Gallai-Edmonds percolation

Kedar Damle, Tata Institute (TIFR)

Based on:

Bhola & KD, preprint

KD, PRB 105 235118 (2022)

Bhola, Biswas, Islam, KD, PRX 12 021058 (2022)

Ritesh Bhola

Sounak Biswas

Mursalin Islam

Sambuddha Sanyal

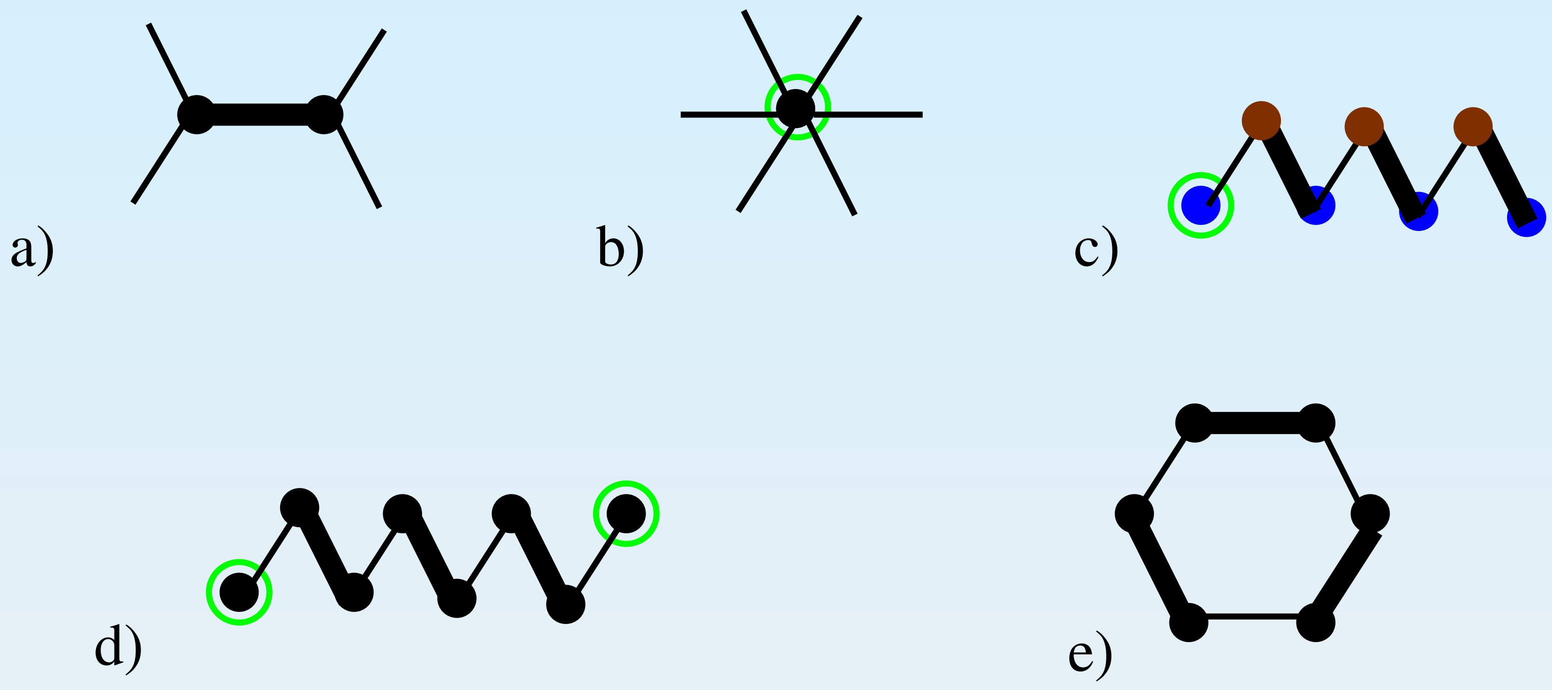
earlier work: Sanyal, KD, Motrunich, PRL 117, 116806 (2016)

Lesik Motrunich

What do we study?

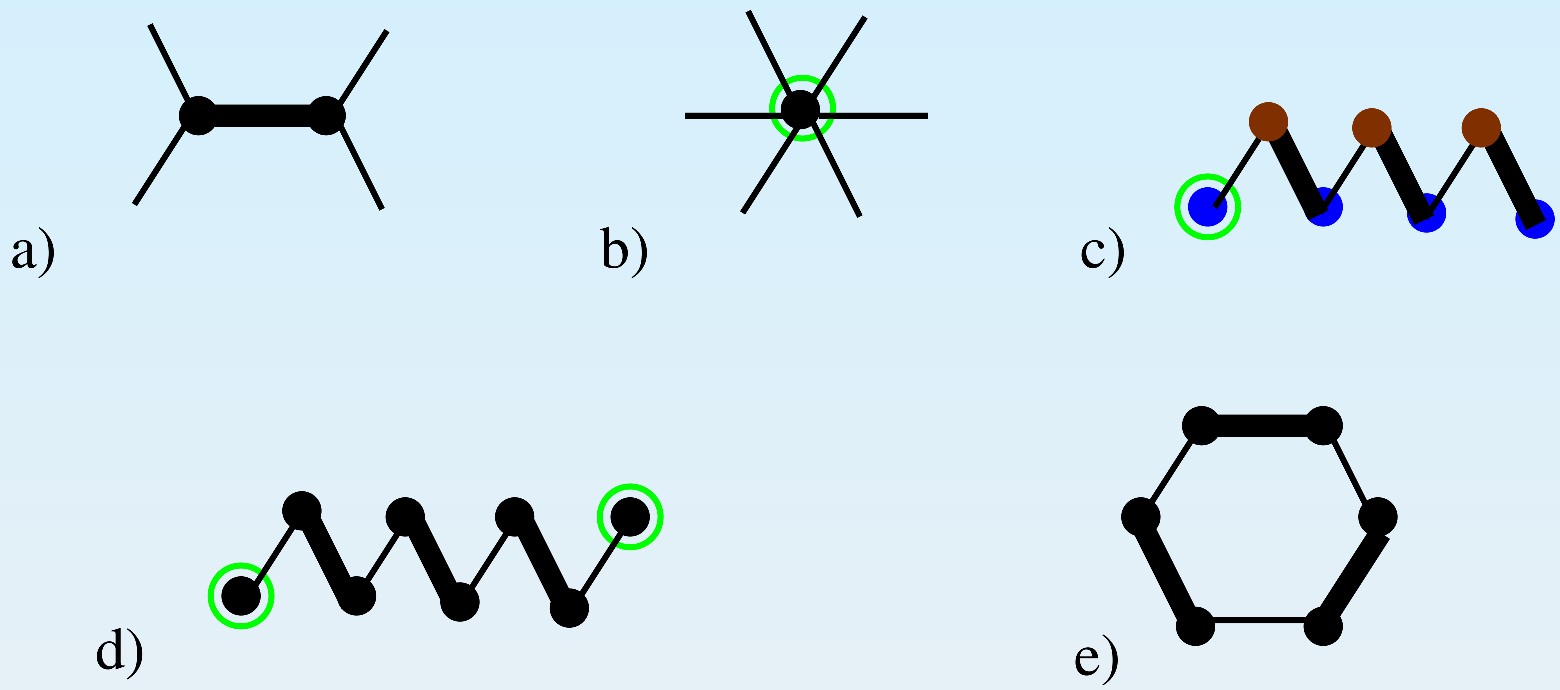
- Maximally packed dimer models on diluted lattices

Glossary



- Vacancy disorder (quenched): Randomly delete sites of a lattice
- Matchings: attempt to pair each vertex with exactly one neighbour via hard-core dimers on link.
- Unmatched vertex: Location of monomer

Glossary



- Maximum matching aka maximally packed dimer cover: Leave as few vertices unmatched as possible.
- Alternating and augmenting paths
- Matching maximum iff no augmenting path
- Perfect matching aka fully packed dimer cover: No vertex left unmatched
- Ensemble of maximum matchings: Maximally-packed dimer model

The questions:

- Number of monomers in the maximally packed dimer model (as function of vacancy density)?
- Where do these monomers live?

Why ask these questions?

Answers have implications for:

- Particle-hole symmetric quantum percolation at band center (in bipartite case)
- Collective Majorana fermion excitations of networks of localised Majorana modes
- Vacancy-induced local moments in short valence-bond spin liquids
- Many-body localized phases of quantum dimer models with vacancy disorder.
- Nonmagnetic impurity induced Curie tails in Kitaev magnets

What do we find?

- Percolation phenomena of monomer-carrying “R-type” regions (deep within geometrically percolated phase of host lattice)
- Sublattice symmetry breaking transition within percolated phase (in d=3 bipartite case)
- Unusual “zero-half” threshold behaviour at percolation transition (in non bipartite case)

Some generalities:

- Quenched disorder matters (often)
- Particles scatter and diffuse (may be anomalously...)
- Matter-waves scatter and localise (sometimes weakly...)

Classical transport of fluid

- Simplest setting: Porous random medium
- Random geometry of medium determines fluid transport
- Paradigm of percolation
- More generally: Diffusion in presence of random potentials

Localization of matter waves

- Anderson localization of electrons in dirty metals
- Localization of quasiparticles in dirty superconductors
- Symmetries of disordered Hamiltonian matter (e.g. in random matrix theory)

Anderson, Ramakrishnan, Abrahams, Thouless, Dyson, Wegner, Mehta...

Simplest lattice model: Disordered tight-binding Hamiltonian

- $\sum_{j \in i} t_{ij}\psi_j + V_i\psi_i = \epsilon\psi_i$ for all i
- ϵ is the energy of the particle described by wave function ψ_i
- t_{ij} are 'hopping amplitudes' for particle to hop from site i to site j
- V_i are values of external potential at sites i
- Allowed ϵ : Eigenvalues of matrix of $t_{ij} + V_i\delta_{ij}$

Vacancy disorder

- Random dilution of the underlying lattice
- Models missing atoms in crystal structure
- Also natural if substitutional impurities correspond to missing orbital (binary alloys)

Quantum percolation

- Anderson localisation meets geometric percolation (Kirkpatrick-Eggarter ’72, Shapir-Aharony-Harris ’82…)
- Vacancy disorder
- No external random potentials
- Can the quantum electron fluid be localised even when the corresponding classical fluid diffuses from end to end?

Simplest case: bipartite lattice with hopping and vacancy disorder

- Particle hopping on a randomly diluted bipartite lattice (binary alloy)
- (Possibly random) hopping amplitudes between nearest neighbour sites
- Bipartite symmetry: State with energy ϵ has partner at energy $-\epsilon$
- Symmetry broken by random potentials, next-neighbour hopping — left out here.

The “Gade-Wegner” problem

- $\epsilon = 0$ is special
- What is the asymptotic low-energy behaviour of $\rho(\epsilon)$?

“Gade-Wegner problem”

Hopping disorder: Singular tail of low-energy states. DOS has ‘modified Gade-Wegner’ scaling.

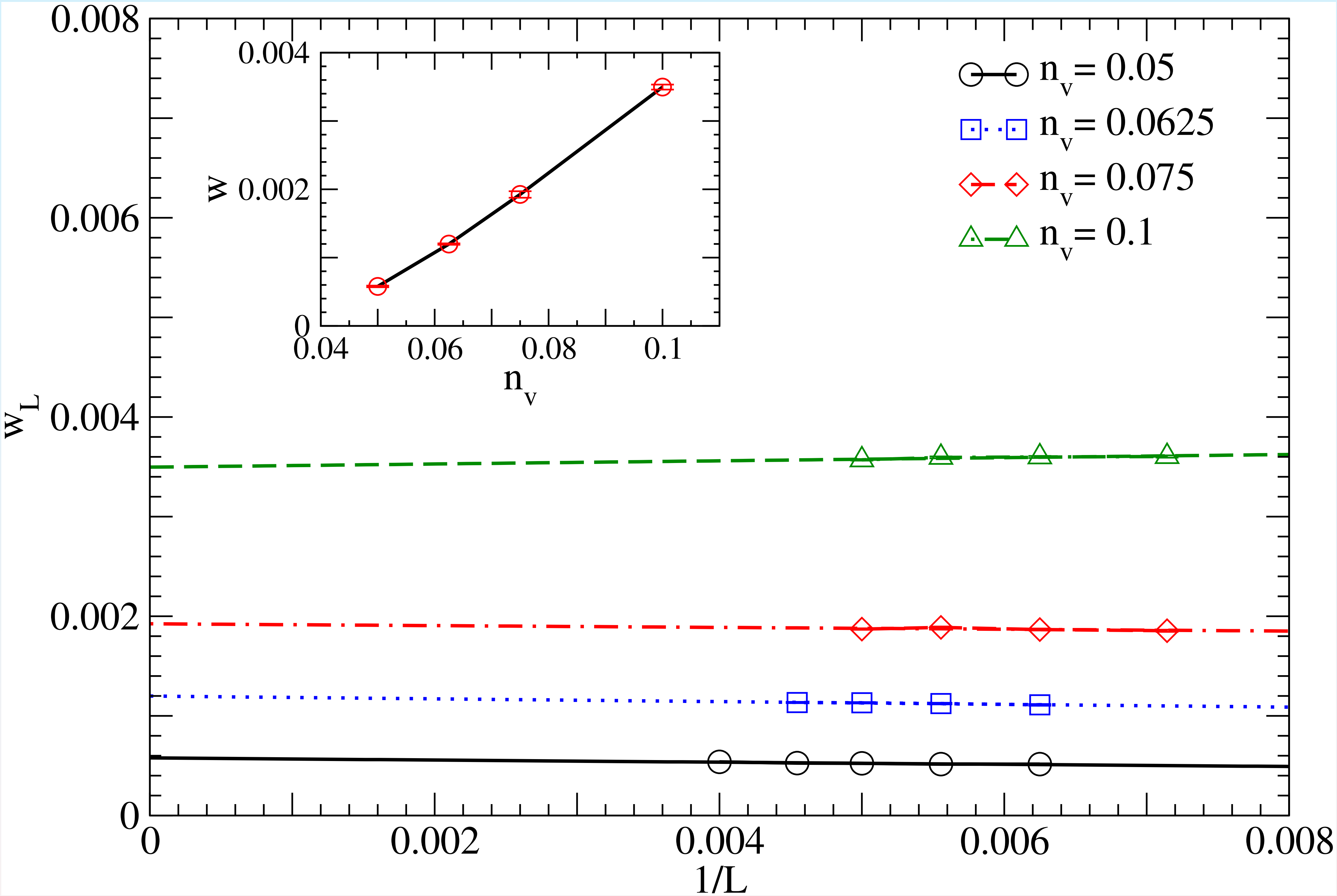
(Gade-Wegner ’91, Motrunich, KD, Huse ’02, Mudry-Ryu-Furusaki ’03)

