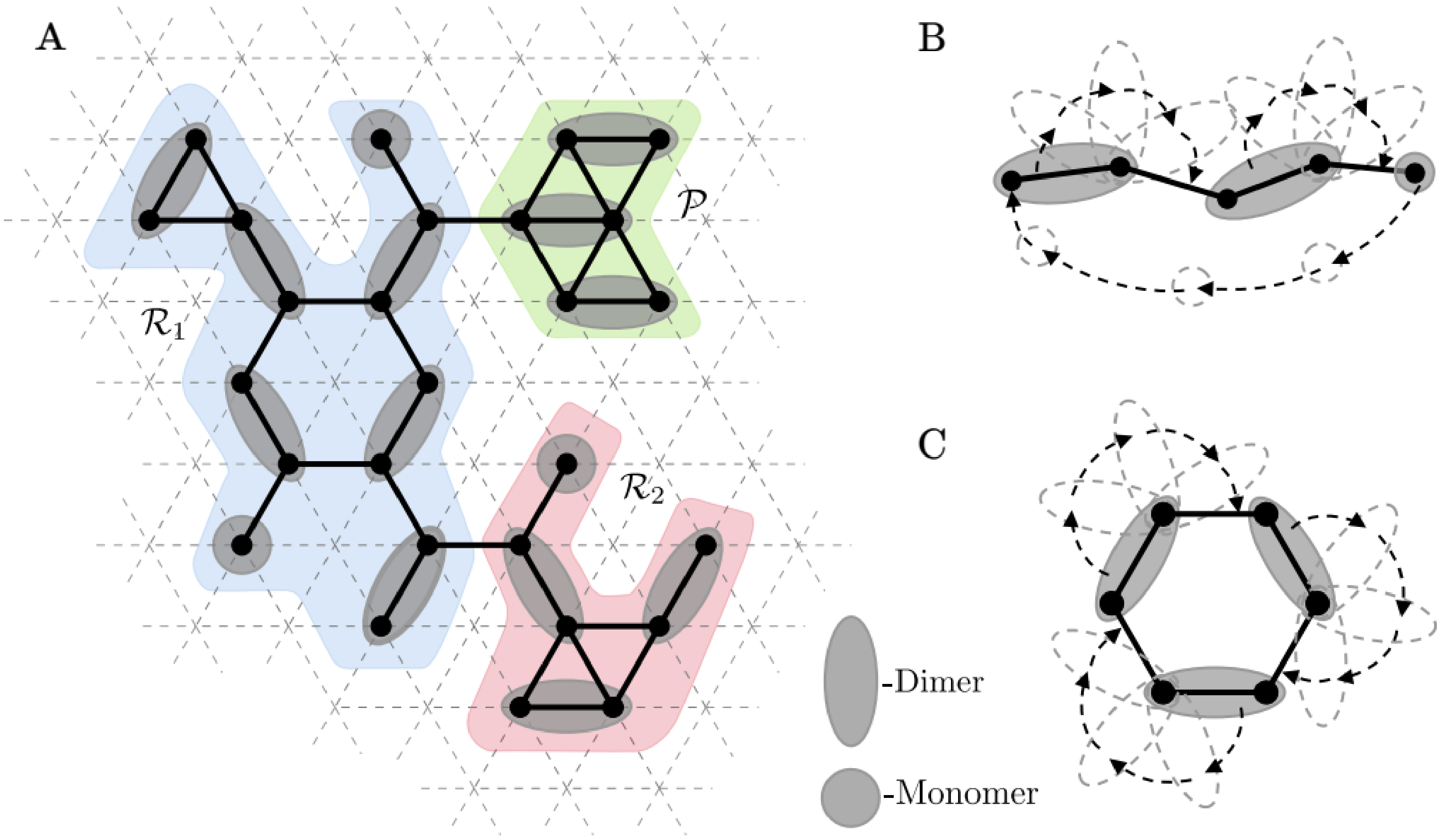
Likely: “R-type samples” have spin-glass order but not “P-type” samples

Weak vacancy disorder in pinned vortex lattice state of p+ip superconductors will also lead to similar effects

At a minimum: Strong violations of thermodynamic self-averaging in the thermal conductivity

Likewise for weak vacancy disorder in triangular lattice Majorana spin liquids

The setting: Maximum-density dimer packings of diluted lattices



Some conclusions (from pictures):

Pure case:

Most regular lattices have nonzero entropy density of fully-packed dimer coverings

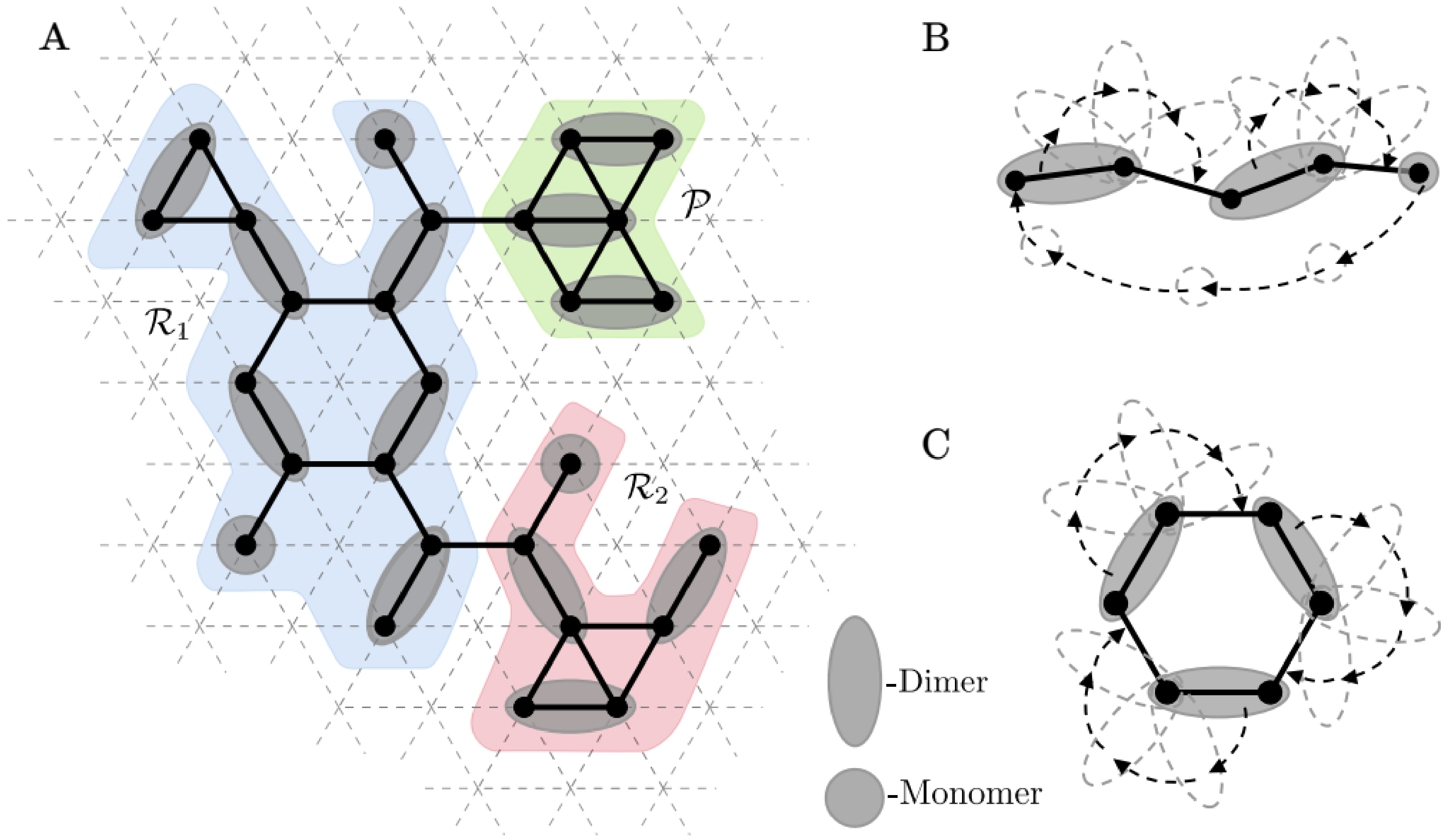
(if bipartite, require |A|=|B| of course)

Weak vacancy disorder or bond dilution:

Typically have nonzero density of monomers in any maximum-density dimer packing

(and nonzero entropy density of such packings)

Constraints on maximum-density dimer packings



More conclusions (from pictures):

Consequences of hard-core and maximum-density constraints:

Constrained kinematics: ring-exchange or monomer-hopping

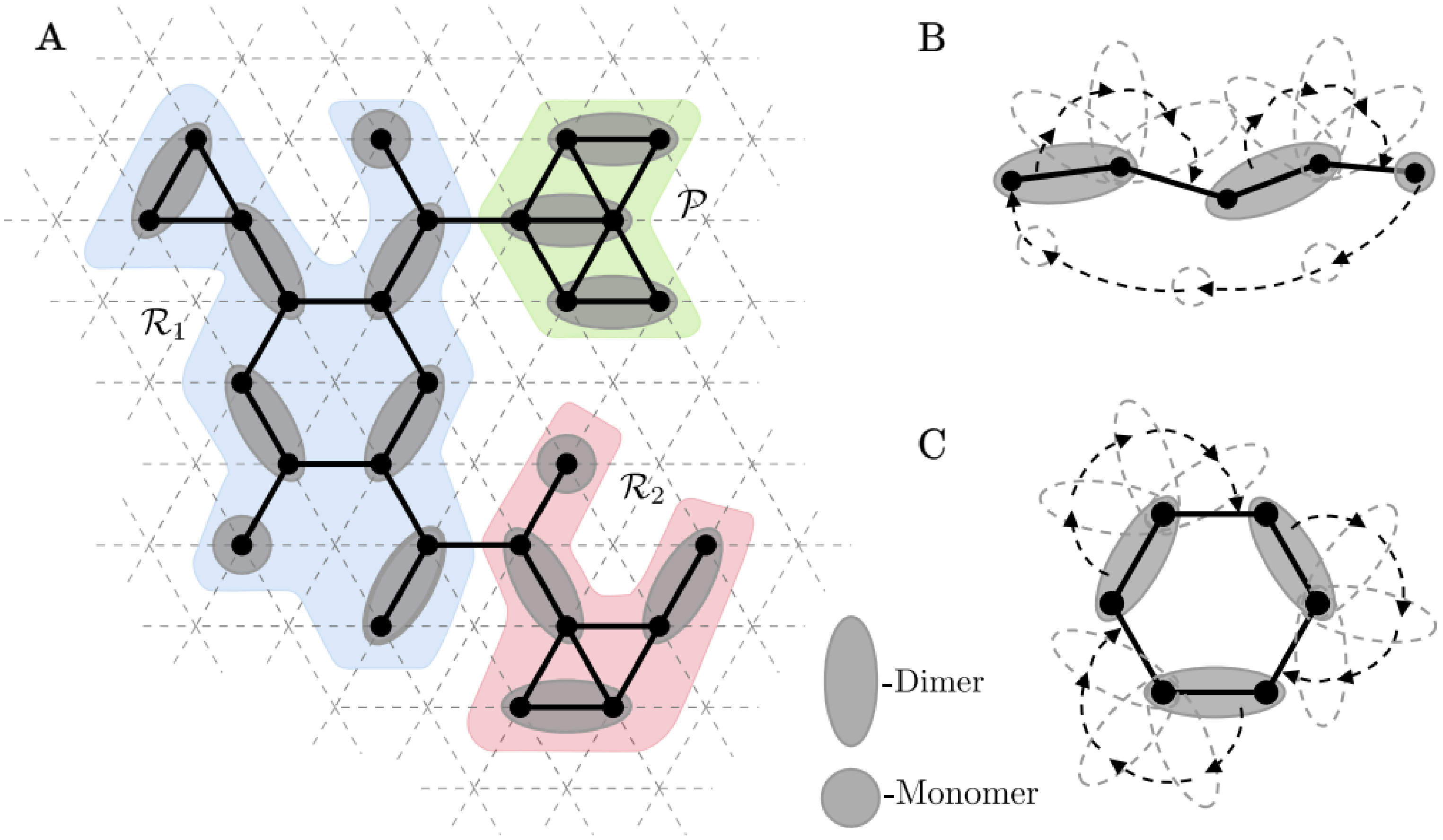
Constraint on links of ring-exchange and monomer-hopping process paths:

Each such link must be occupied by a dimer in at least one such dimer packing

Constraint on monomer and dimer motion:

Monomers confined to well-defined regions of disordered lattice. Other regions fully-packed.