Vacancy disorder: **Very slow crossover** to ‘modified Gade-Wegner’ scaling **+ zero mode density**

(Sanyal, KD, Motrunich ’16)

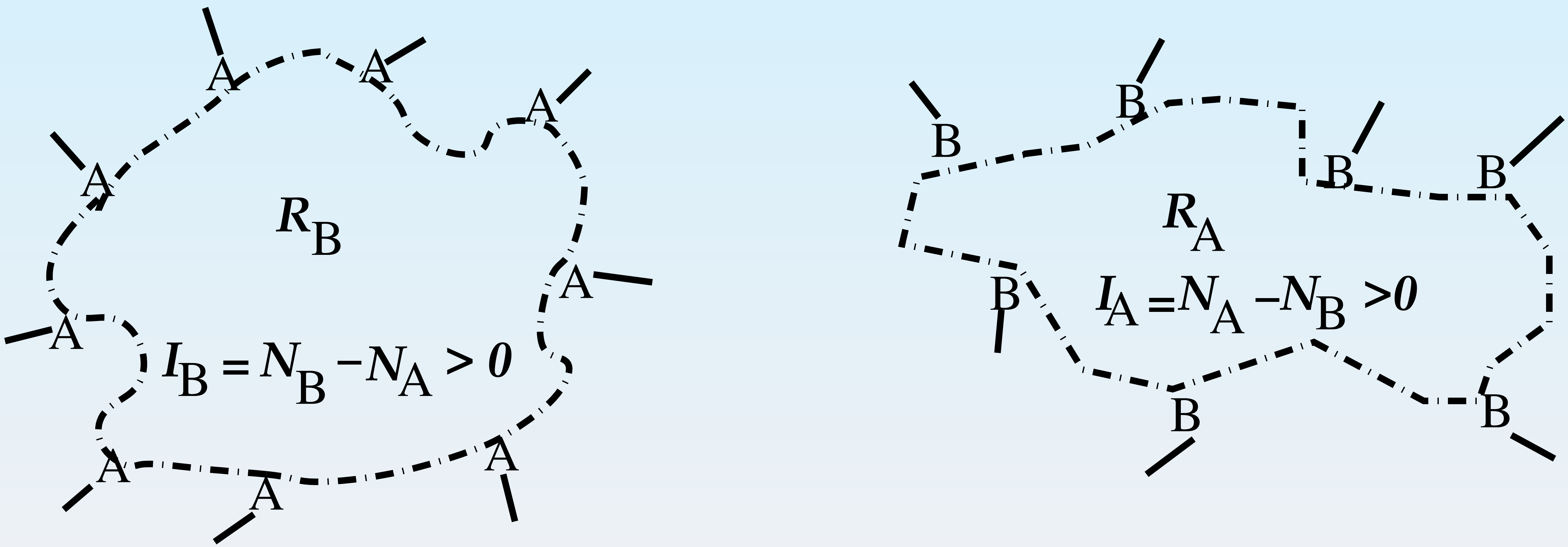
(Also earlier: Willans-Chalker-Moessner ’11, Ostrovsky et al ’14, Hefner et al ’14)

Vacancy-induced nonzero density of zero modes



(Sanyal, KD, Motrunich, PRL '16)

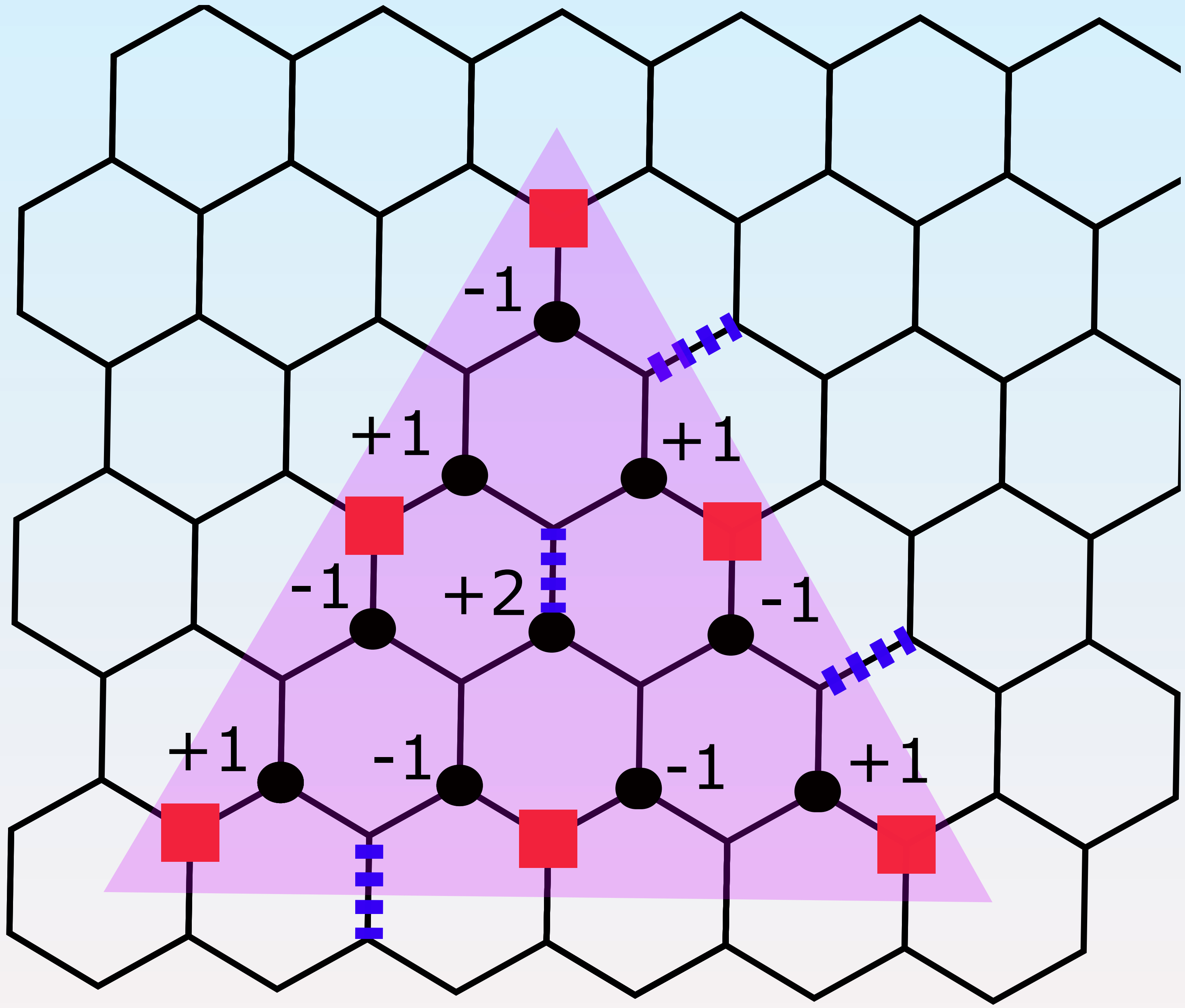
Our idea: R-type regions hosting zero modes



Constraint on zero-energy wavefunction: $\psi_A : \sum_{A \in B_0} t_{AB_0} \psi_A = 0$ $\qquad \psi_B : \sum_{B \in A_0} t_{BA_0} \psi_B = 0$

(Sanyal, KD, Motrunich, PRL '16)

Example of R-type region



Rigorous lower bound on density of zero modes on diluted honeycomb lattice
(Sanyal, KD, Motrunich, PRL '16)

Major puzzle remained:

- Actual density of zero modes much larger than lower bound
- What dominates?

Enter maximum matchings

- Key idea: Disorder-robust zero modes only depend on connectivity, not hopping strengths.
- R-type regions rely on local sublattice imbalance between A and B type site densities.
- Suggests thinking in terms of matchings a.k.a dimer covers
- Places that cannot be covered by dimers host wavefunctions

Switch gears: Majorana networks

$$H_{\text{Majorana}} = \frac{i}{4} \sum_{rr'} a_{rr'} \eta_r \eta_{r'}$$

with $a_{rr'} = -a_{r'r}$

$\{\eta_r, \eta_{r'}\} = 2\delta_{rr'}$

$c = \frac{1}{2}(\eta_r + i\eta_{r'})$ is a canonical fermion

Modes can serve as resource for quantum computing

(Read & Green, Ivanov, Kitaev, Biswas, Sau, Alicea…)

Question: Collective Majorana excitations?

- Mixing amplitudes are non-idealities, expected to cause dephasing in quantum computing schemes
- But: zero energy collective excitations of network are Majorana fermions, can also serve as quantum computing resources.
- Interesting question: Are there protected zero energy collective excitations of network?
- Equivalently: Zero energy eigenvectors of pure imaginary anti symmetric matrix $ia_{rr'}$?
- In bipartite case, previous “construction” yields collective Majorana excitations of network

What about general networks?

- No two-sublattice decomposition, no sublattice imbalance, so how do we think?
- Key point: Nonbipartite lattices have odd-perimeter plaquettes (e.g. triangles), and each such isolated plaquette hosts a protected zero mode of $ia_{rr'}$
- Zero modes of full network somehow built from linear combinations of these?

Making all this precise: Bipartite case

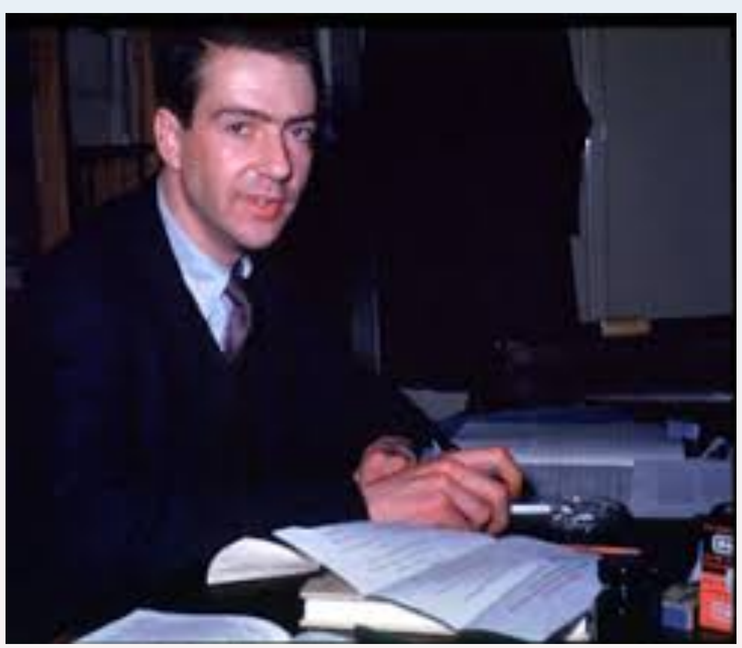
Counting zero modes from maximum matchings

AIP | The Journal of Chemical Physics

Some Studies in Molecular Orbital Theory I. Resonance Structures and Molecular Orbitals in Unsaturated Hydrocarbons

H. C. Longuet-Higgins

(1950)



Chemist - structure of diborane)

Physicist - (co)advisor of Peter Higgs)

Pioneer in:

Cognitive science and computer vision

‘Machine-intelligence (aka AI!)

Computer music...

Longuet-Higgins (restated)

- Number of monomers in any maximum matching of bipartite graph gives number of topologically-protected zero modes of corresponding tight-binding model

"nonzero-defect" generalization of "Tutte's Theorem" for bipartite graphs (Edmonds)

But what do the zero modes “look” like?

- What do we mean by “look” like?
- Our interest: Consequences for basis invariant Green functions and for transport at particle-hole symmetric chemical potential (e.g. undoped graphene)
- What we really want: A general way of identifying “all possible” R-type regions
- In other words: A choice of “maximally localised” basis for the zero-mode subspace

Partial answer from Longuet-Higgins:

- Global statement from Longuet-Higgins: Set of all sites that host a monomer in at least one maximum matching form support of all topologically-protected zero mode wavefunctions
- Clearly, we want more:
- General algorithm for identifying all R-type regions?
- Maximally-localised basis for zero mode subspace?

Our progress: A local statement

Brings into play classic result from graph theory

COVERINGS OF BIPARTITE GRAPHS

A. L. DULMAGE AND N. S. MENDELSOHN

Can. J. Math. 10: 517, 1958

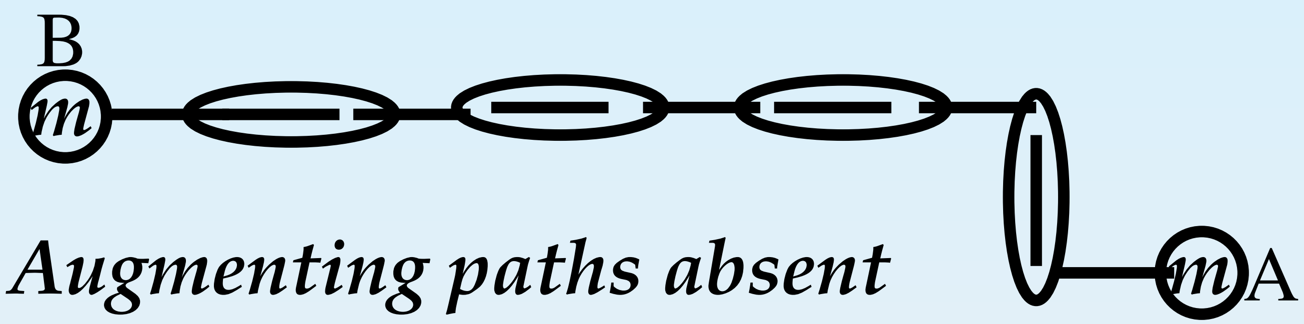
Use structure theory of Dulmage-Mendelsohn to construct non-overlapping ‘complete’ set of R-type regions.

`R-type’ regions of lattice host monomers in maximum matchings and zero modes of a quantum particle

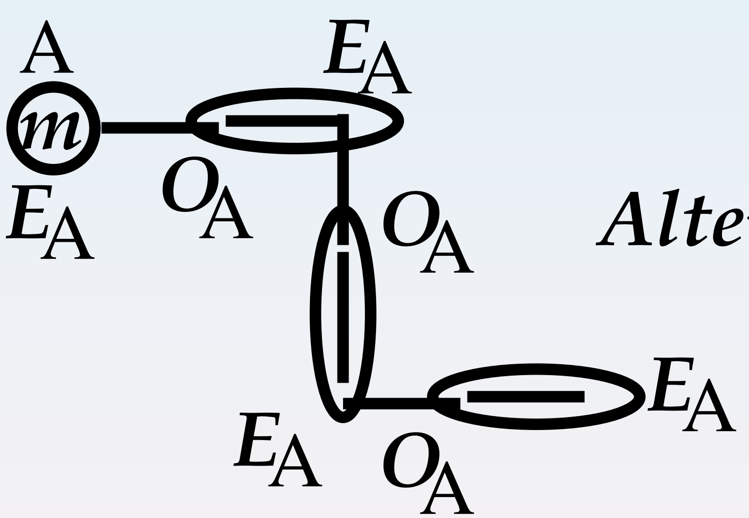
Zero temperature two-terminal conductance is zero if R-type regions don’t percolate

Matchings, augmenting paths, and alternating paths

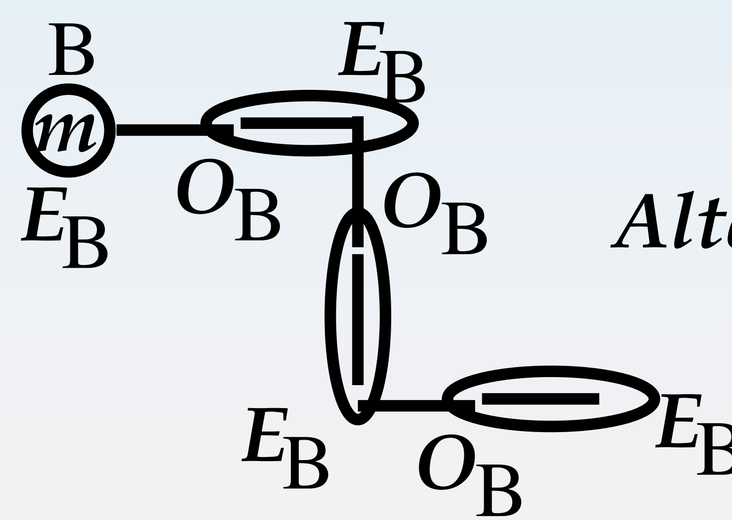
In any maximum matching M:



Augmenting paths absent



Alternating path



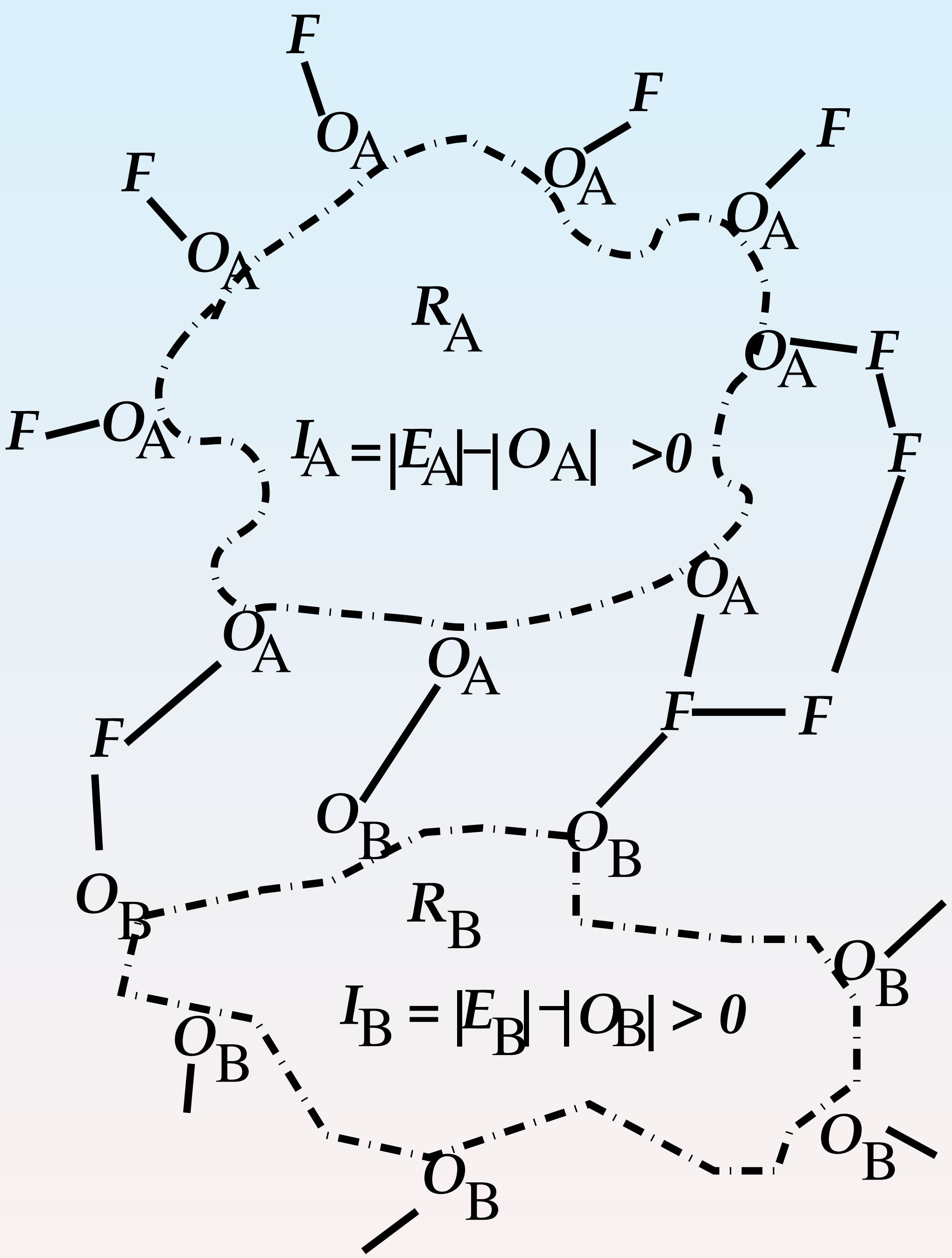
Alternating path

A useful version of the DM decomposition

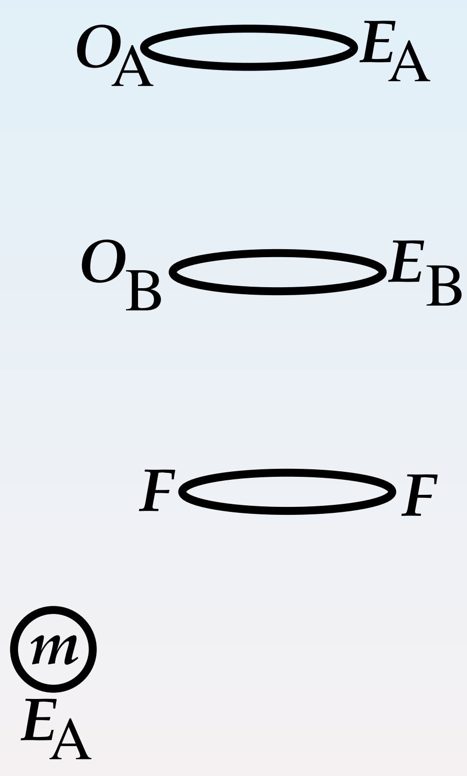
Even, odd and unreachable sites from any one maximum matching M

- Even: Reachable by even-length alternating paths from monomers of M (including length zero)
- Odd: Reachable by odd-length alternating paths from monomers of M
- Unreachable: Not...
- Decomposition:
- $C_A : E_A \cup O_A$
- $C_B : E_B \cup O_B$
- $P : U_A \cup U_B$

Key: Connected components of C_A and C_B



In any maximum matching:

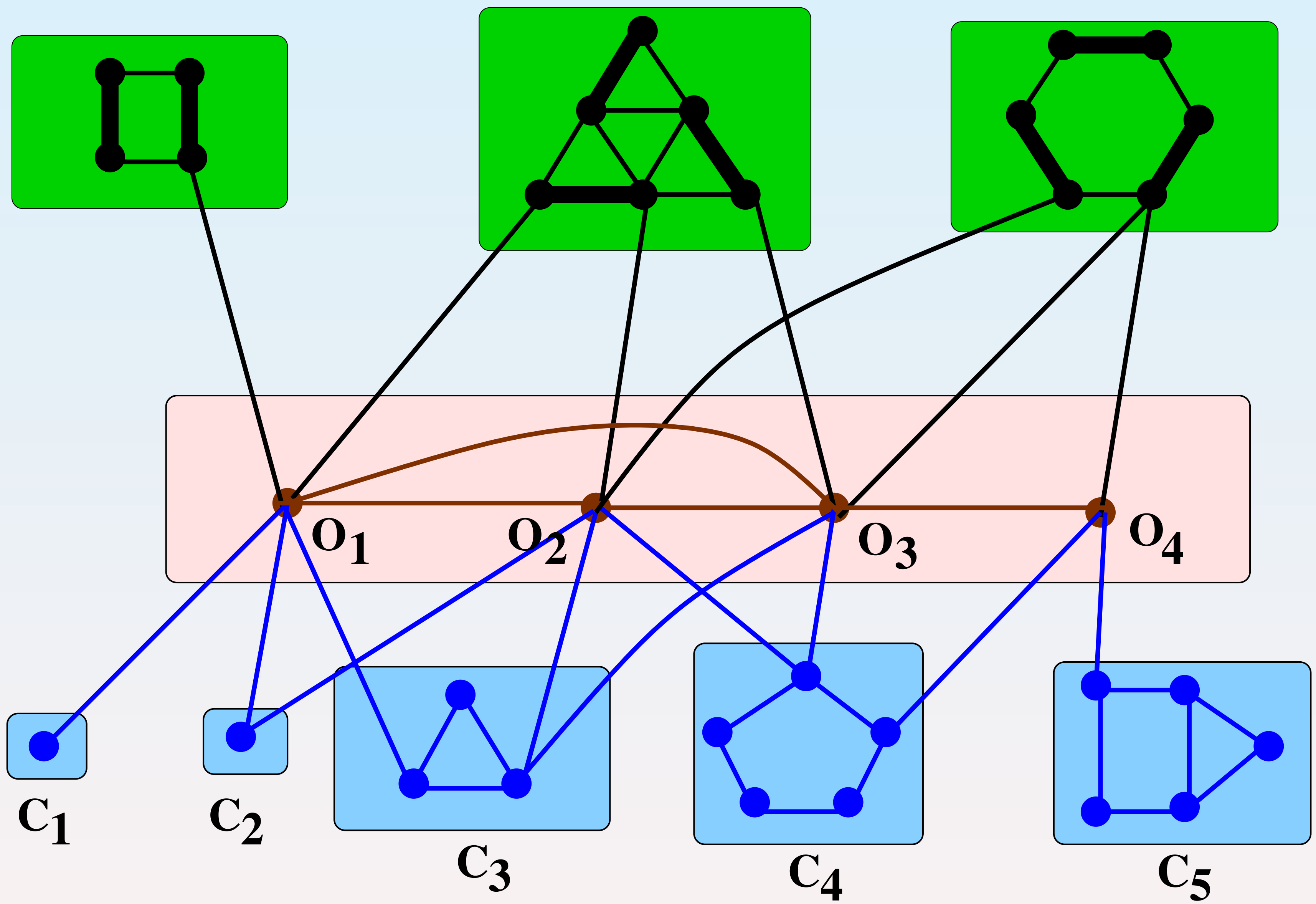


Connected components are R-type regions

- Each R_i^A (R_j^B) hosts I_i^A (I_j^B) topologically-protected zero modes with wavefunctions confined to the region.
- Provides alternate ‘local’ proof for correspondence between monomers of maximum matchings and zero modes of adjacency matrix
- Proof gives topologically-robust construction of a maximally-localised basis for zero modes