Geometry of monomer-carrying and fully-packed regions



Another conclusion (from pictures):

Boundaries of monomer-carrying $\mathcal{R}$-type, fully-packed $\mathcal{P}$-type regions:

Some “forbidden” links of disordered lattice can never be occupied by a dimer in any such packing

Boundaries of these regions demarcated by the “forbidden” links

These regions are properties of disordered lattice, not any one maximum-density packing

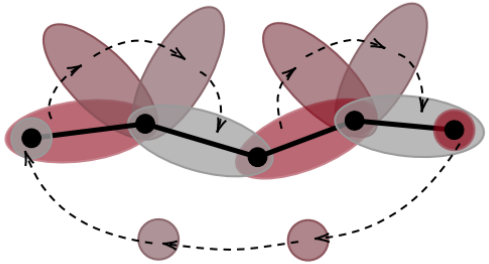
T. Gallai 1963,'64

Formal justification

J. Edmonds, 1965

PATHS, TREES, AND FLOWERS

JACK EDMONDS



Prescription:

Pick favorite maximum-density dimer packing

Explore forest of alternating paths starting from all monomers

Label vertices e (even) if they can be reached along an even-length path of this forest

Label vertices u (unreachable) if they cannot be reached along any paths of this forest

Label vertices o (odd) otherwise (i.e. can be reached by odd-length path but not even-length path)

Gallai-Edmonds Theory

T. Gallai 1963,’64

J. Edmonds, 1965

PATHS, TREES, AND FLOWERS

JACK EDMONDS

Labeling independent of choice of favorite maximum-density dimer packing

Property of underlying disordered lattice

Labeling comes with structural guarantees about disordered lattice

No e – u links possible

Deleting e – o organizes all e vertices into odd-cardinality connected components: “Blossoms”

Labeling also comes with guarantees about ensemble of maximum-density dimer packings

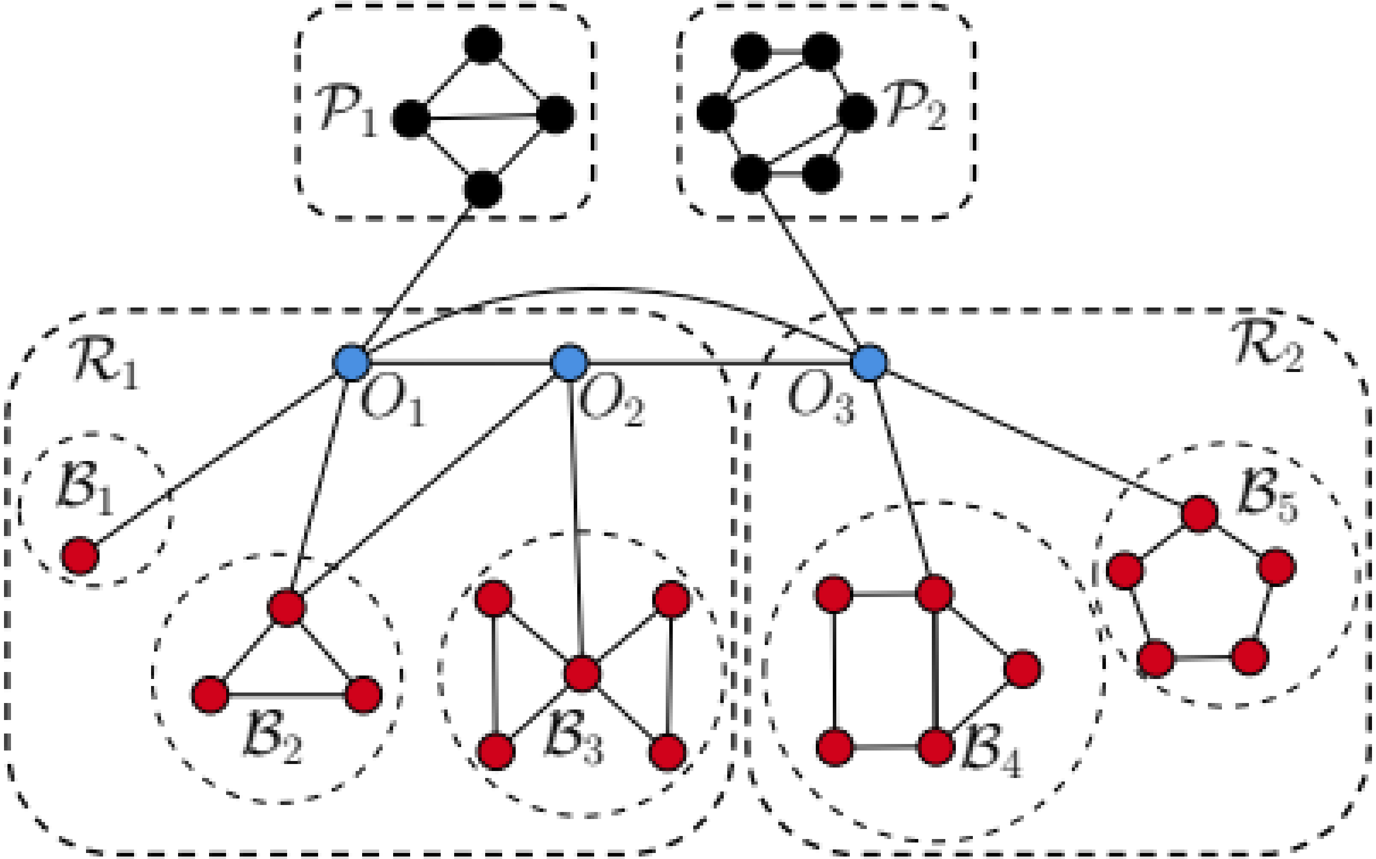
All u vertices connected to another u vertex by a dimer

All o vertices connected to some e vertex by a dimer

All monomers live on blossoms, no blossom has more than one monomer on it.

Construction of $\mathcal{R}$-type and $\mathcal{P}$-type regions

Key observation: o – o and o – u links are the "forbidden" links. Delete!



KD, PRB 105 235118 (2022)

Significance of $\mathcal{R}$-type and $\mathcal{P}$-type regions: Take 1

Quantum monomer-dimer models

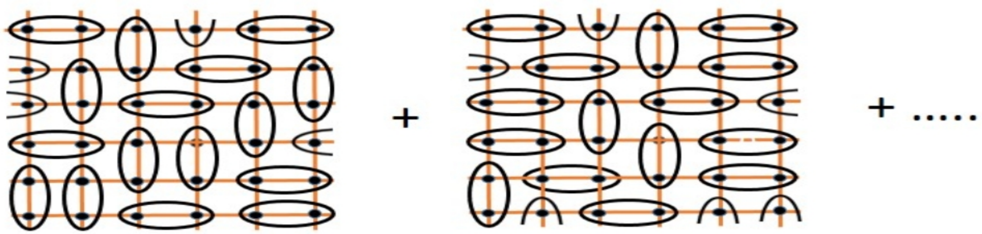
Monomer-hopping and ring-exchange processes cannot cross boundaries

All eigenstates of quantum/classical monomer-dimer models factorize

(for any dimer-interactions along flippable loops, but short-range monomer interactions)

Implies: If all regions small, area law entanglement in the middle of the many-body spectrum

Significance of $\mathcal{R}$-type and $\mathcal{P}$-type regions: Take 2



Short-range resonating valence bond spin liquid

Non-magnetic impurities (vacancy disorder) in short range resonating valence bond spin liquids

Within Rokhsar-Kivelson quantum dimer model framework:

Monomers associated with emergent local moments, confined to $\mathcal{R}$-type regions

Emergent local moments are a multi-vacancy effect

Dominant short-range interactions between these local moments also confined

Geometry of $\mathcal{R}$-type regions expected to determine low-energy state and magnetic response

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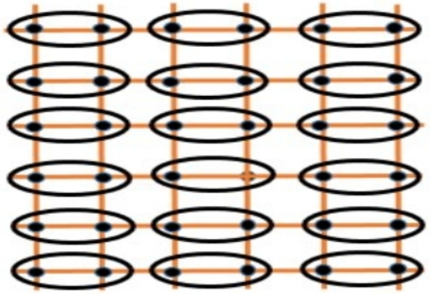
Contrast with lattice symmetry broken VBS state



Valence bond solid (VBS)

spontaneous lattice symmetry broken

Each vacancy individually nucleates a local moment bound to it

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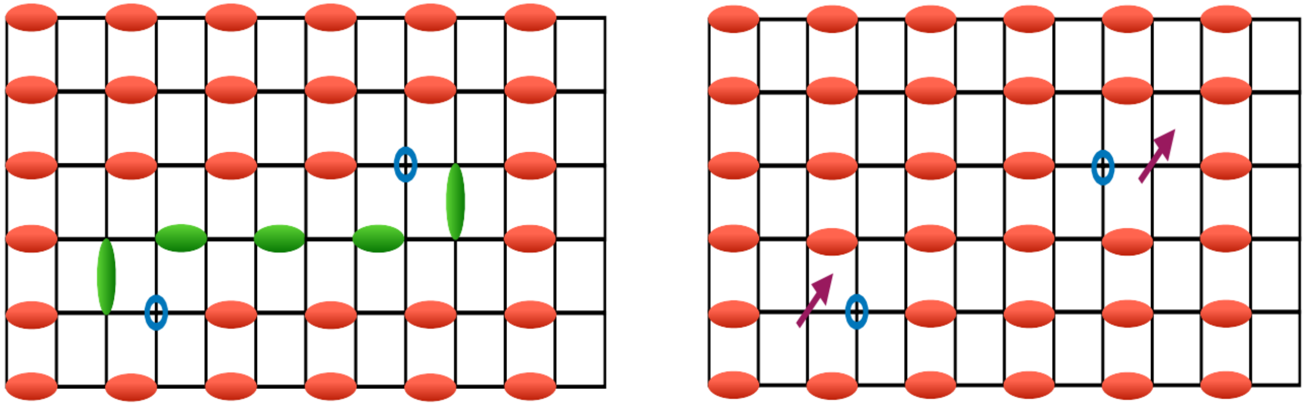
Why the distinction?

Heuristic picture



Also follows from more formal arguments relying on large-N expansions

And via computations (QMC) on model “designer” Hamiltonians with SU(N), O(N) symmetry

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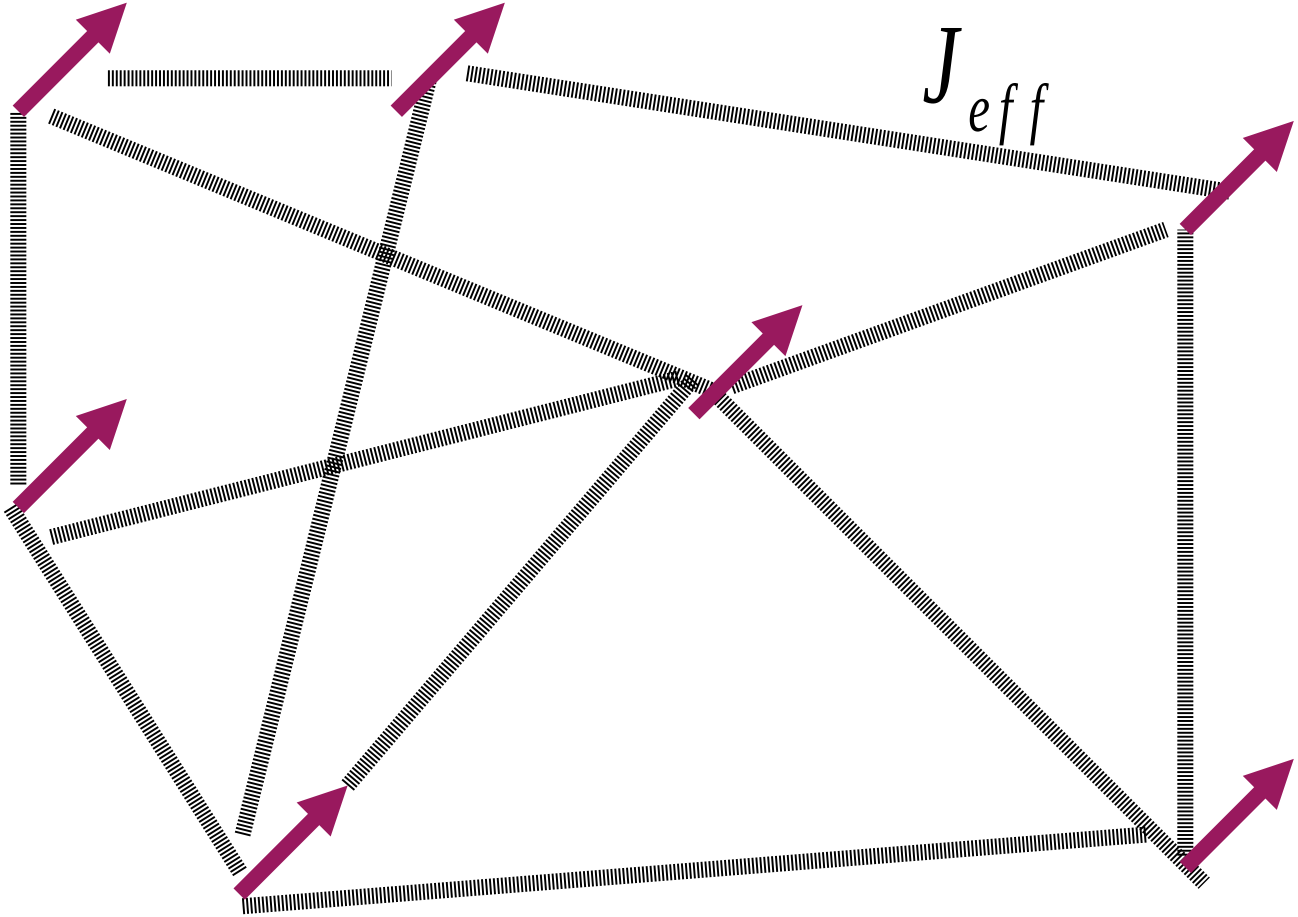
Conclusion: local moment instability of RVB and VBS states



In RVB case, only if

$$w \neq 0$$

In VBS case, even when

$$w = 0 \;\; \text{but} \;\; n_v \neq 0$$

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Significance of $\mathcal{R}$-type and $\mathcal{P}$-type regions: Take 3

Majorana modes of pinned triangular vortex lattice state of p+ip superconductors

Triangular lattice Majorana spin liquids

Effective low-energy Hamiltonian for Majorana excitations

$$
\begin{aligned}
H_{\text{Majorana}} &= \frac{i}{2}\sum_{rr'} \mathcal{A}_{rr'}\eta_r\eta_{r'} \\
\{\eta_r, \eta_{r'}\} &= 2\delta_{rr'} \\
\mathcal{A}_{r'r} &= -\mathcal{A}_{rr'}
\end{aligned}
$$

Topologically-protected collected Majorana zero modes of such “Majorana networks”

Each $\mathcal{R}$-type region hosting $\mathcal{I}$ monomers has $\mathcal{I}$ topologically-protected collective Majorana zero modes

Localized entirely within individual $\mathcal{R}$-type regions.

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With this motivation...