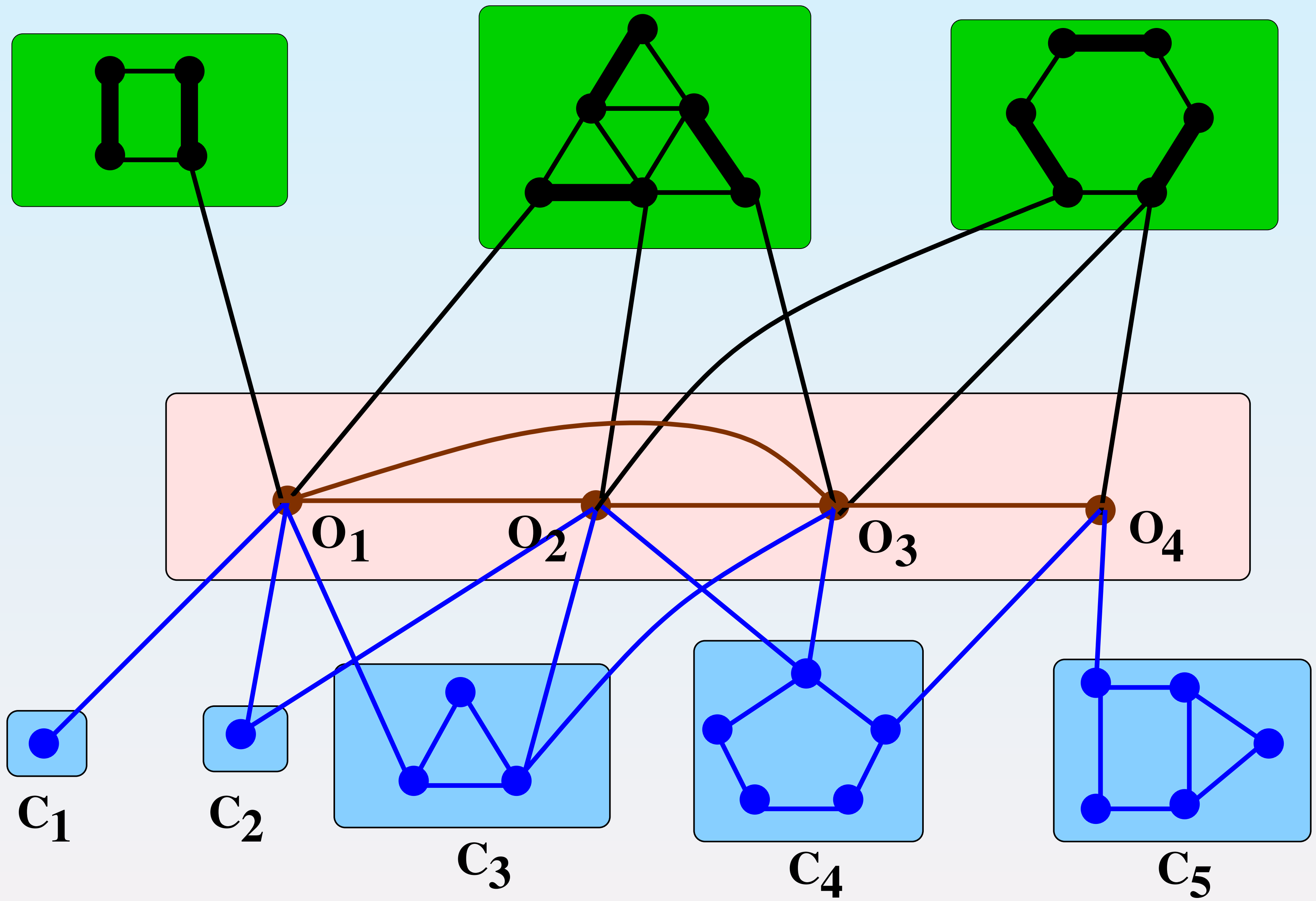
Making things precise for general nonbipartite networks

Gallai-Edmonds decomposition



Auxiliary bipartite graph

Bipartite network of factor critical components and odd sites



(KD, PRB 2022)

R-type regions of auxiliary bipartite graph

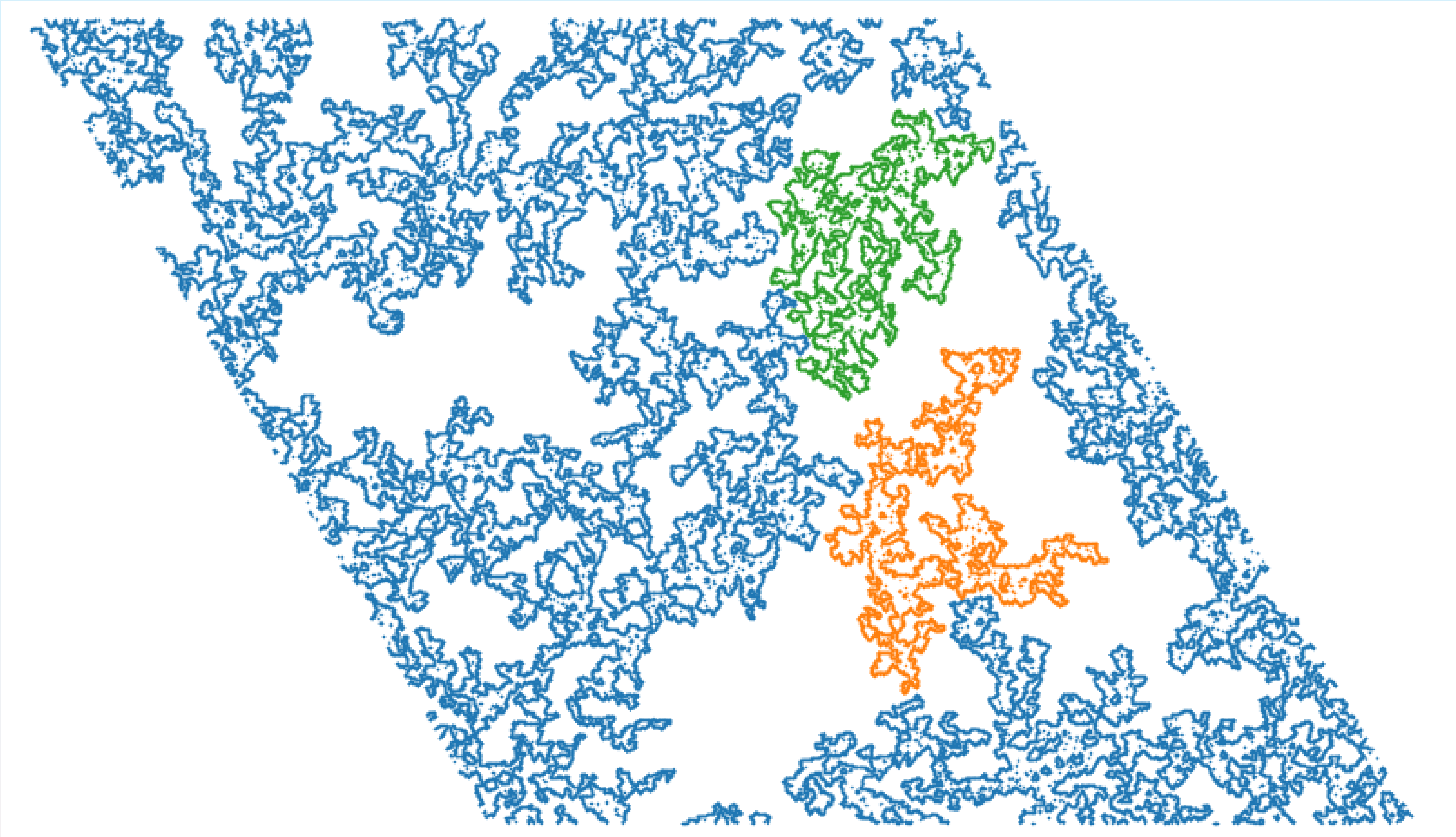
- Auxiliary graph only has R_i^A regions, no free regions, no R_i^B regions.
- Each R_i^A host I_i^A monomers in any maximum matching and I_i^A topologically-protected zero modes of $ia_{rr'}$
- Provides alternate ‘local’ proof for correspondence between total number of monomers of maximum matchings and total number of protected zero modes (earlier proof of Lovasz-Anderson is global)
- Proof gives construction of maximally-localised basis for protected zero mode sub space of $ia_{rr'}$

(KD, PRB 2022)

Upshot: Computational strategy

- Standard algorithms for finding any one maximum matching
- Alternating path tree from each monomer to obtain DM/GE labels of vertices
- Burning algorithm to construct connected components and obtain R-type regions
- Size of R-type regions gives upper bound on localization length of zero-energy Green function
- In bipartite case: compensated disorder ($|A| = |B|$) to avoid nuisance modes
- In nonbipartite case: check that global parity has no effect in large size limit

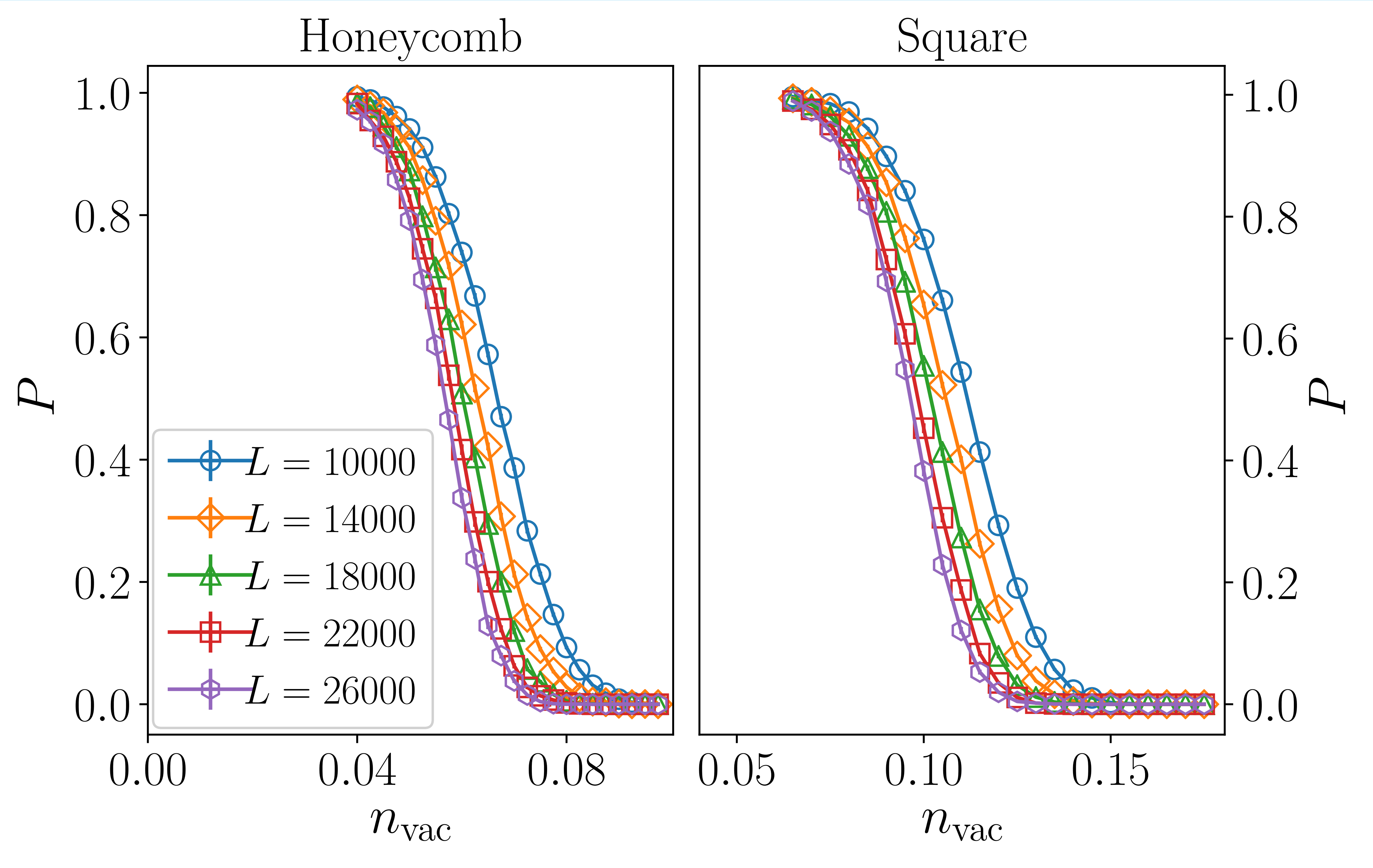
Basic picture in d=2 (for bipartite honeycomb lattice)



Typical R-type regions are BIG at low (~5%) dilution

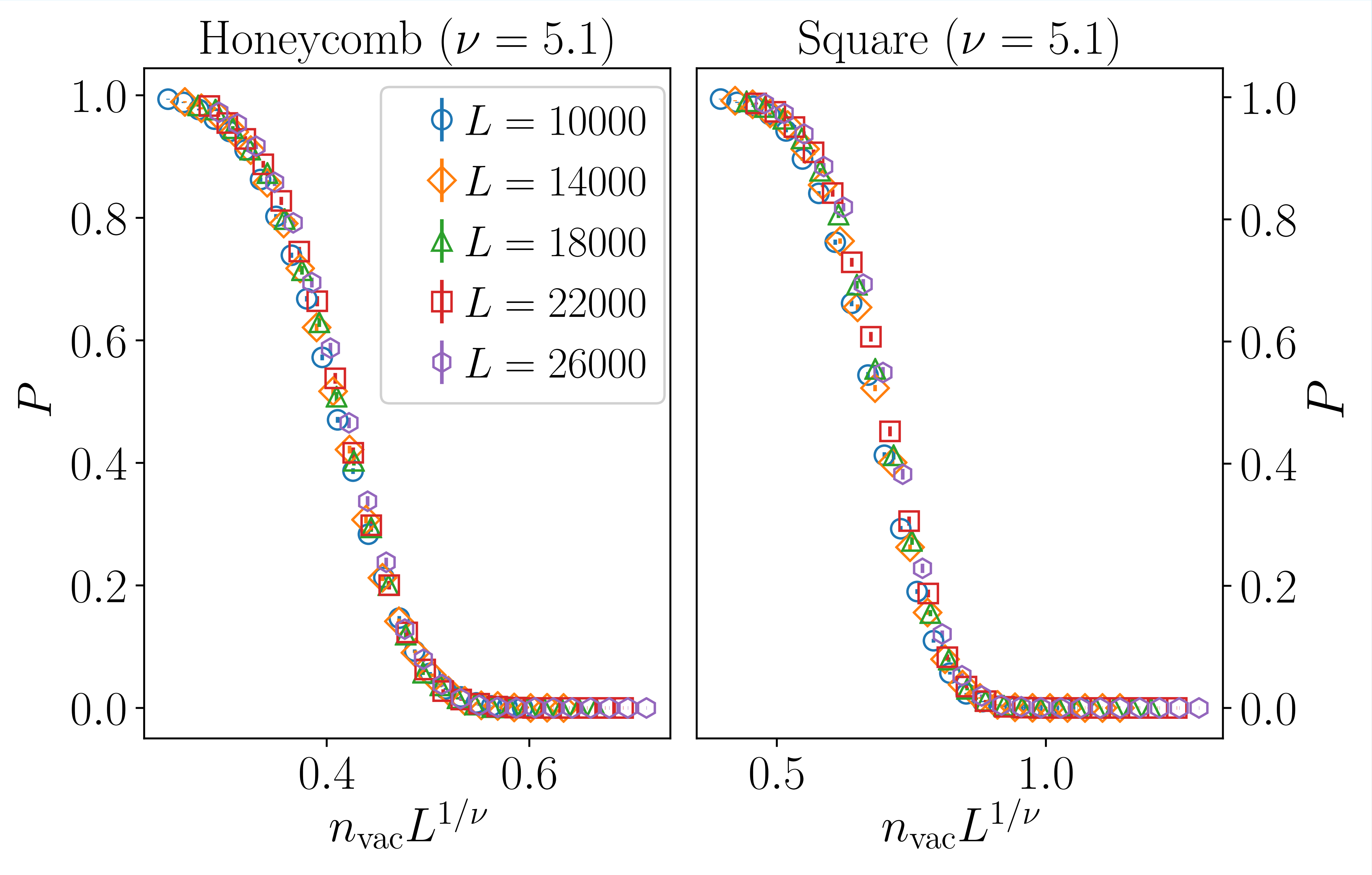
Bhola, Biswas, Islam, KD, PRX 12 021058 (2022)

Incipient percolation: wrapping probabilities



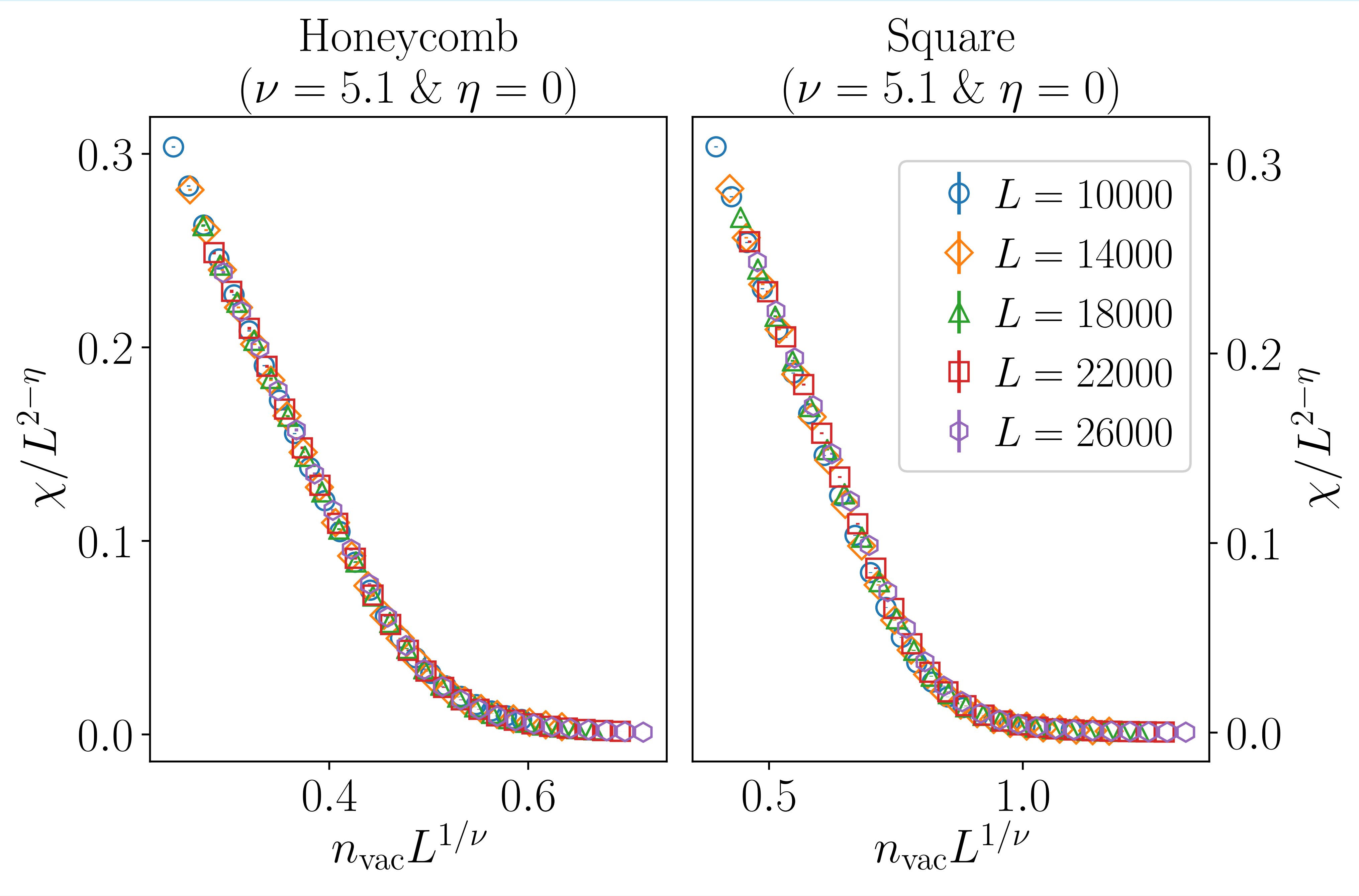
Bhola, Biswas, Islam, KD, PRX 12 021058 (2022)

Universal scaling at zero-dilution critical point



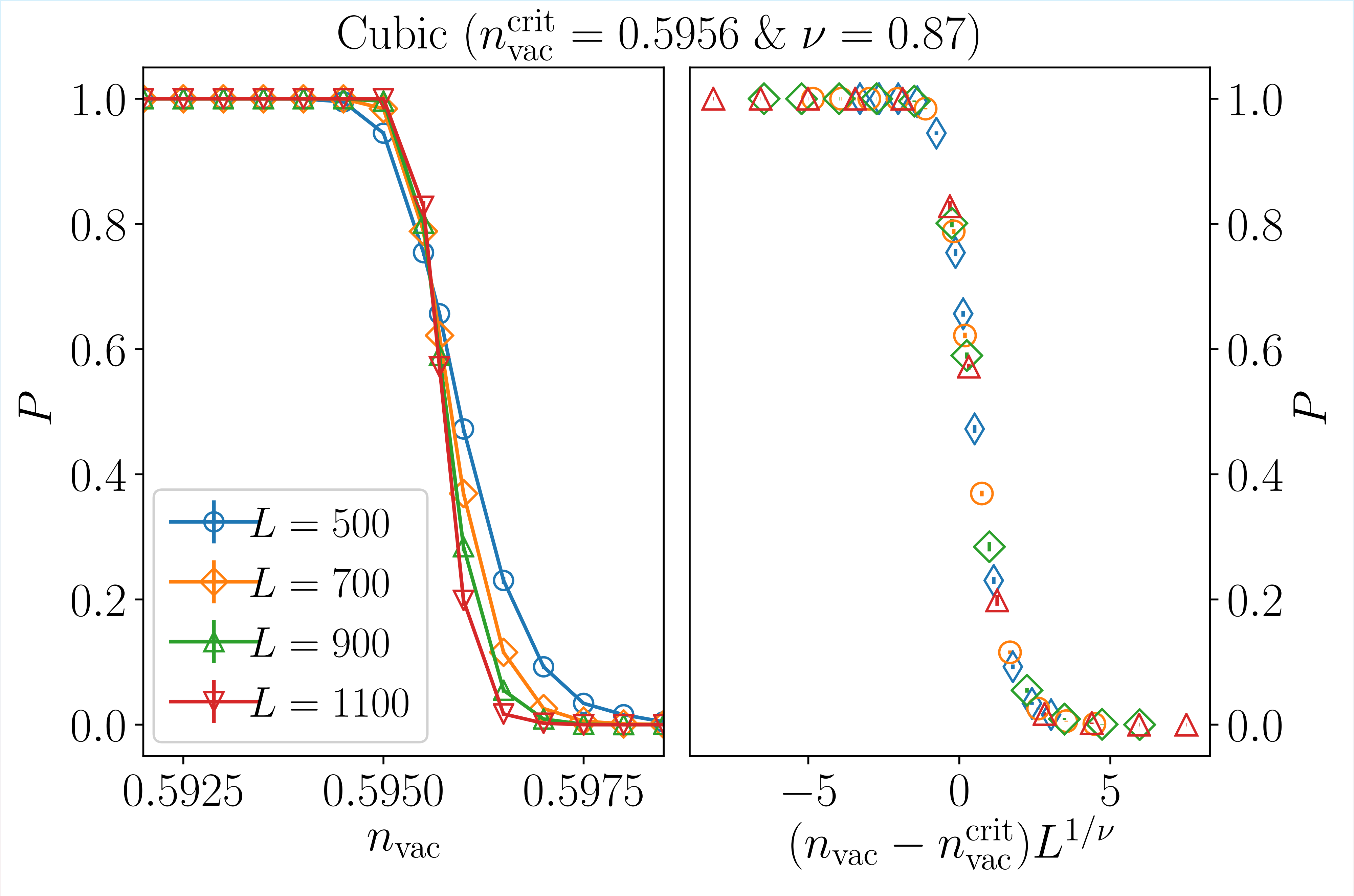
Bhola, Biswas, Islam, KD, PRX 12 021058 (2022)

Universal scaling of mass susceptibility $\chi = \langle m^2 \rangle / L^2$



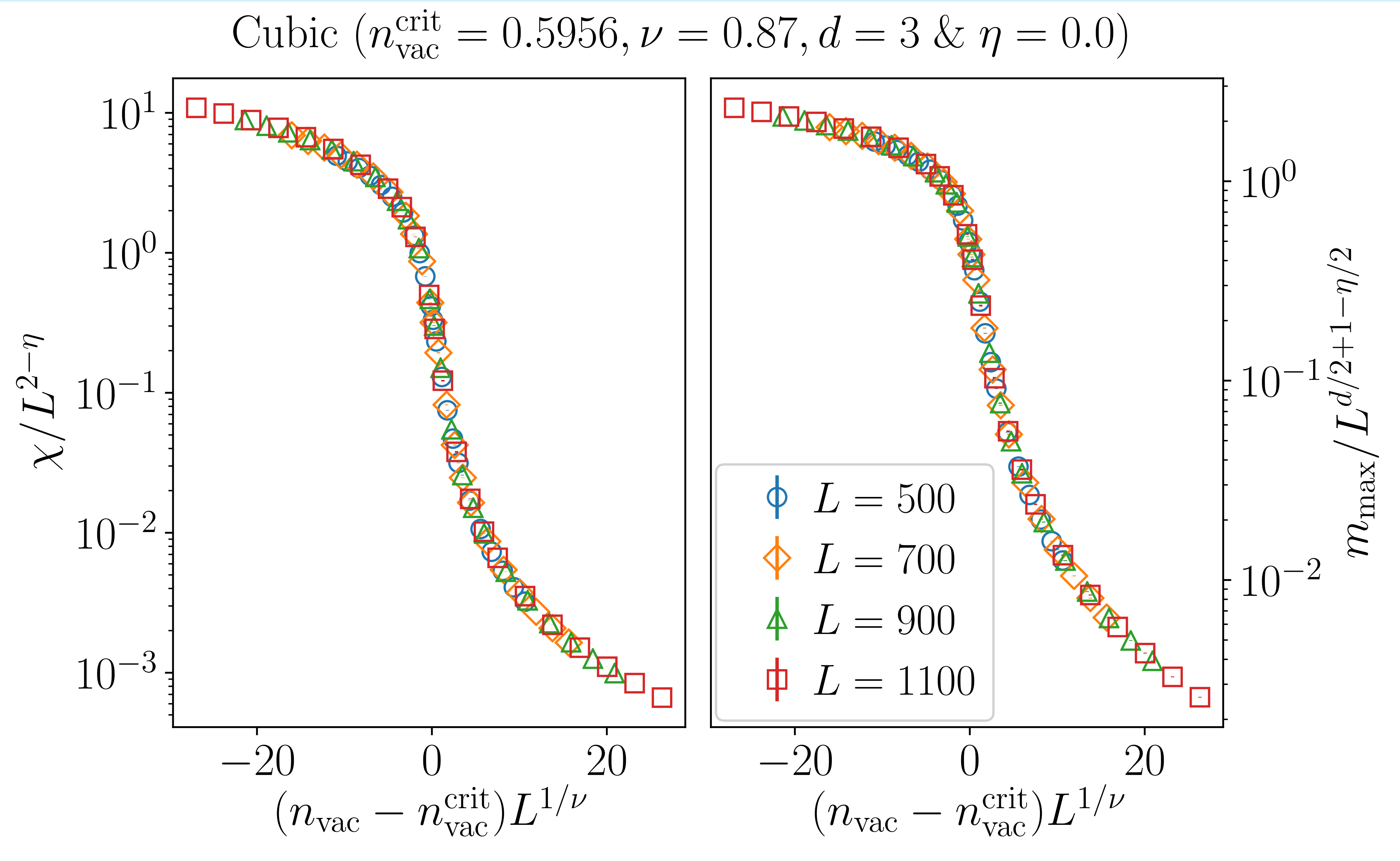
Anomalous exponent η indistinguishable from 0