Computational study of large-scale geometry of monomer-carrying $\mathcal{R}$-type, fully-packed $\mathcal{P}$-type regions

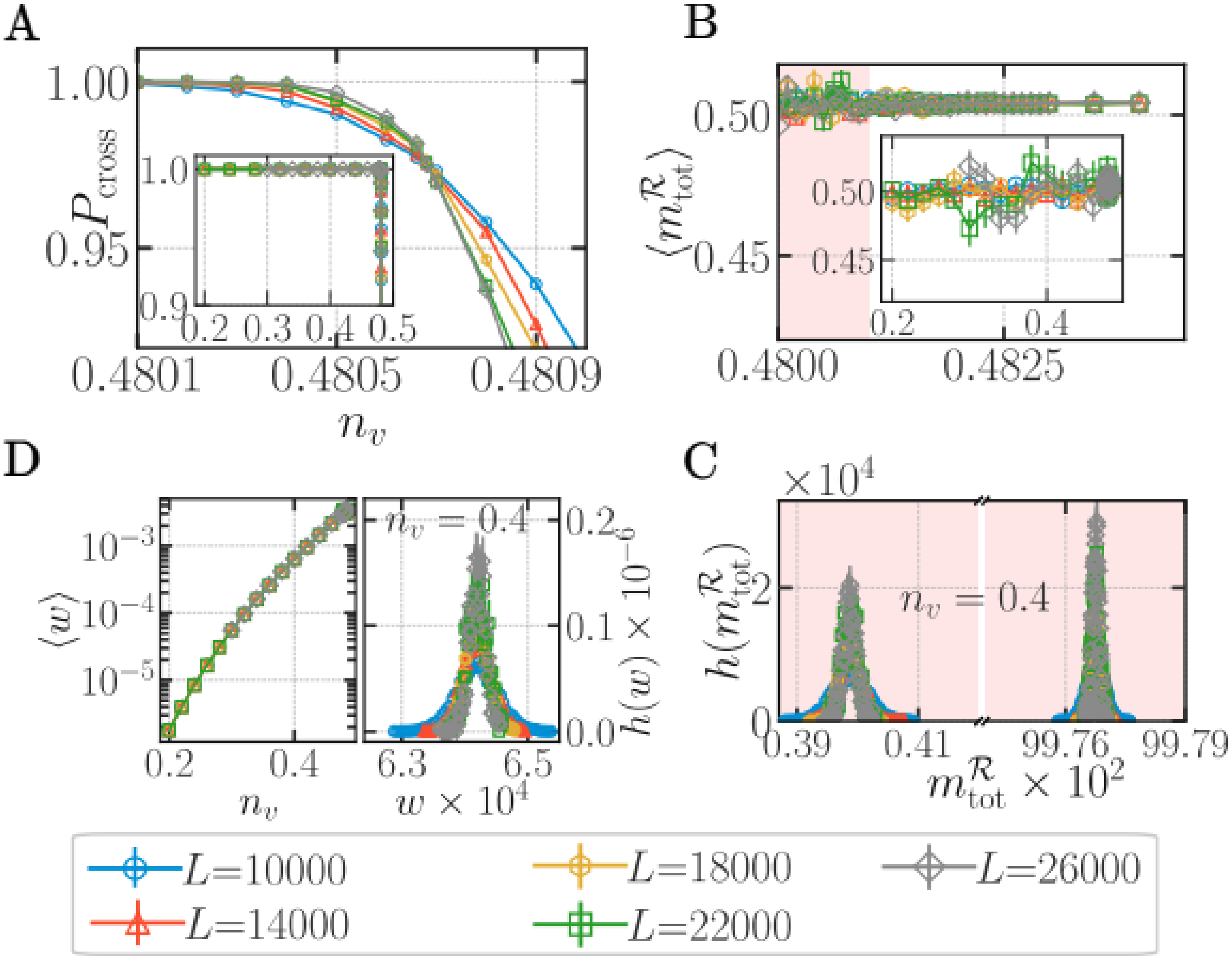
Computationally tractable (but challenging) using Edmonds' polynomial time matching algorithm

Typical regions are large at low dilution: Think in terms of percolation

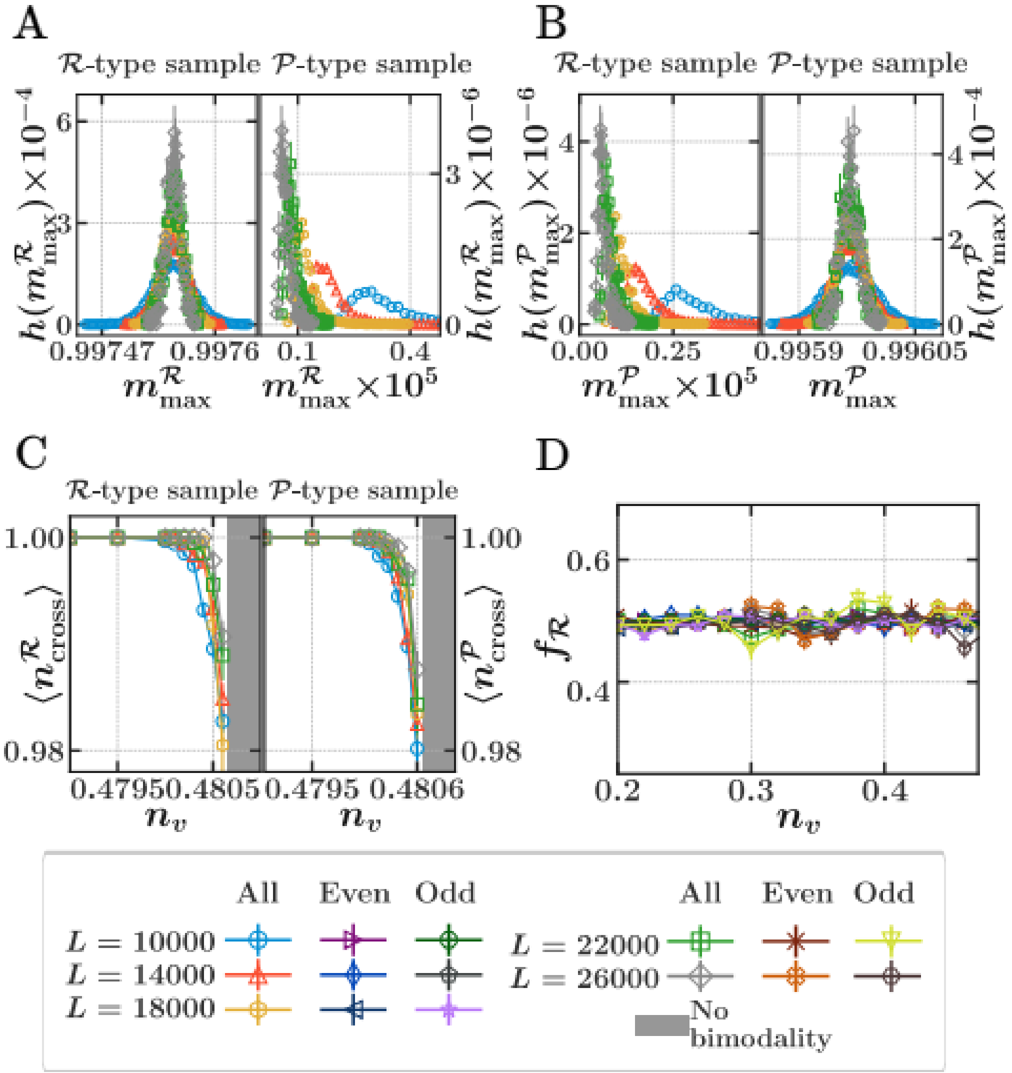
What is percolation? Sharp threshold (as function of some parameter) in end-to-end connectivity of a medium

The "right" yes/no question to ask: Can one walk from one end of a sample, staying within a single region?

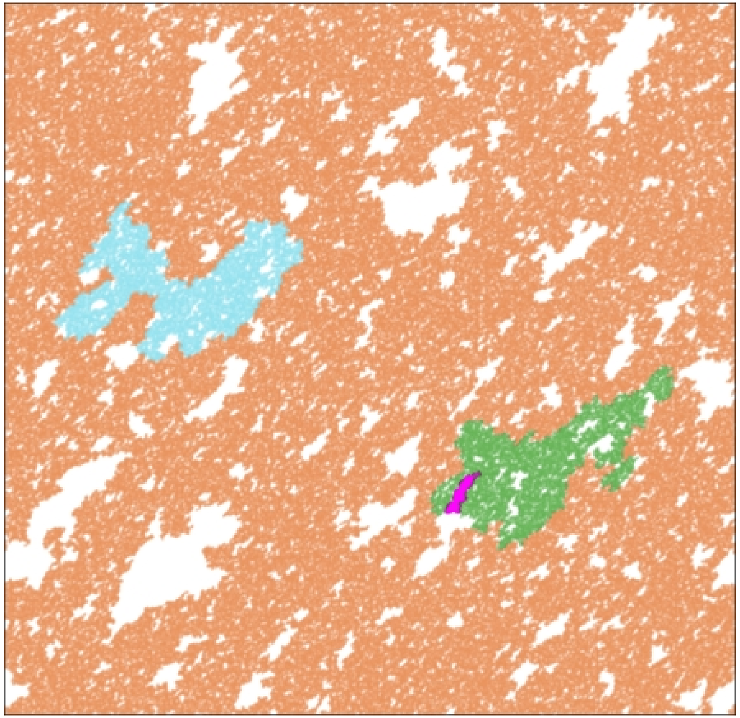
On the diluted triangular lattice



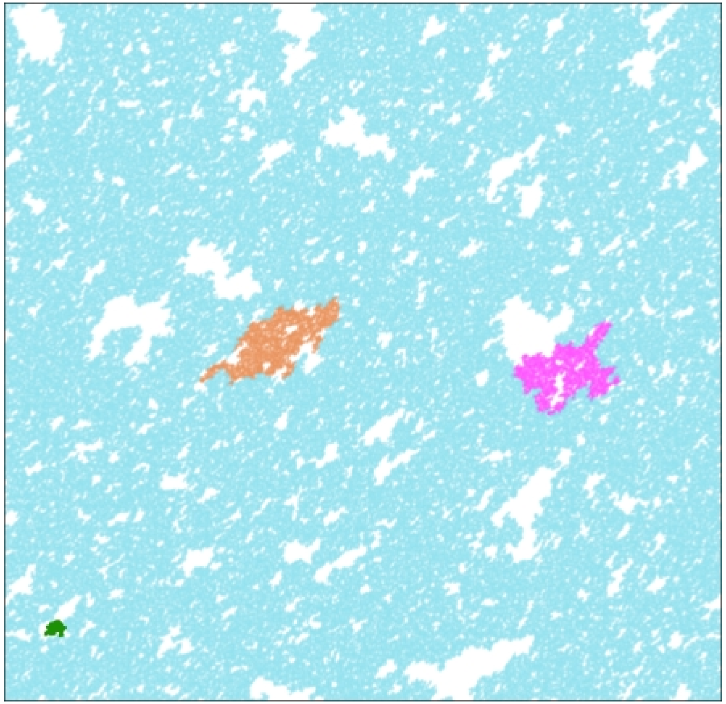
On the diluted triangular lattice



Pictorially on the diluted triangular lattice



R-type sample



P-type sample

On the diluted triangular lattice

$$f_{\mathcal{R}} = 1/2$$

Why???

Does this suggest some emergent symmetry between monomer-carrying and fully-packed regions

Again: Parity of largest geometric cluster plays no role!

On the diluted triangular lattice

Violation of even the weak form of "central dogma" at low vacancy concentration:

Monomers delocalized in half the samples, localized to O(1) regions in the other half!

All samples identically prepared, randomly diluted, with the exact same density of vacancies

On the diluted triangular lattice

Suggests extreme sensitivity of large-scale geometry to micro-scale details of disorder configuration

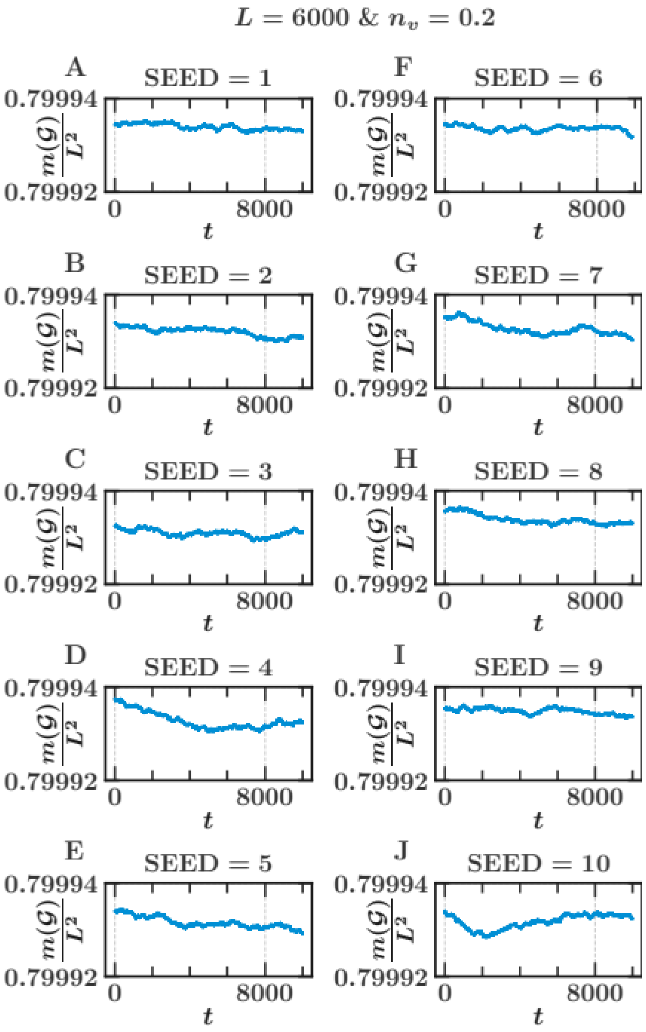
Can we quantify this?

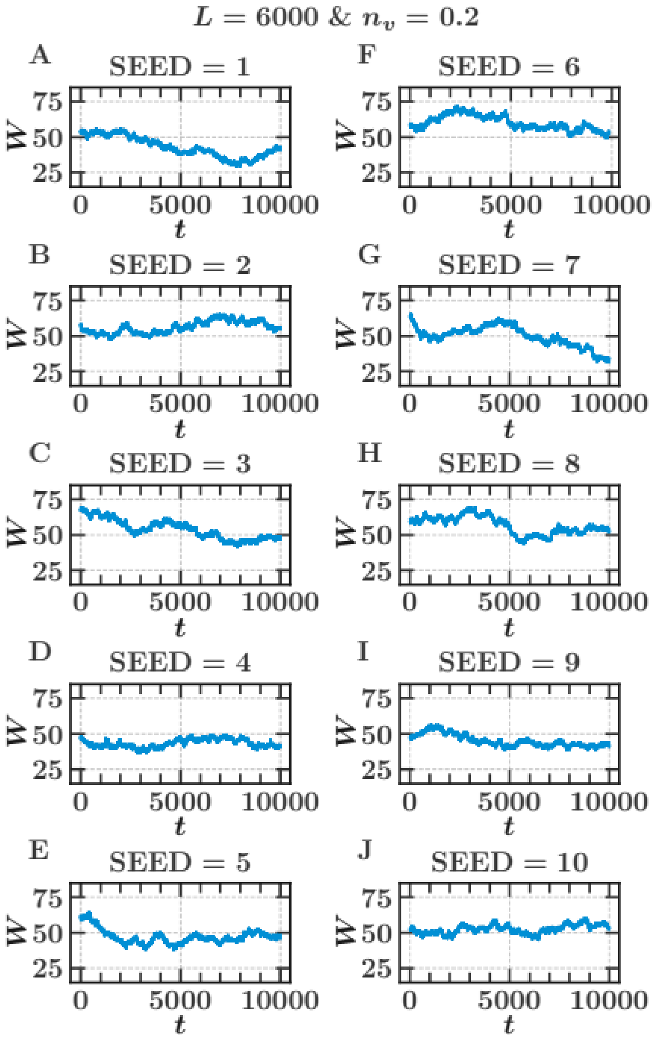
Model dynamics: Set vacancies in motion and watch what happens!