Cubic lattice: percolation transition



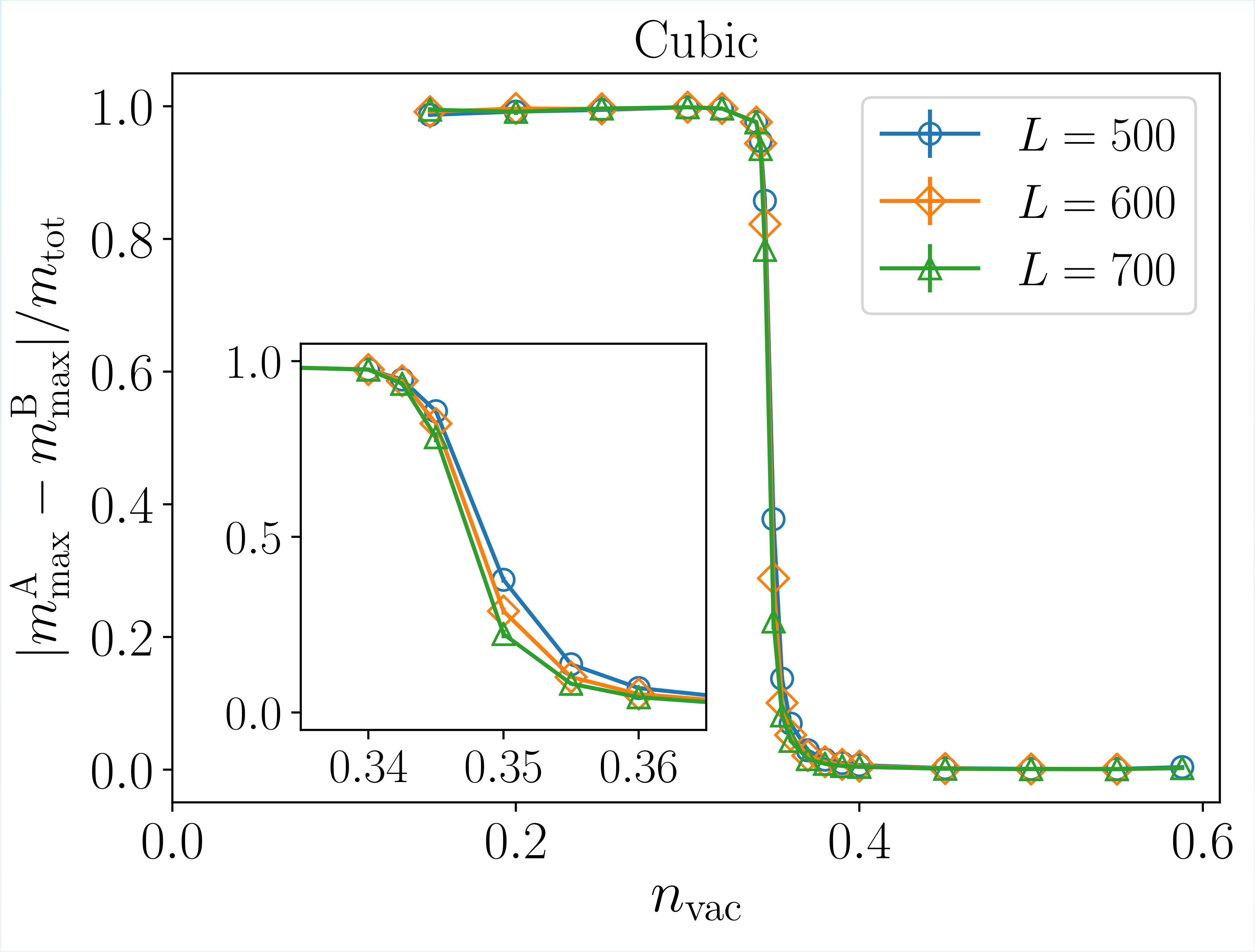
Bhola, Biswas, Islam, KD, PRX 12 021058 (2022)

Cubic lattice: susceptibility scaling



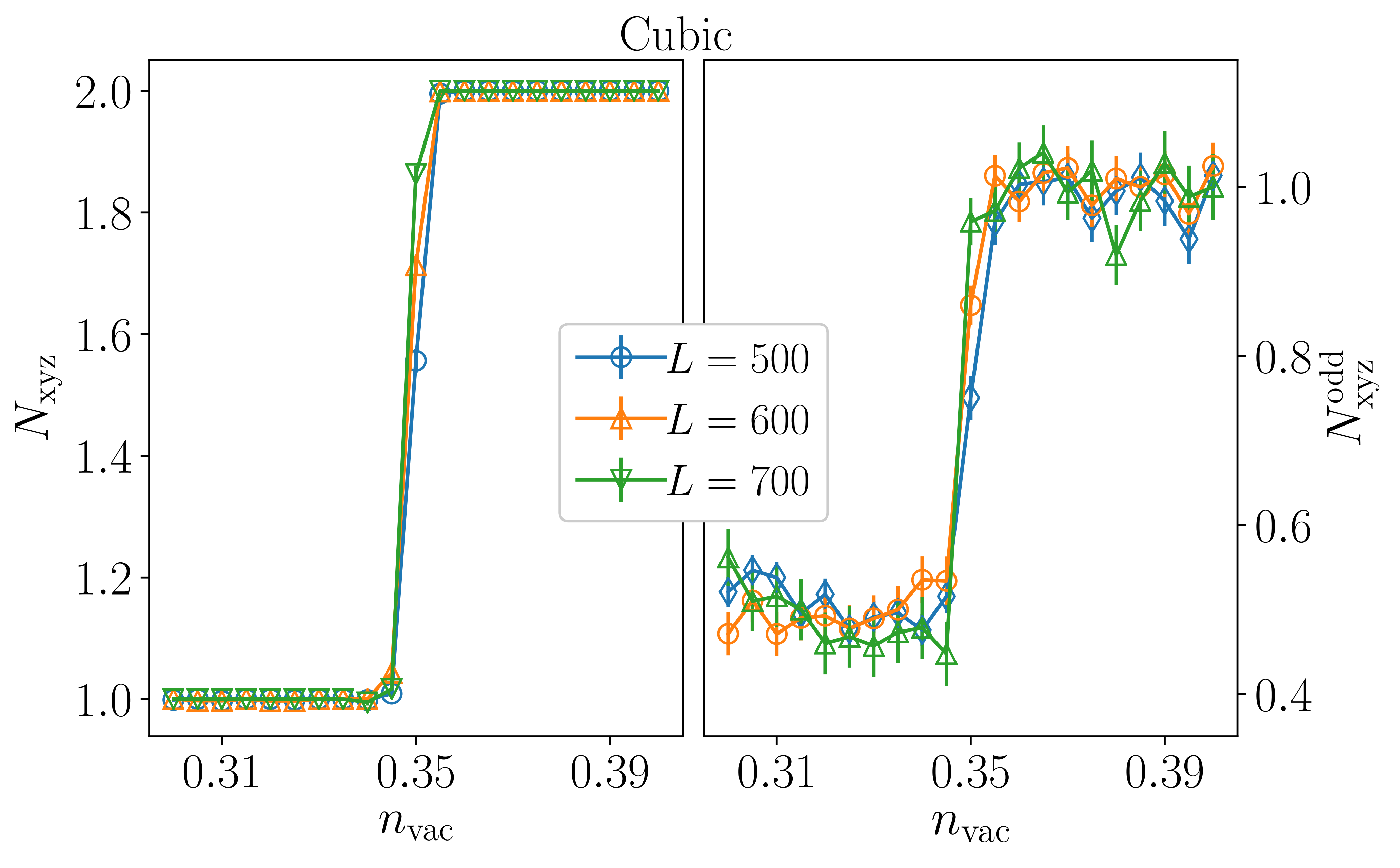
Bhola, Biswas, Islam, KD, PRX 12 021058 (2022)

Cubic lattice: sublattice symmetry breaking



Bhola, Biswas, Islam, KD, PRX 12 021058 (2022)

Sublattice symmetry breaking: Transition from two to one percolating clusters



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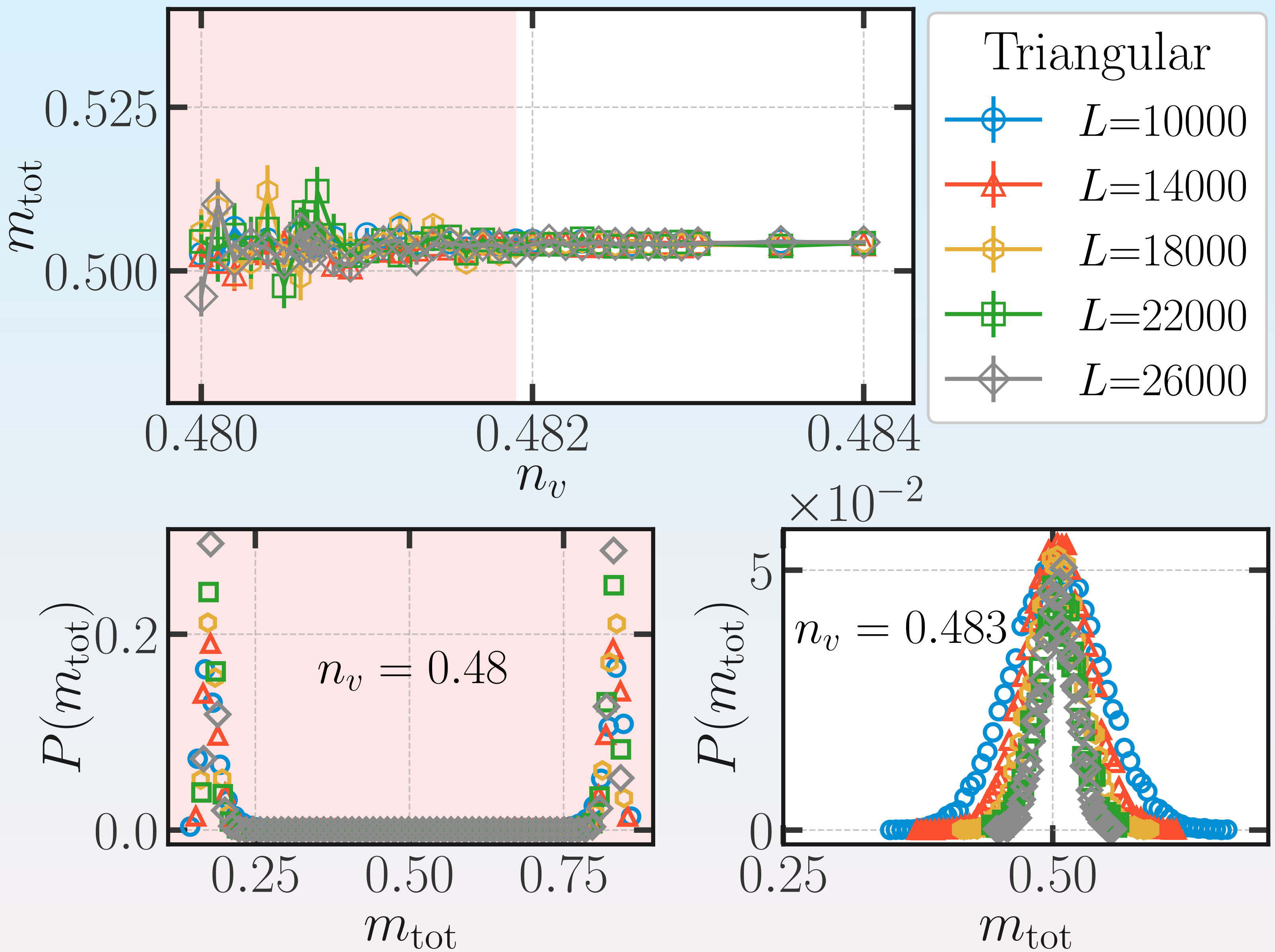
Triangular lattice: Basic picture