Small fraction of vacancies exchange position with neighboring surviving site at each time step

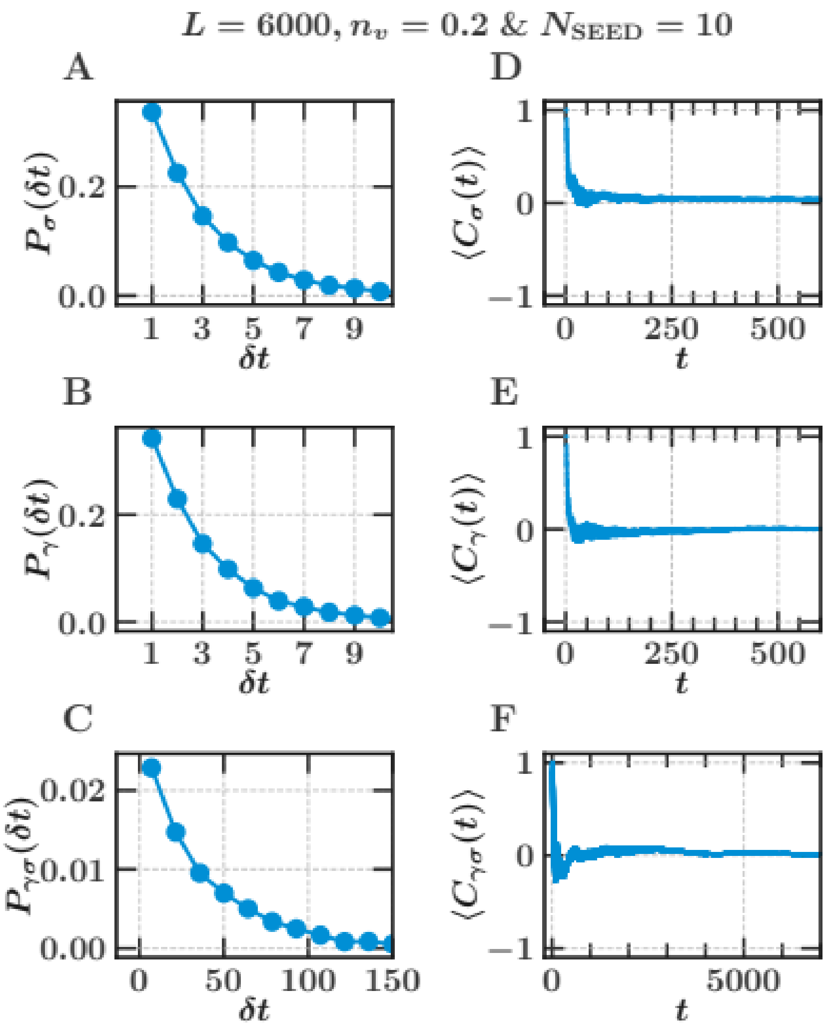
How does the large-scale geometry of these regions react?

Dynamics doesn't disturb underlying lattice much





Yet: Large-scale geometry of monomer-carrying/fully-packed regions responds chaotically



Some predictions for observable effects

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Likewise for weak vacancy disorder in triangular lattice Majorana spin liquids

On the theory side...

Similar phenomena on other non-bipartite lattices:

Checked: Percolation transition and low-dilution phase essentially the same on the Shastry-Sutherland lattice

A totally baroque phase diagram in three dimensions

Interesting from the vantage point of percolation theory

Exception: Site-diluted kagome lattice: A striking result

$w = 0$ in the thermodynamic limit of the diluted kagome lattice with nonzero vacancy density

Short-range RVB state stable to vacancy disorder on kagome lattice (!)

Generally true on all claw-free lattices (pyrochlore lattice, star lattice etc)

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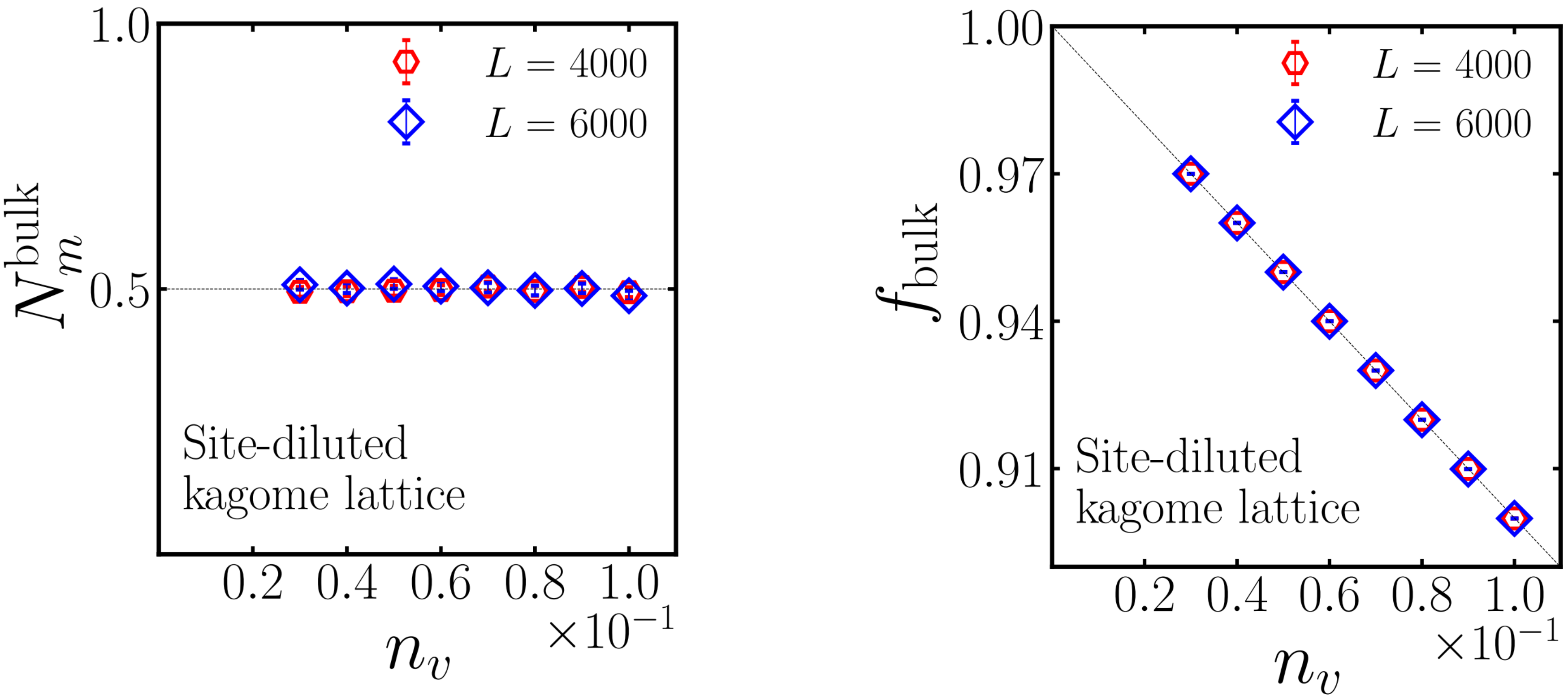
Bhola, KD, in preparation

Explicit computation:



Any maximum matching has at most 1 monomer in each connected component of lattice(!)

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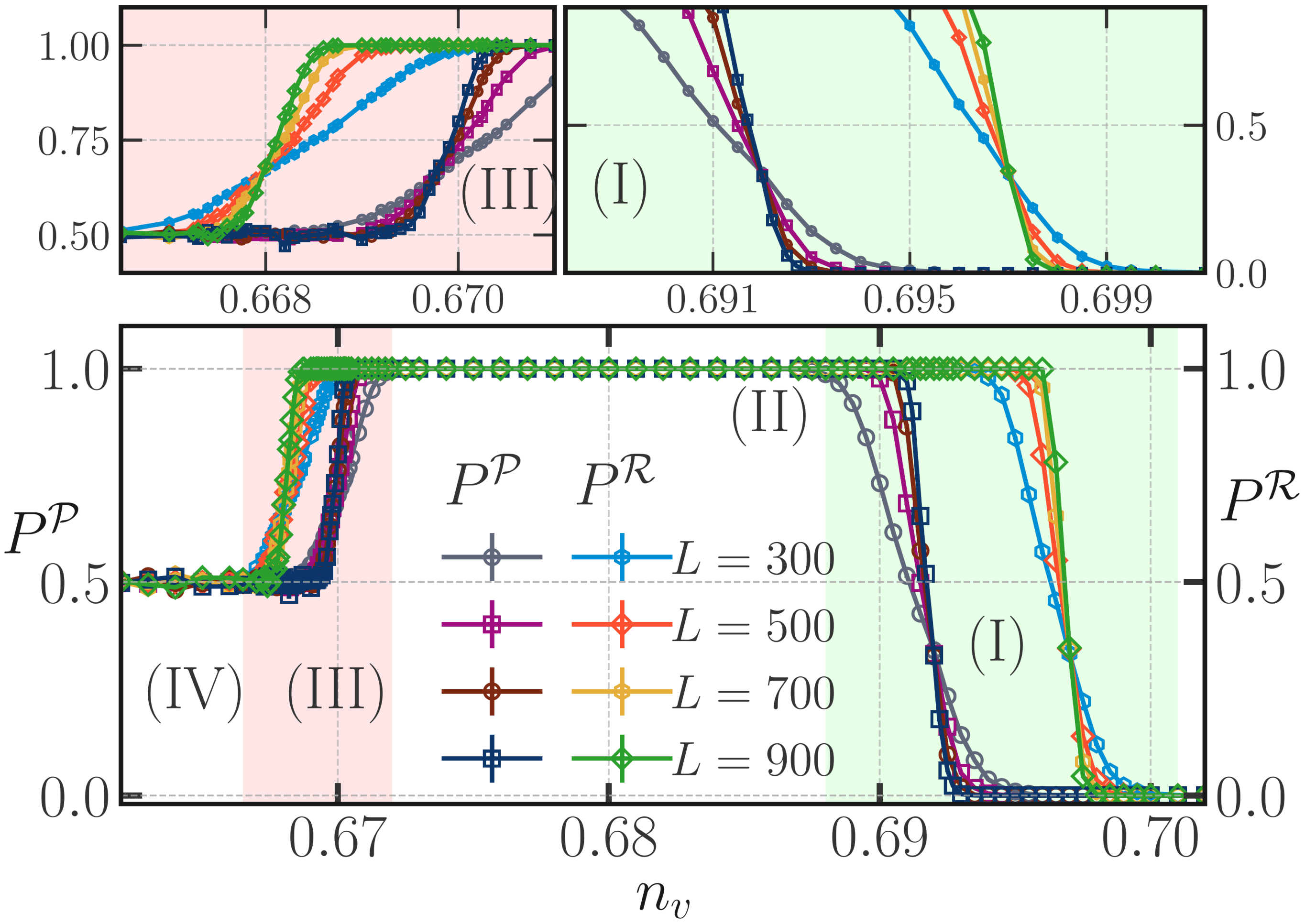
3D Phase diagram via wrapping probabilities



Acknowledgements

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Discussions on

vacancies in sRVB, VBS, & AFM states: S. Bhattacharjee, L. Balents, S. Sachdev, A. Sandvik

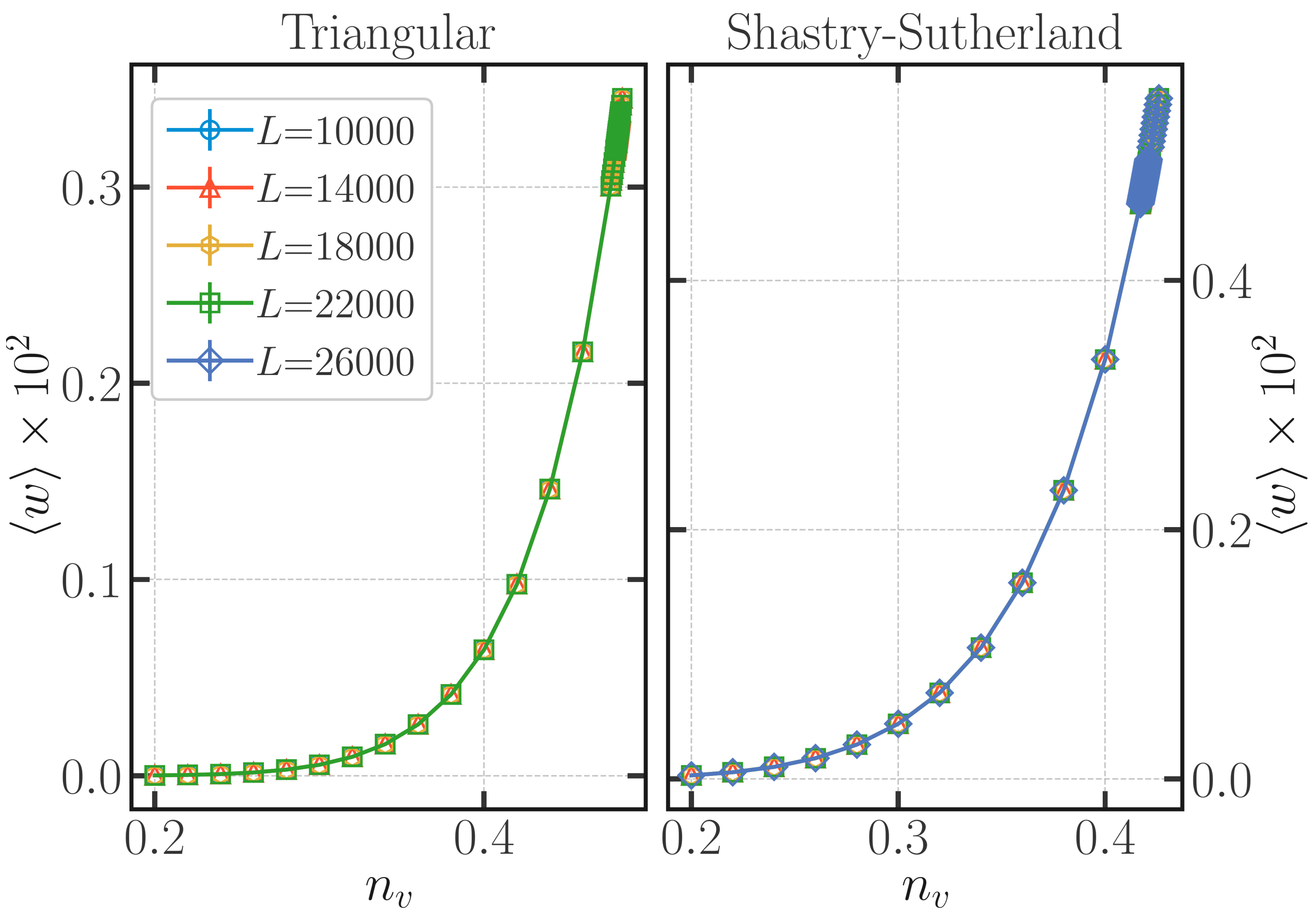
Percolation: Deepak Dhar, Subhajit Goswami

Other collaborations (2010-20) on disorder effects: Fabien Alet, Argha Banerjee, Sylvain Capponi, Pranay Patil, Arnab Sen

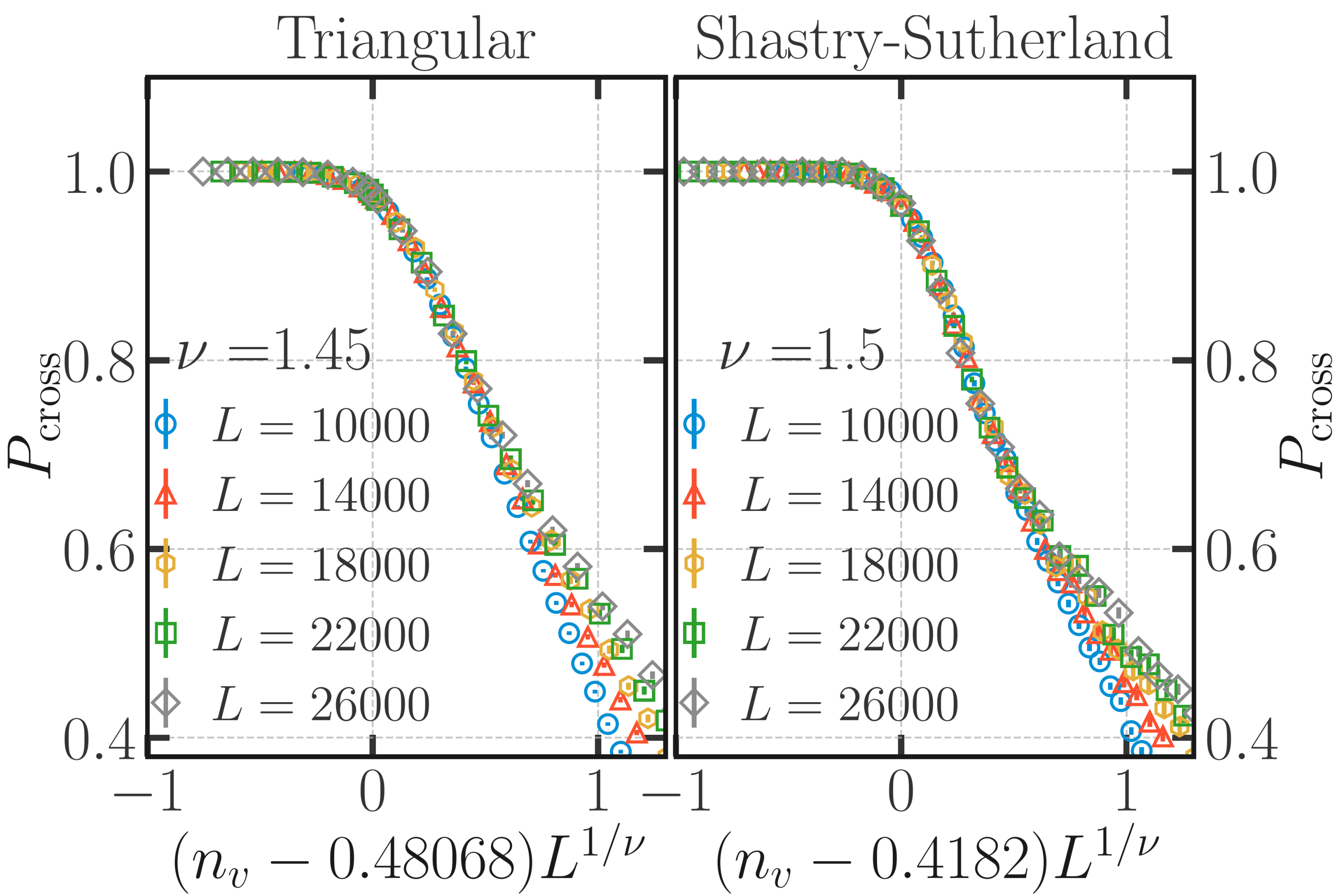
Computing cluster related: K. Ghadiali and A. Salve (DTP SysAds)

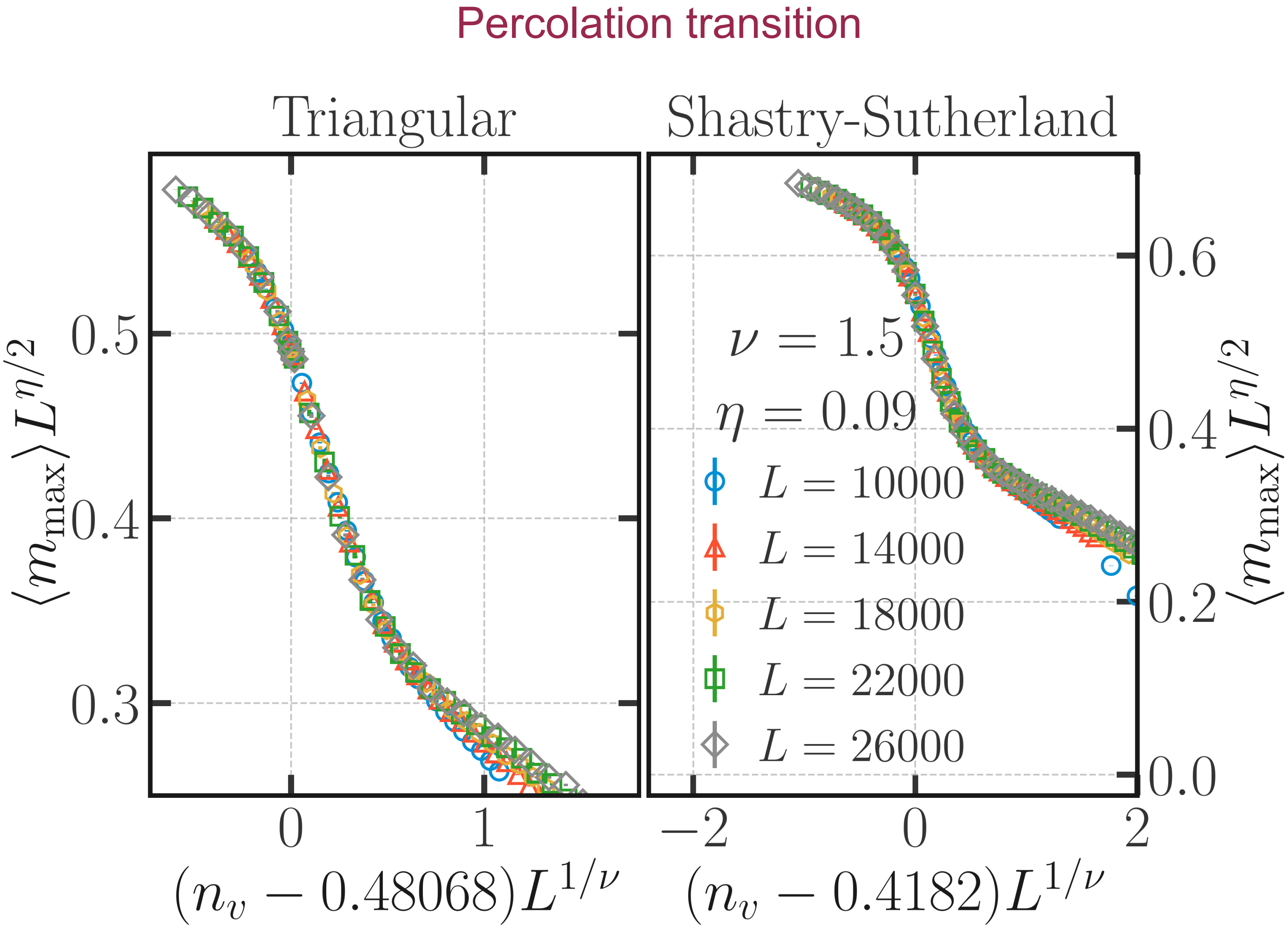
Supplementary information slides follow...

Monomer density



Percolation transition




Percolation transition
Triangular
Shastry-Sutherland
$\langle m_{\max} \rangle L^{\eta/2}$
0.5
0.4
0.3
0.6
0.2
0.0
$\nu = 1.5$
$\eta = 0.09$
$L = 10000$
$L = 14000$
$L = 18000$
$L = 22000$
$L = 26000$
0
1
−2
2
$(n_v - 0.48068)L^{1/\nu}$
$(n_v - 0.4182)L^{1/\nu}$

Quantum dimer model framework

Rokhsar and Kivelson: Effective Hamiltonian living in subspace of singlets spanned by nn VB

$$H_{QDM} = -t\left(|=\rangle\langle \parallel | + | \parallel \rangle\langle = | \right) + \ldots$$

More generally: Ring-exchange kinetic terms on “flippable” plaquettes, and local interactions

Additional terms incorporate the effect of matrix elements to further-neighbor singlet states

Z2 spin liquid example: Triangular QDM

Triangular lattice: Moessner-Sondhi (within QDM framework):

Triangular lattice QDM has truly quantum disordered phase

Short-range spin correlations, valence bond correlations, genuine Z2 spin liquid

(also for kagome lattice)

Monomers correspond to “emergent” local moments in spin system

Each monomer corresponds to a disorder-induced “emergent” local moment
(purely kinematic effect, independent of VBS vs RVB nature of ground state)

Signature: Large intermediate temperature range with Curie tail in susceptibility

Quenched below scale set by residual interactions

$$\chi_{\mathrm{imp}} \sim \frac{\mathcal{C}}{T} \text{ for } J_{\mathrm{eff}} \ll T \ll J$$

$$\mathcal{C} \propto w$$

But wait: This conclusion seems to rely too much on having only nearest-neighbor singlets?
Does it hold for more generic short-range RVB liquid?

To answer: large-N route to quantum dimer model

$$
\begin{aligned}
H &= J \sum_{\langle rr' \rangle} \vec{S}_r \cdot \vec{S}_{r'} + \ldots \\
&= -J \sum_{\langle rr' \rangle} \left(\mathcal{P}_{rr'} - \frac{1}{4} \right) + \ldots
\end{aligned}
$$

Enlarge symmetry group:

$$
H = -\frac{J_m}{N} \sum_{\langle r_1 r_2 \rangle} \sum_{\alpha,\beta=1}^{N} |\alpha\rangle_{r_1} |\alpha\rangle_{r_2} \langle\beta|_{r_1} \langle\beta|_{r_2} + \cdots \ ,
$$

Affleck, Read, Sachdev, Auerbach, Coleman, Sandvik, Kawashima, Beach, Kaul...(1988 - now)

What's the enlarged symmetry?

$$\begin{aligned}
\mathcal{A}_{\alpha\beta}(r) &= -i(|\alpha\rangle_r\langle\beta|_r - |\beta\rangle_r\langle\alpha|_r) \;\; \forall \;\; \text{pairs} \;\; \alpha < \beta \\
\mathcal{S}_{\alpha\beta}(r) &= (|\alpha\rangle_r\langle\beta|_r + |\beta\rangle_r\langle\alpha|_r) \;\; \forall \;\; \text{pairs} \;\; \alpha < \beta \\
\mathcal{Q}_{\alpha\alpha}(r) &= (|\alpha\rangle_r\langle\alpha|_r - 1/N) \;\; \forall \;\; \alpha = 1 \ldots N-1
\end{aligned}$$

$$\mathcal{A}^{\text{tot}}_{\alpha\beta} = \sum_r \mathcal{A}_{\alpha\beta}(r)$$

O(N) symmetry on any arbitrary lattice

Bipartite case: Enhanced "staggered" SU(N) symmetry

$$\mathcal{S}^{\text{tot}}_{\alpha\beta} = \sum_r (-1)^r \mathcal{S}_{\alpha\beta}(r)$$

$$\mathcal{Q}^{\text{tot}}_{\alpha\alpha} = \sum_r (-1)^r \mathcal{Q}_{\alpha\alpha}(r)$$

Large N limit in pure case

Any perfect (fully packed) dimer cover is a ground state (each dimer interpreted as singlet state)

Leading 1/N corrections: Captured precisely by QDM Hamiltonian with ring-exchange

Higher orders in 1/N: Additional local terms in QDM Hamiltonian

(Affleck, Read, Sachdev, Kaul...)

Recover the same QDM framework---without nearest-neighbor singlet assumption.

Large-N limit in disordered case

Any maximum matching now gives a large-N ground state

1/N corrections: QDM Hamiltonian with ring-exchange + monomer kinetic energy terms

Higher orders in 1/N: Additional local terms in QDM Hamiltonian

Presumably: residual interactions between local moments...(?)

Maximally-packed QDM Hamiltonian valid description of disorder effects in short-range RVB liquid

Key claims that need computational test

Isolated vacancies do not seed local moments in RVB states, but do so in VBS states.