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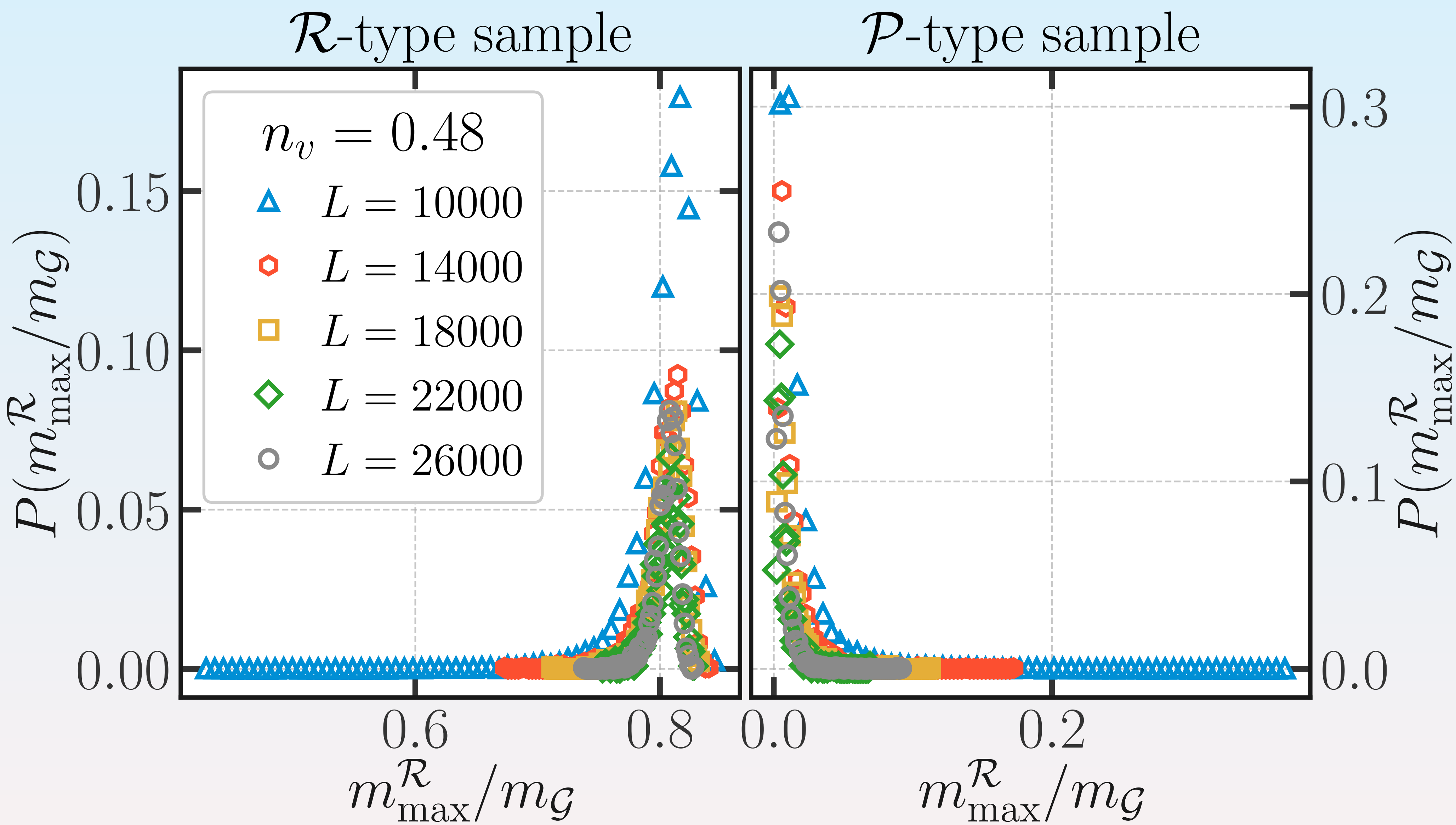

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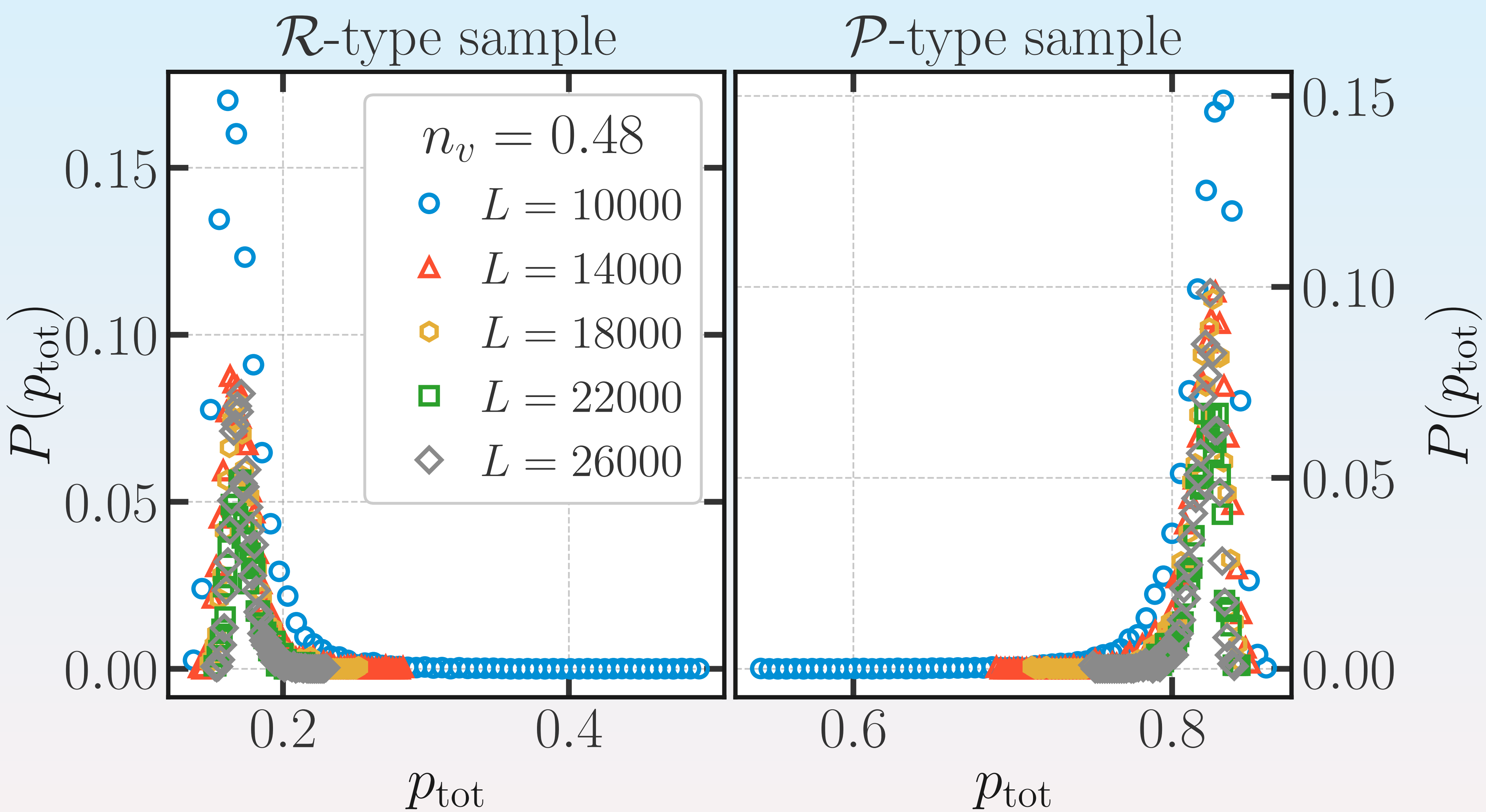
Triangular lattice: basic picture



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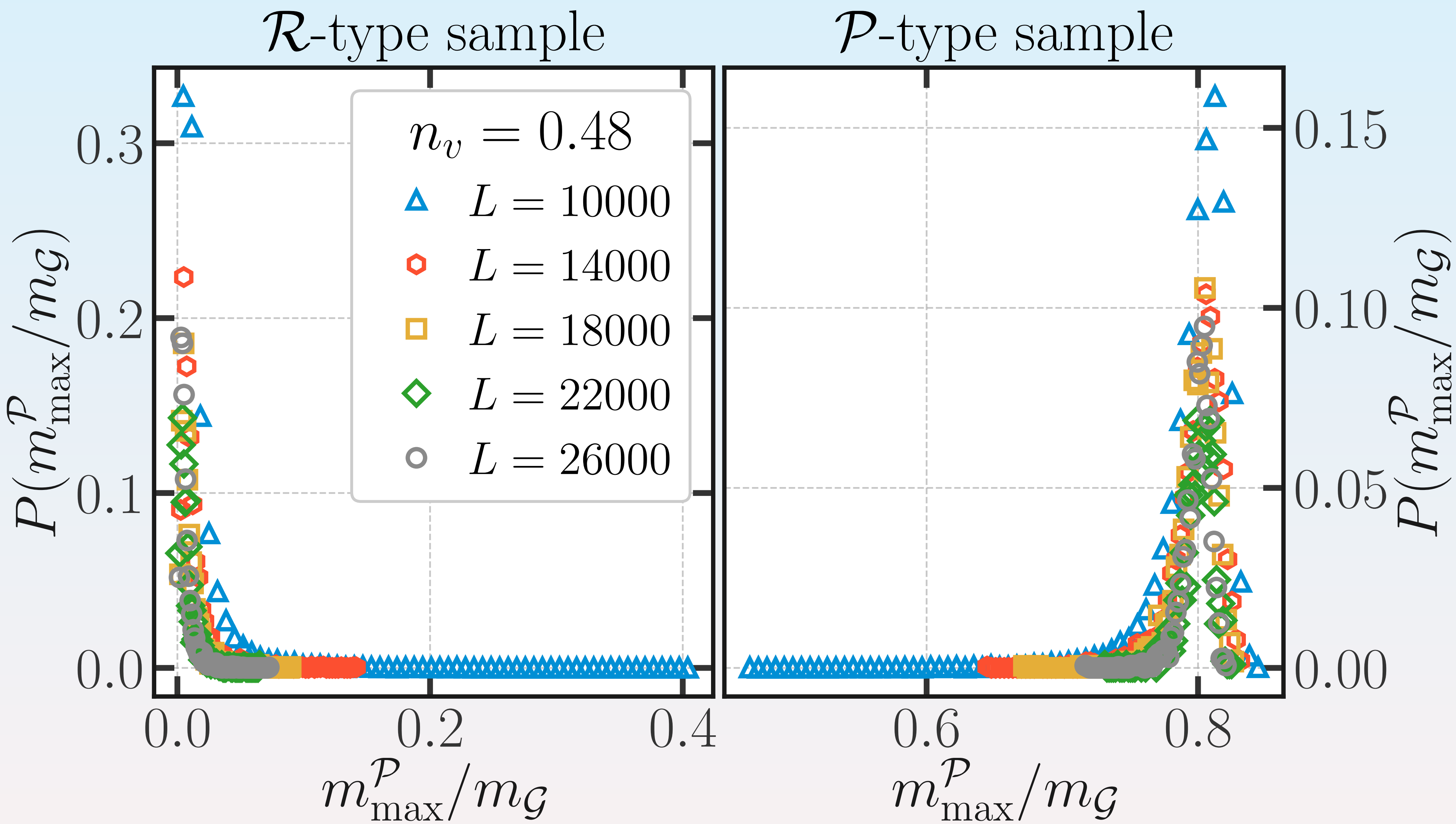

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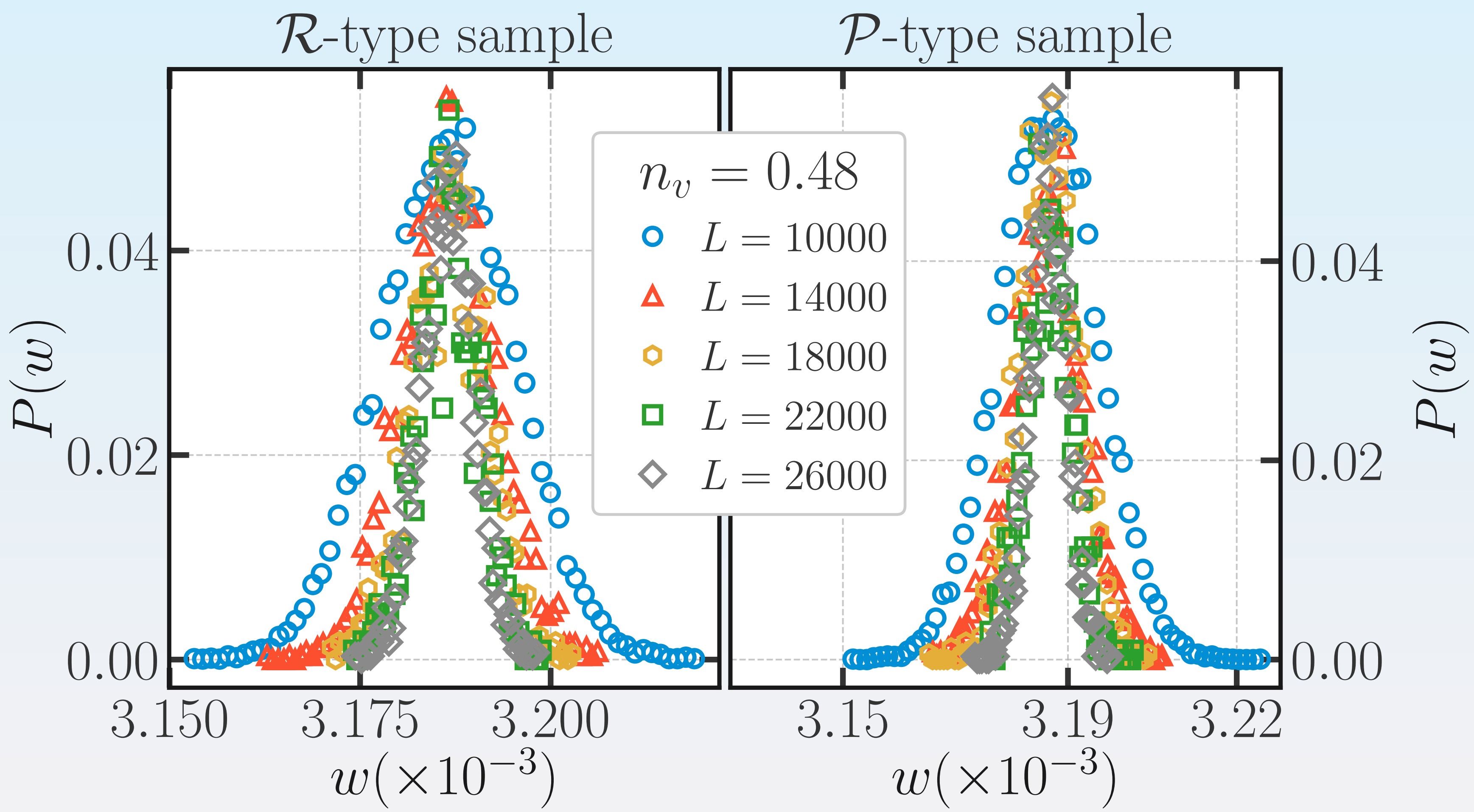
Triangular lattice: basic picture