Monomer-carrying regions of lattice correspond to local moments in both kinds of states

Primer: Computational tests

O(N) models on non-bipartite lattices, SU(N) models on bipartite lattices

Ideal unified test: $\chi^{\mathcal{A}}$ (runs into computational difficulties)

For SU(N) systems, equivalent to checking: $\chi^{\mathcal{Q}}$

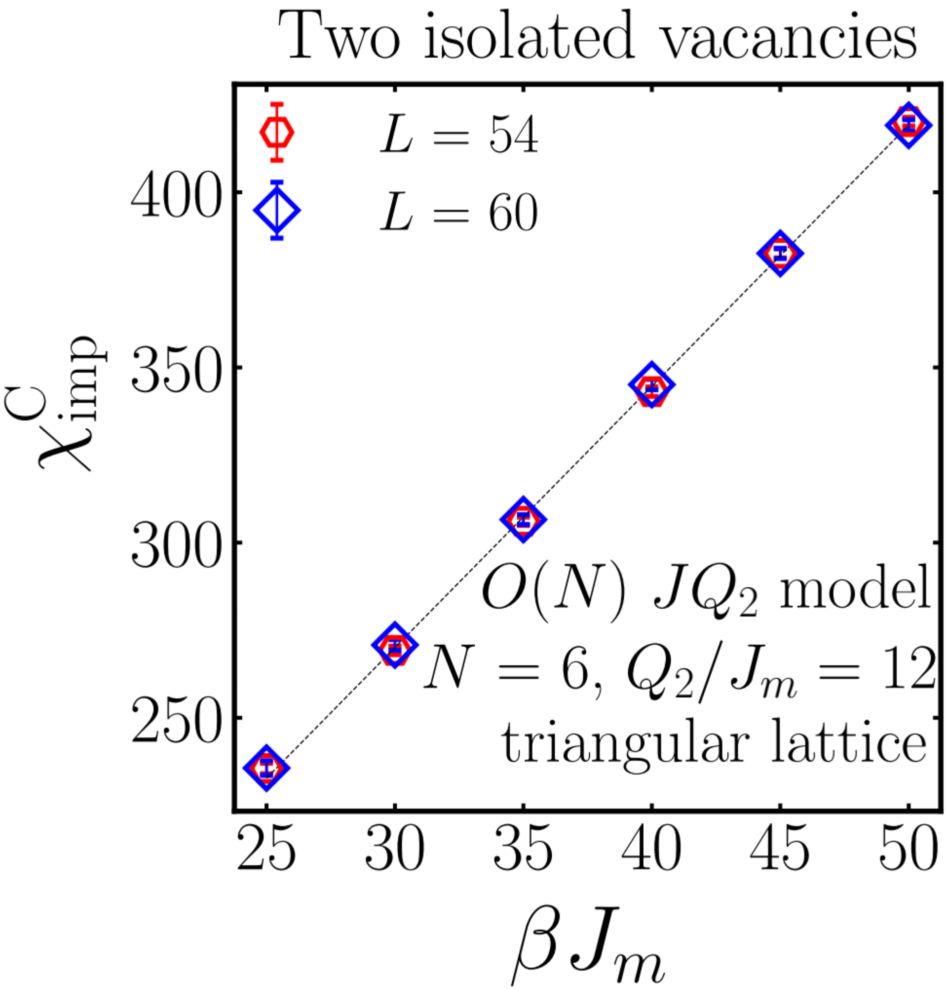
This is not defined for nonbipartite O(N) models

For O(N) systems, can instead check: χ^{C}

$$C^{\text{tot}}_{\alpha\alpha} = \sum_r \mathcal{Q}_{\alpha\alpha}(r)$$

expected to be equivalent for $J_m \gg T \gg J_{\text{eff}}$

Test results (nonbipartite): VBS state



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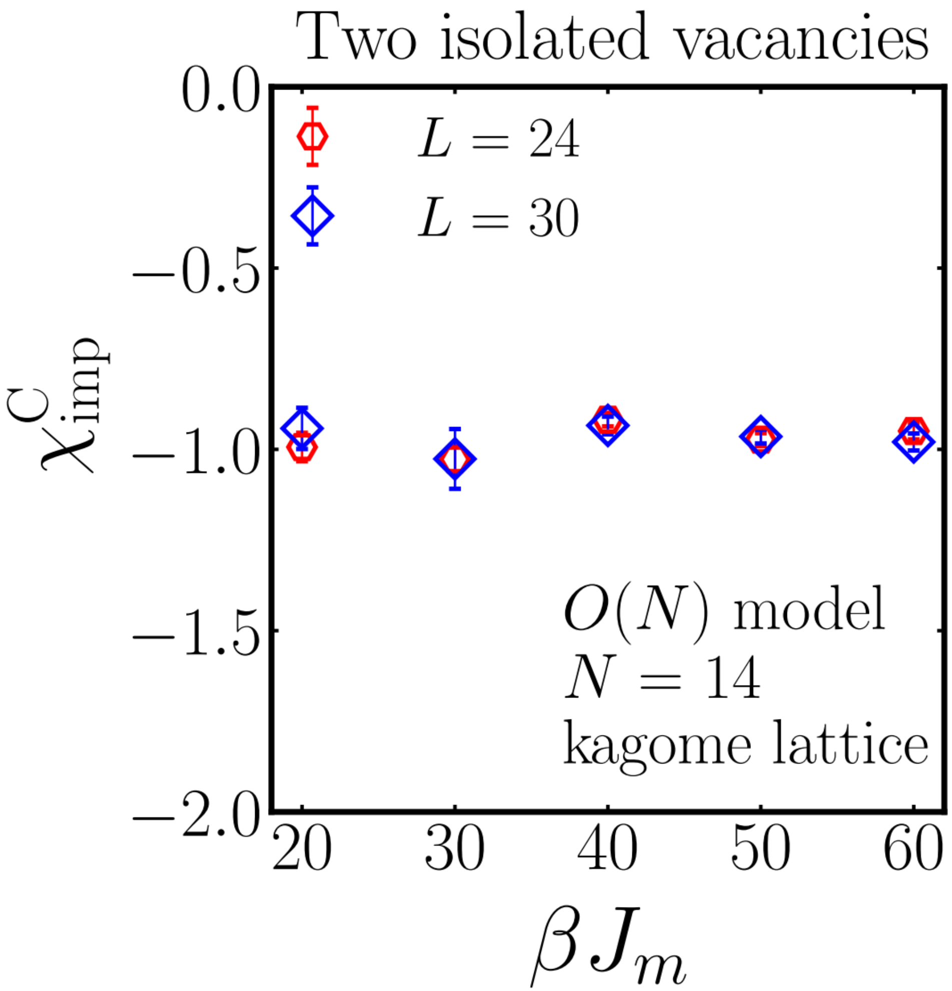

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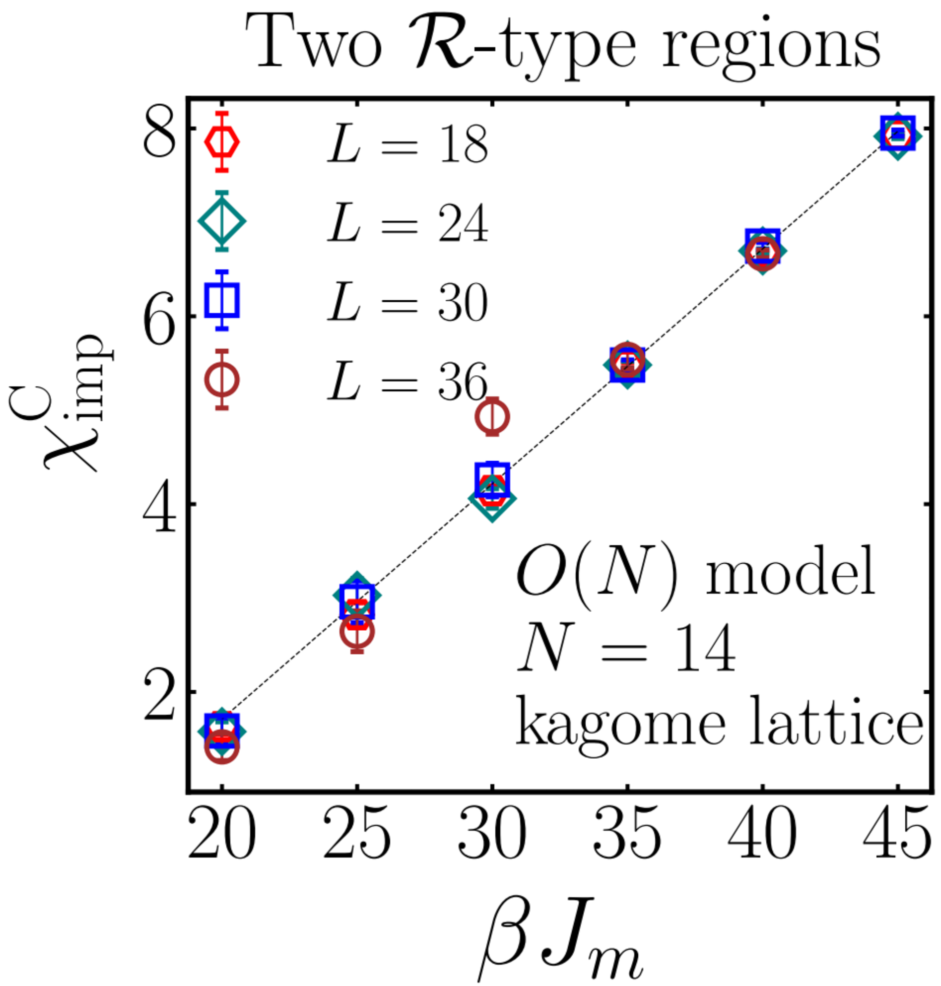

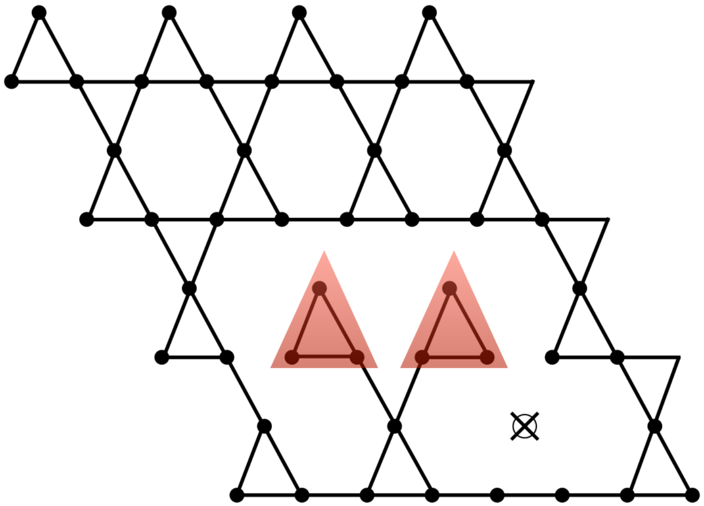

Note: deleted bonds, not sites

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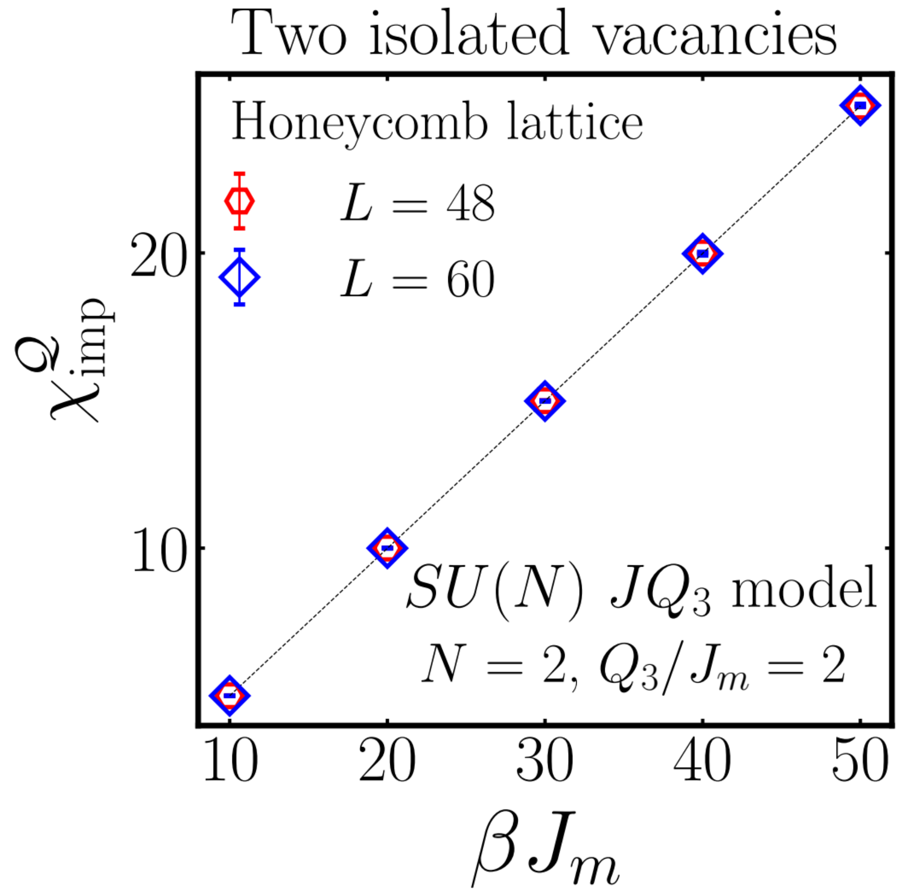
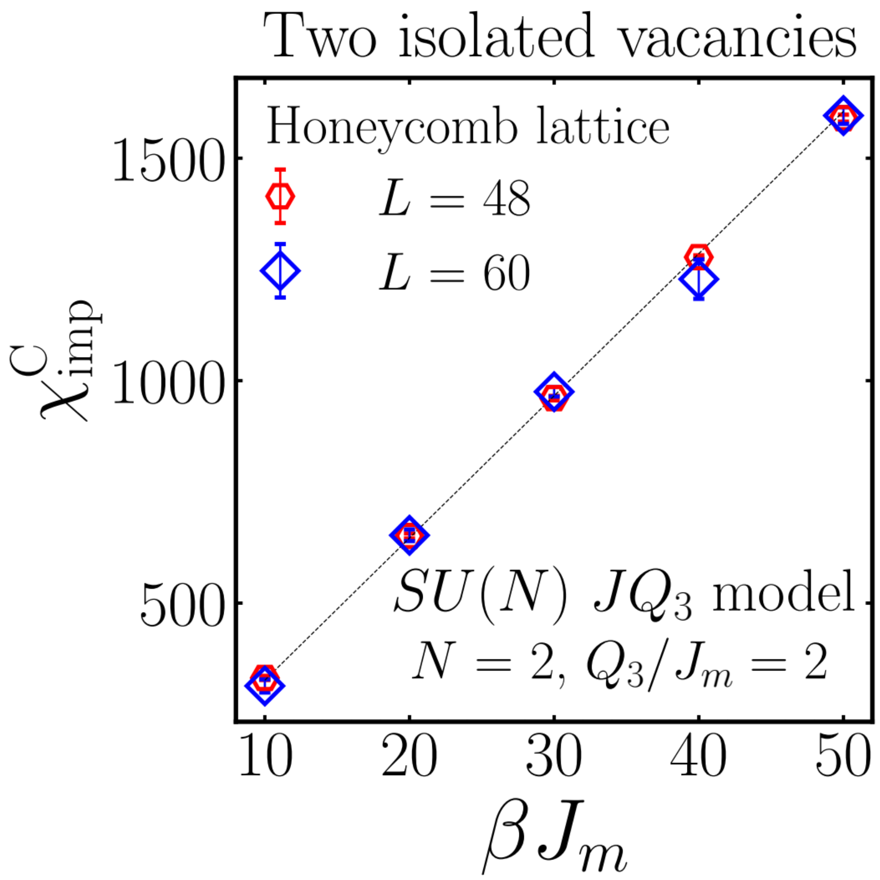
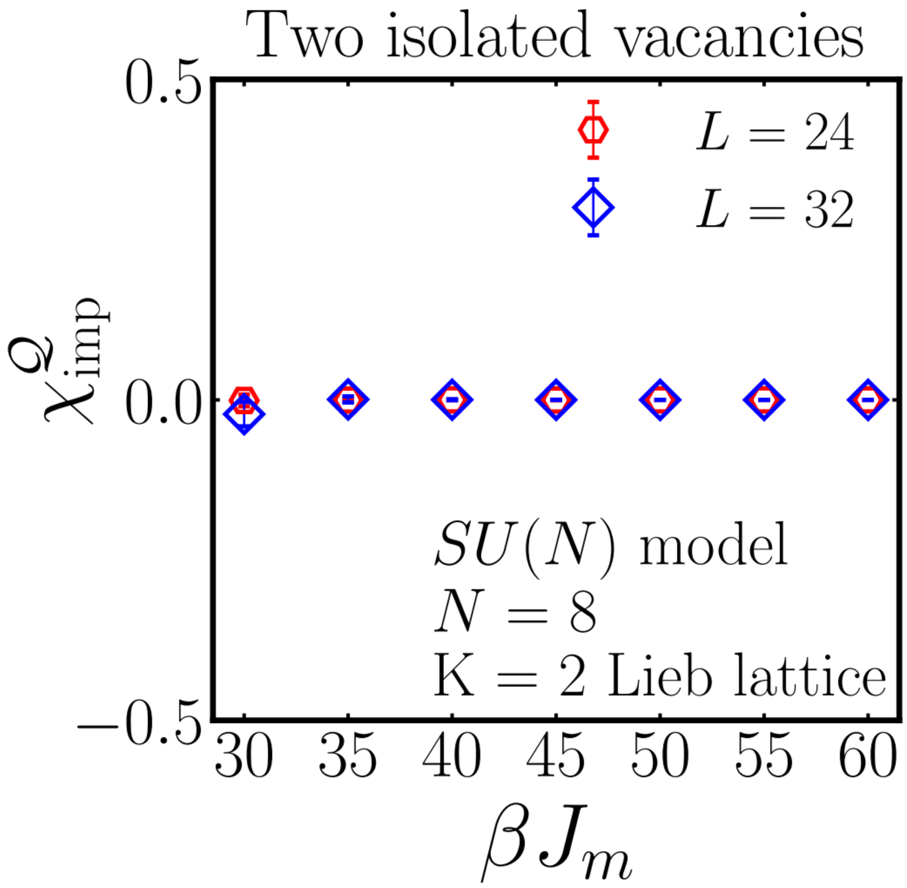
Test results: VBS state (bipartite)

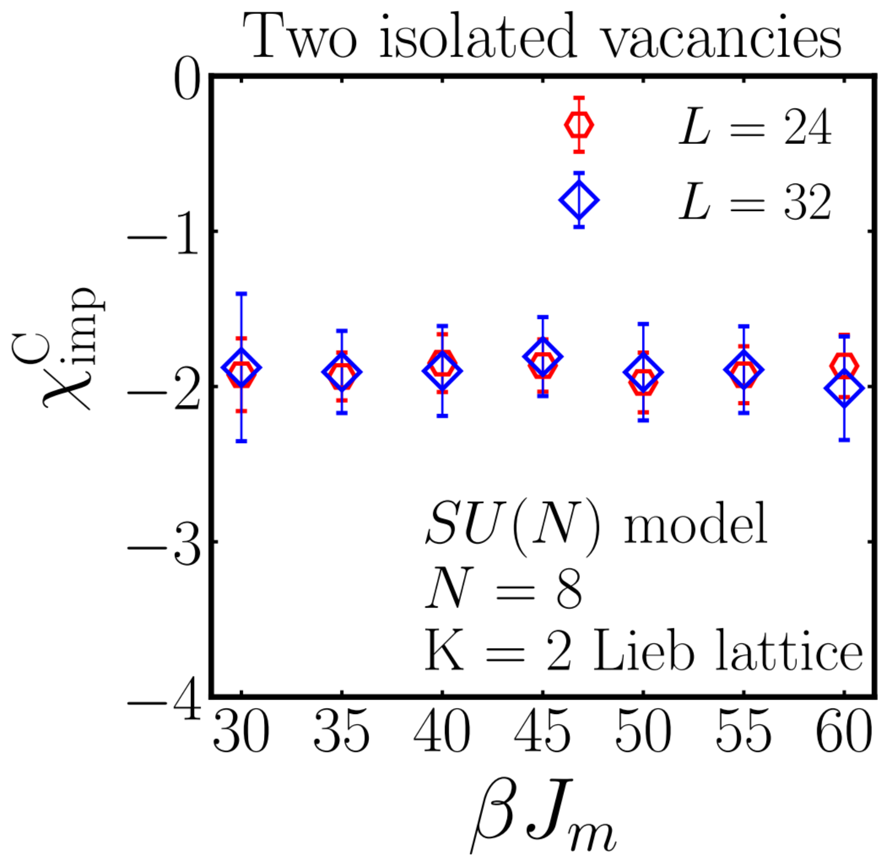




Ansari, Kundu, KD, in preparation

Test results: RVB state (bipartite)





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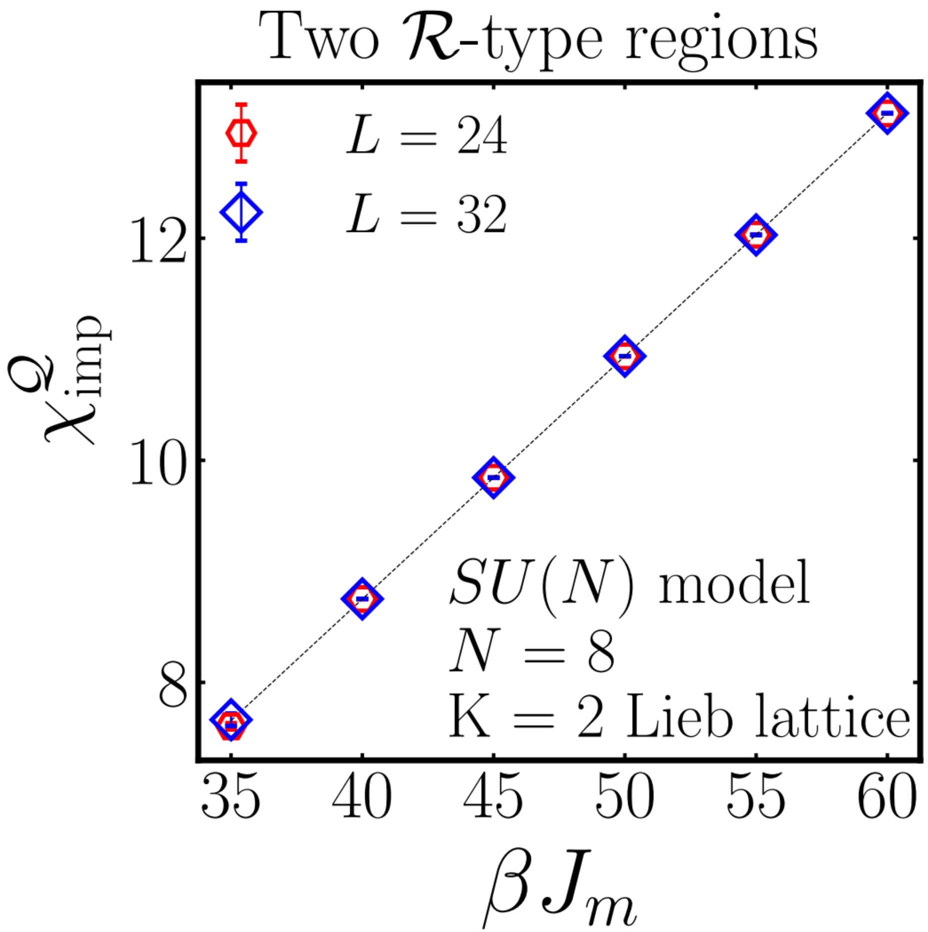

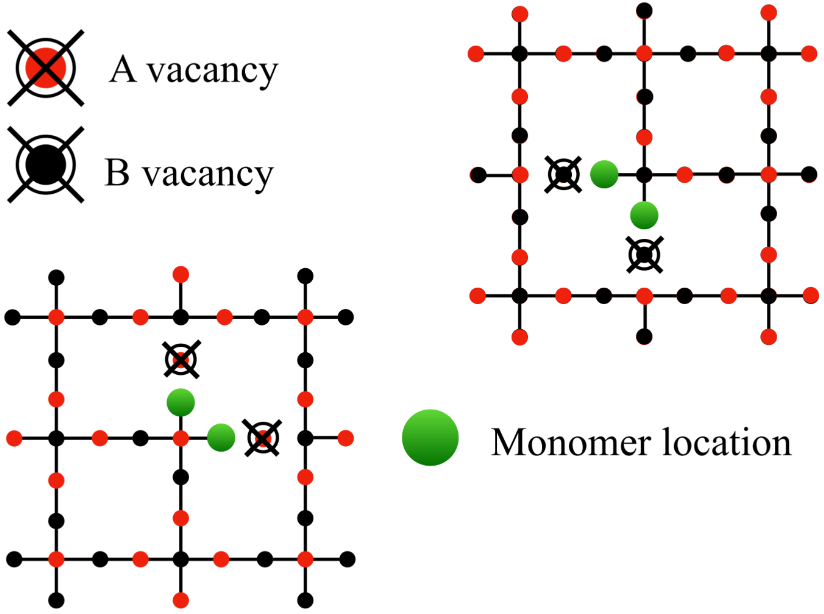

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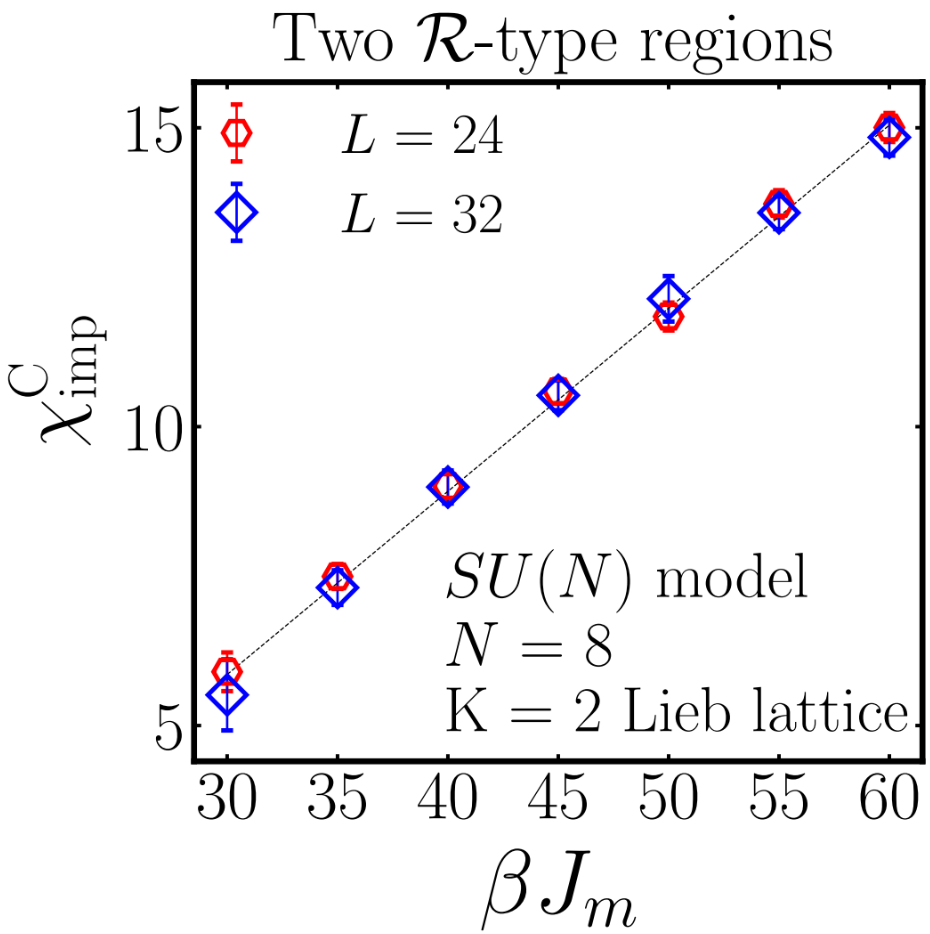
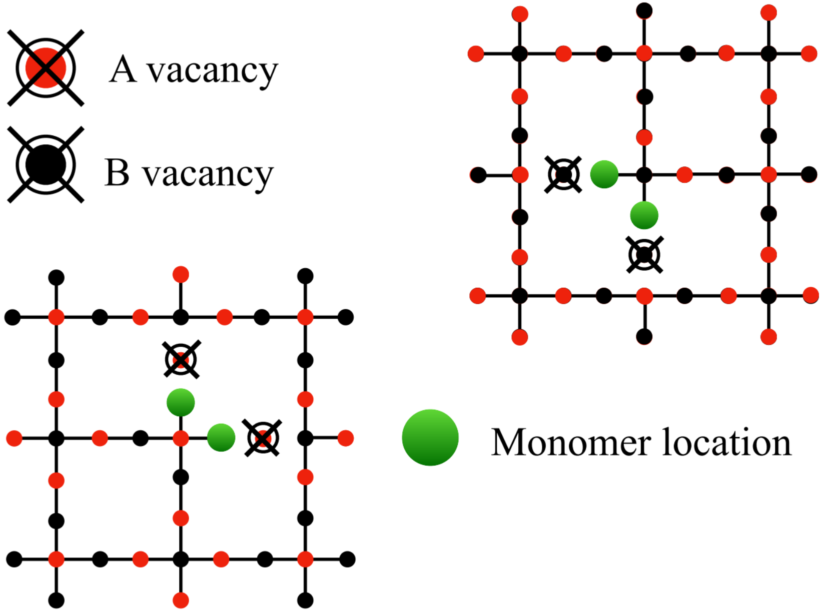
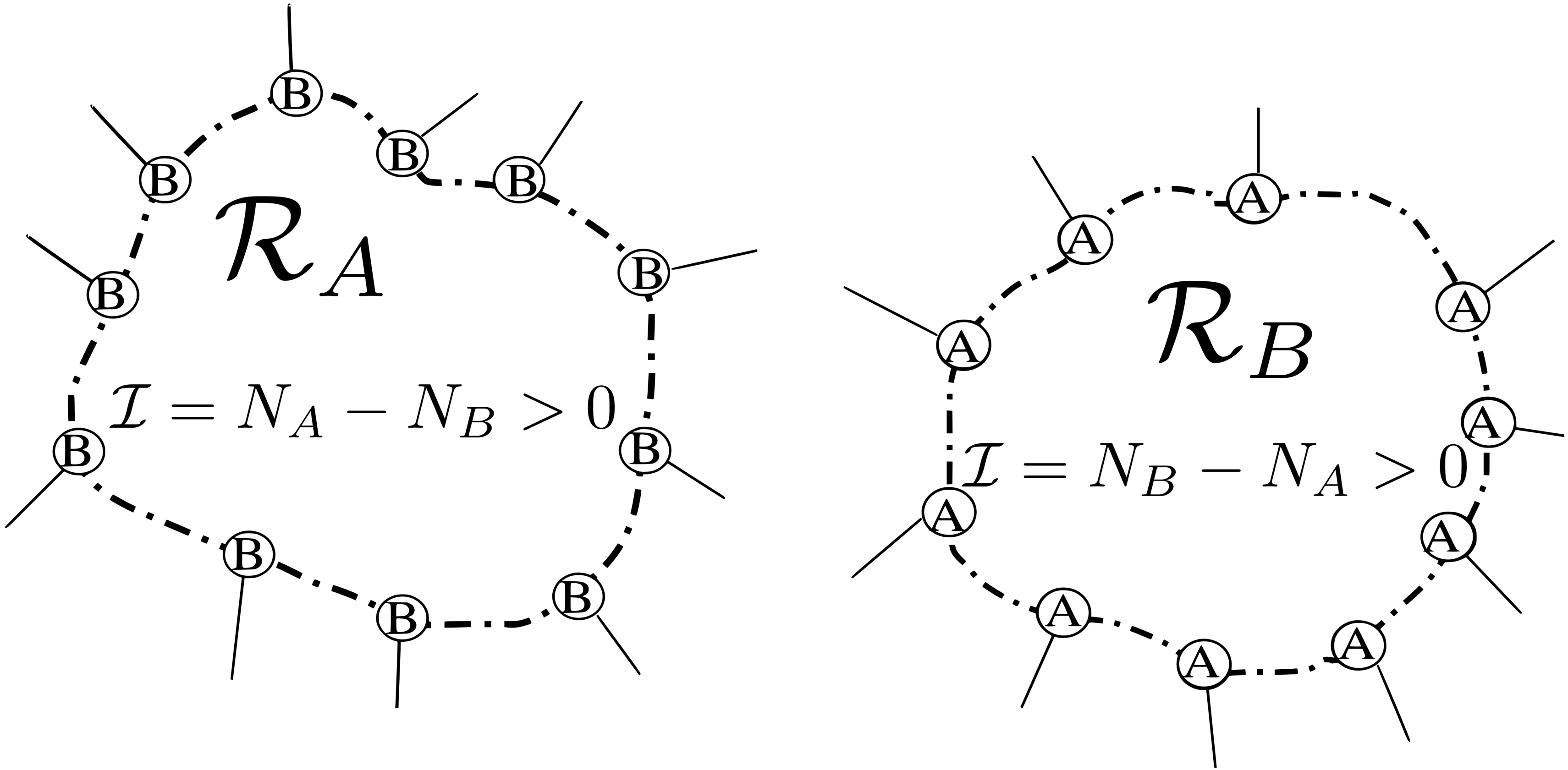




Ansari, Kundu, KD, in preparation

R-type regions in bipartite case: General mechanism

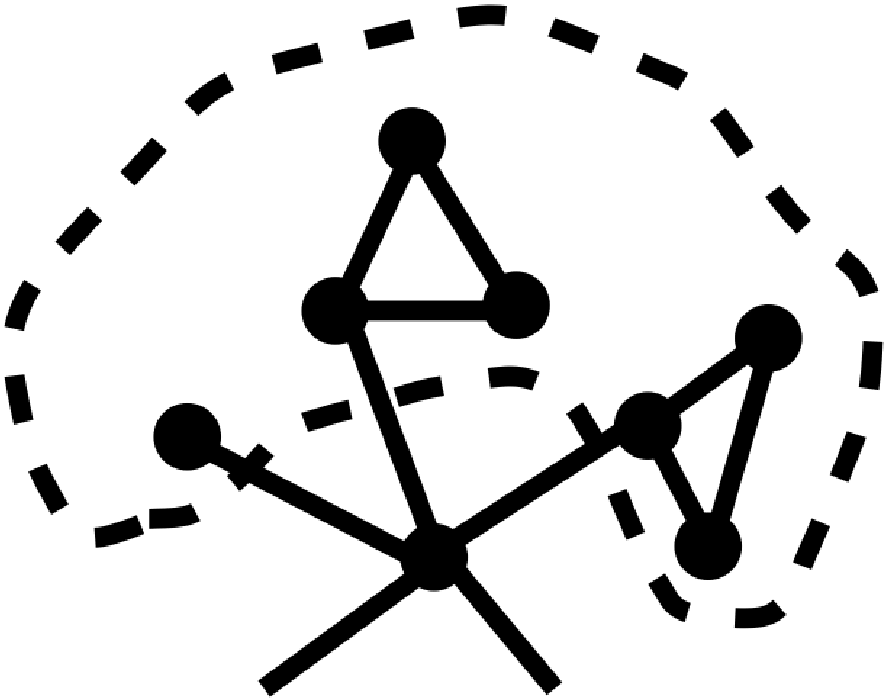


These regions traps $\mathcal{I}$ monomers each (local statement)

Sanyal, KD, Motrunich PRL 2016

Bhola, Biswas, Islam, KD, PRX 12, 021058 (2022)

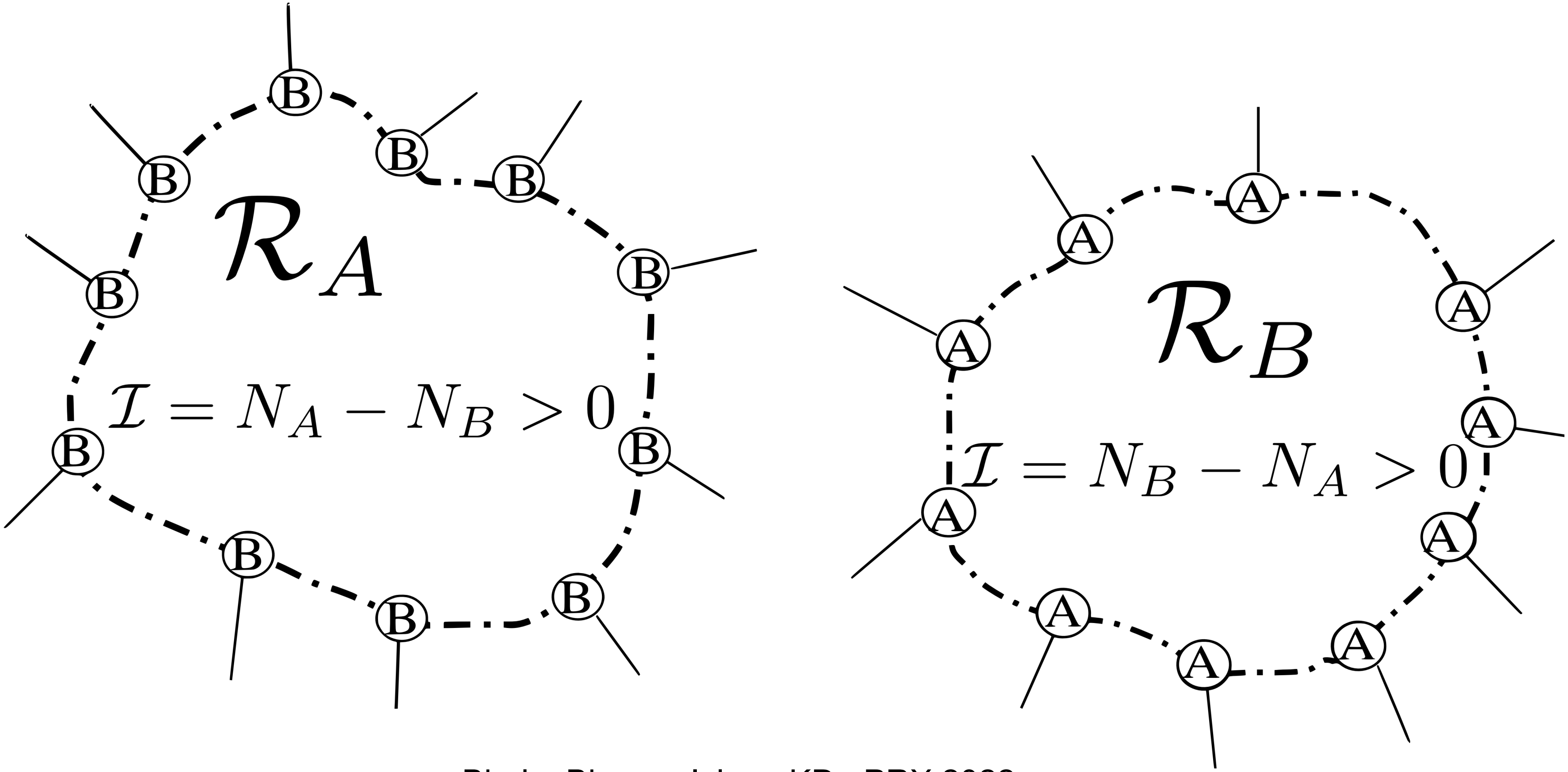
R-type regions in nonbipartite case: General mechanism



This region traps two monomers (local statement)

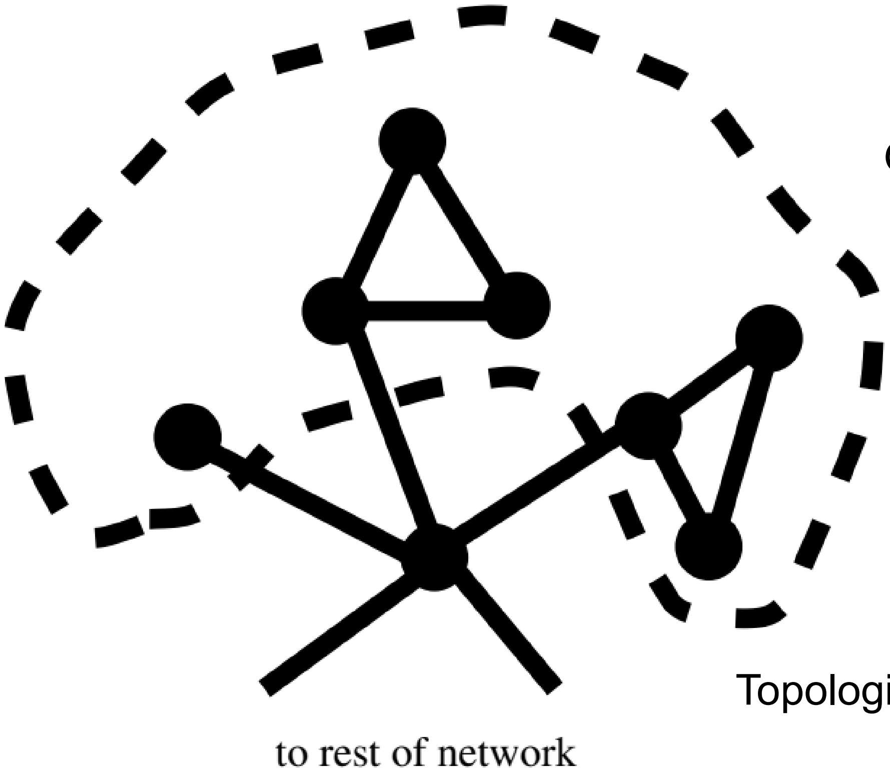
KD, PRB 2022

Aside: R-type regions host topologically-protected zero modes (bipartite case)



Bhola, Biswas, Islam, KD, PRX 2022

Aside: Basic picture for collective Majorana modes (general case)



Odd cyles in isolation guaranteed to have zero mode.

Many such isolated modes mix inside “R-type” region

Topologically protected collective Majorana modes survive

KD, PRB 2022

Gels well with a theorem of Lovasz

ON DETERMINANTS, MATCHINGS, AND RANDOM ALGORITHMS

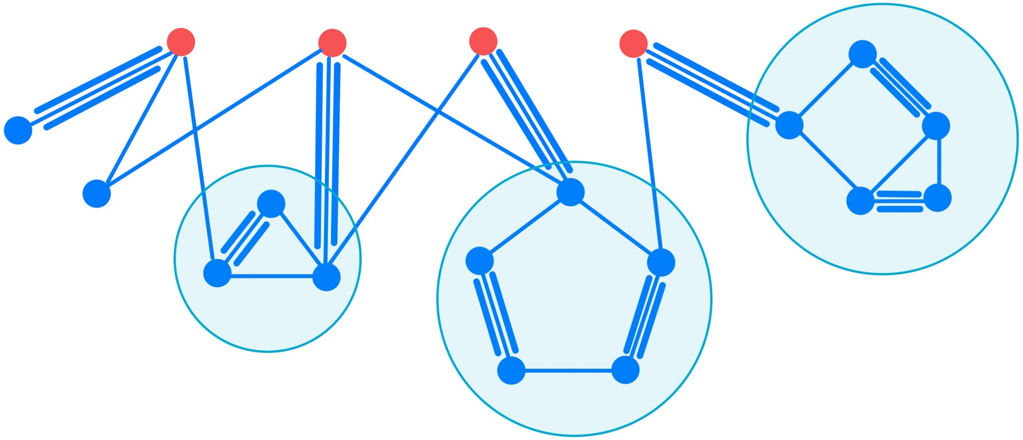
by L. Lovász*

Fund. Comp. Th. 1979

Monomer number = number of topologically protected zero modes of $\mathcal{A}_{rr'}$

Constructing R-type regions in general case

- Each blossom hosts 1 (would-be) monomer.
- Number of monomers in each R-type region of auxillary bipartite graph fixed, independent of maximum matching



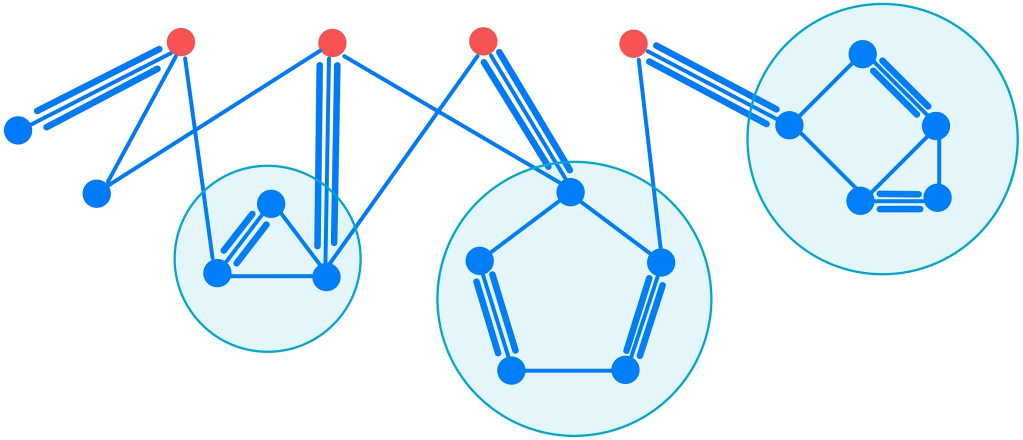
R-type region in bipartite auxiliary graph

KD, PRB 2022

Aside: also constructs the zero modes

Alternate “local” proof of Lovasz’s Thm:

- Each blossom hosts 1 (would-be) mode.
- Number of monomers in each R-type region of auxillary bipartite graph fixed, determines number of collective zero modes.



R-type region in bipartite auxiliary graph

KD, PRB 2022