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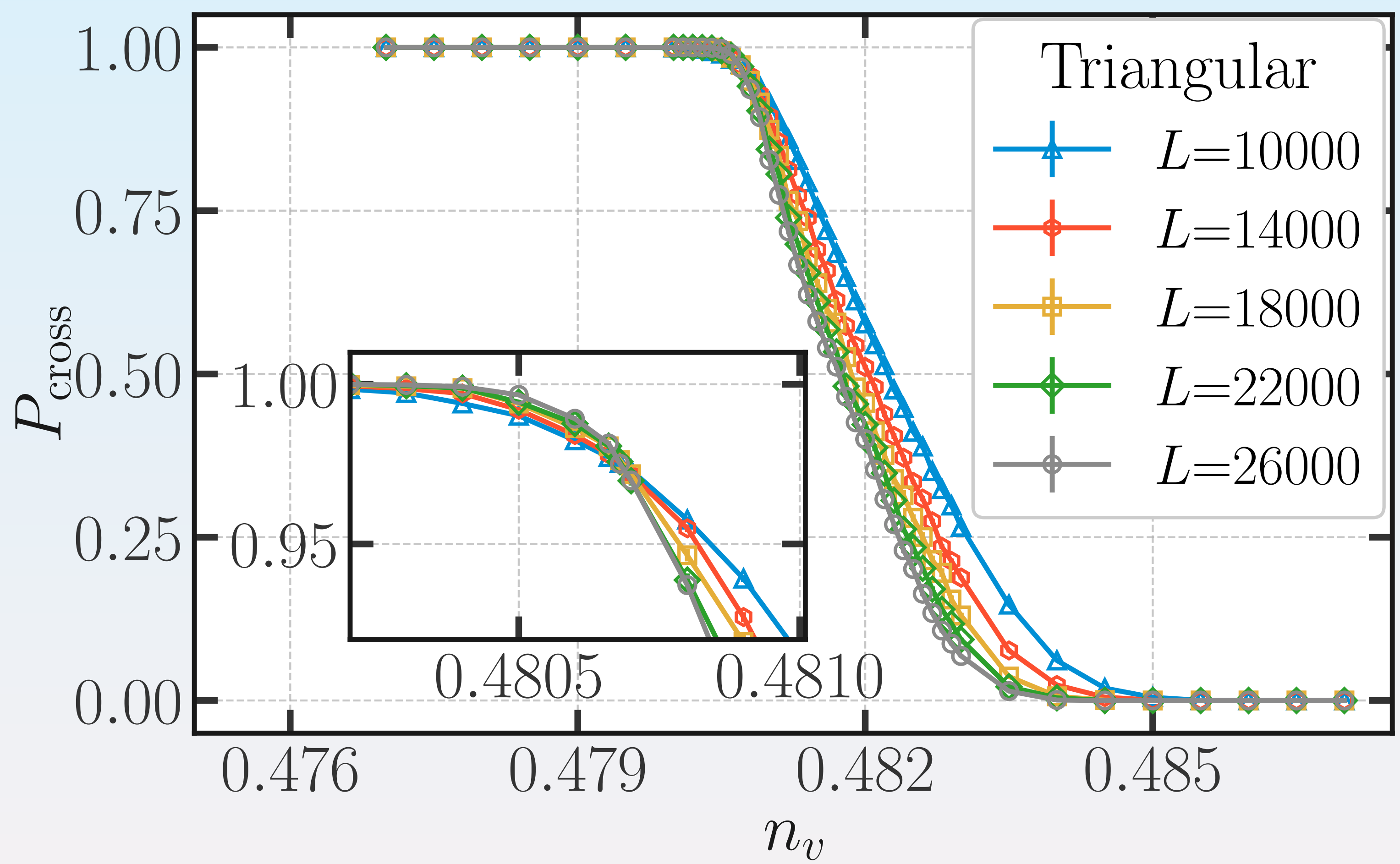
“Zero-half” percolation threshold

R-type samples undergo percolation transition, but not P-type samples



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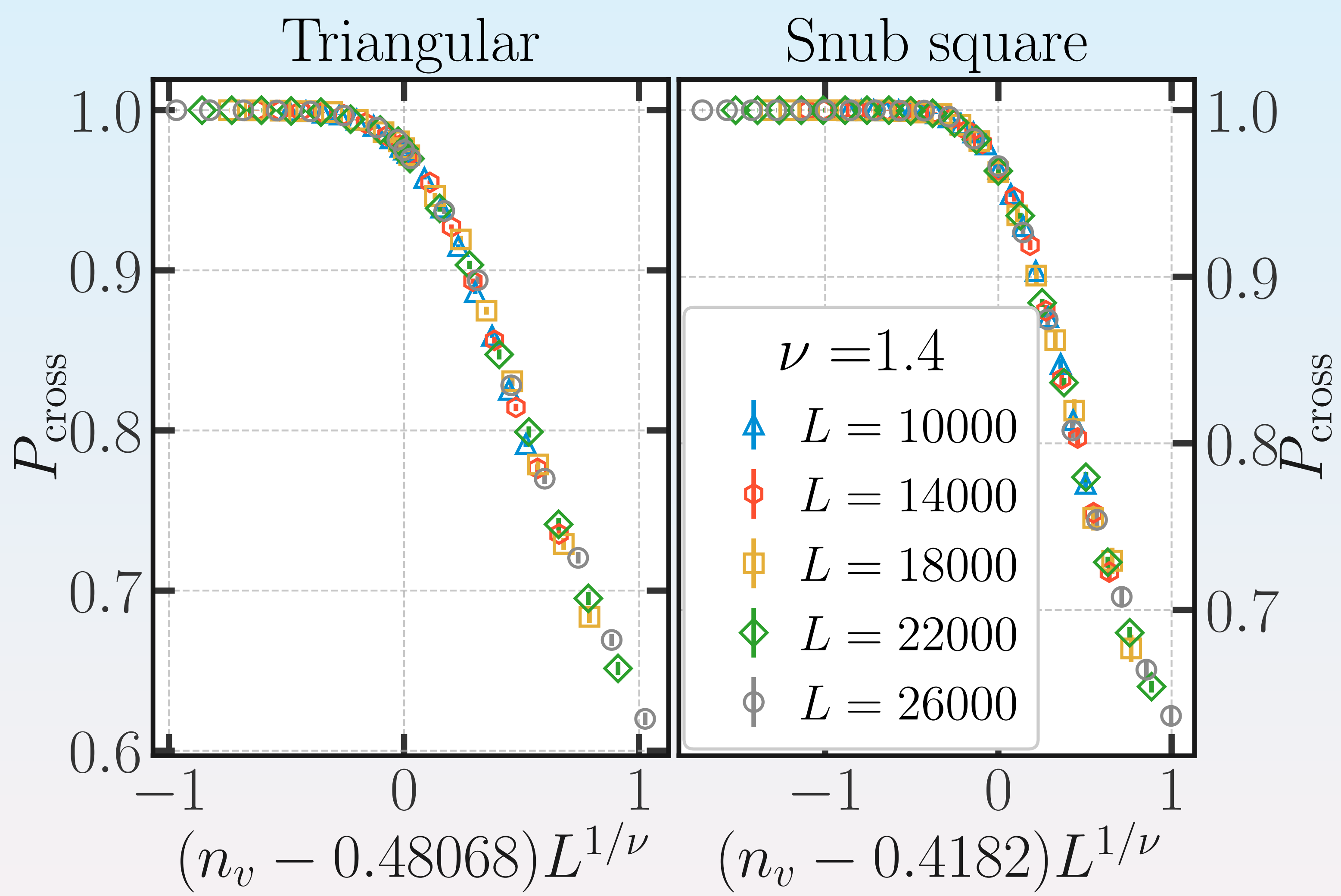
Scaling at percolation transition

Zero-half percolation threshold shows universal scaling



Bhola & KD, preprint

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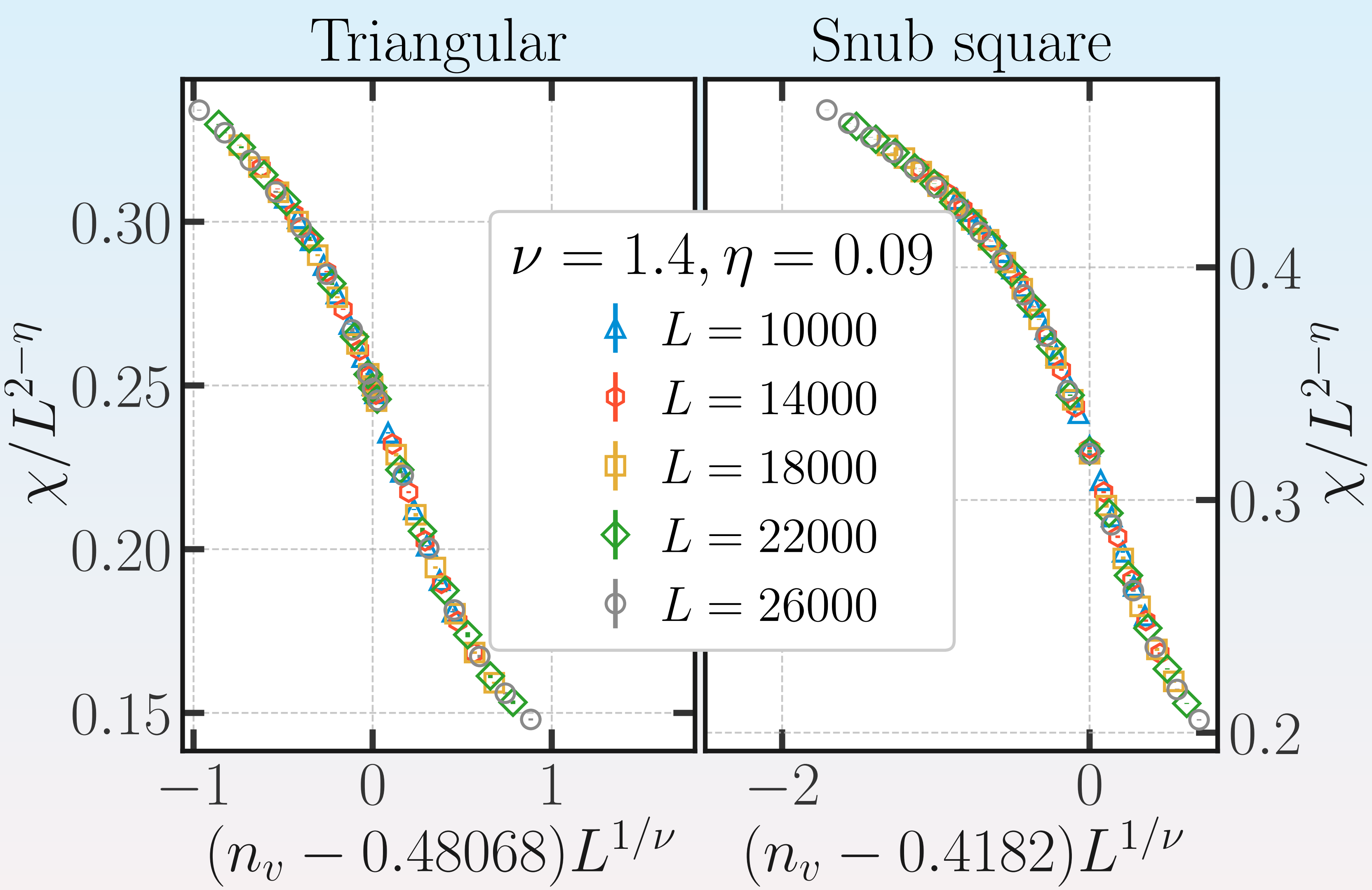


Bhola & KD, preprint

Summary a la Wodehouse

- Patient perseverance produces ***percolative paradigms***!

Acknowledgements

- T. Kavitha and A. Mondal for introduction to graph decompositions
- Computational resources of DTP-TIFR
- Discussions with D. Sen, D. Dhar, Mahan Mj, J. Radhakrishnan, S. Roy, and many others...
- Research at TIFR supported by DAE, and in part by Infosys-Chandrasekharan Random Geometry Centre @ TIFR and SERB (JC Bose Fellowship)

Percolation

- Study end-to-end connectivity of a porous medium
- Can you go from one end to other?
- Answer changes as function of porosity
- Simplest model: Randomly diluted regular lattice (graph)

Broadbent and Hammersley, Percolation processes I, Crystals and Mazes (1957)

Precise question about the random geometry

Crossing probability?

- Consider two dimensional square grid or honeycomb net or three dimensional cubic lattice of linear size L
- Remove fraction n_{vac} of sites and delete links to removed sites.
- $P_w(n_{\text{vac}}, L)$: Probability that one can 'walk' from left end to right end along existing sites and links.
- How does this behave as a function of n_{vac} and L?

Sharp threshold behaviour

Property of thermodynamic limit

- In $d = 2$ and in $d = 3$, $L \to \infty$ limit characterised by sharp threshold behaviour as function of n_{vac}
- Percolation transition
- Simplest geometric example of a thermodynamic phase transition
- For $n_{\mathrm{vac}} < n_{\mathrm{vac}}^{\mathrm{crit}}$, $P_w \to 1$ as $L \to \infty$
- For $n_{\mathrm{vac}} > n_{\mathrm{vac}}^{\mathrm{crit}}$, $P_w \to 0$ as $L \to \infty$

Approach to thermodynamic limit

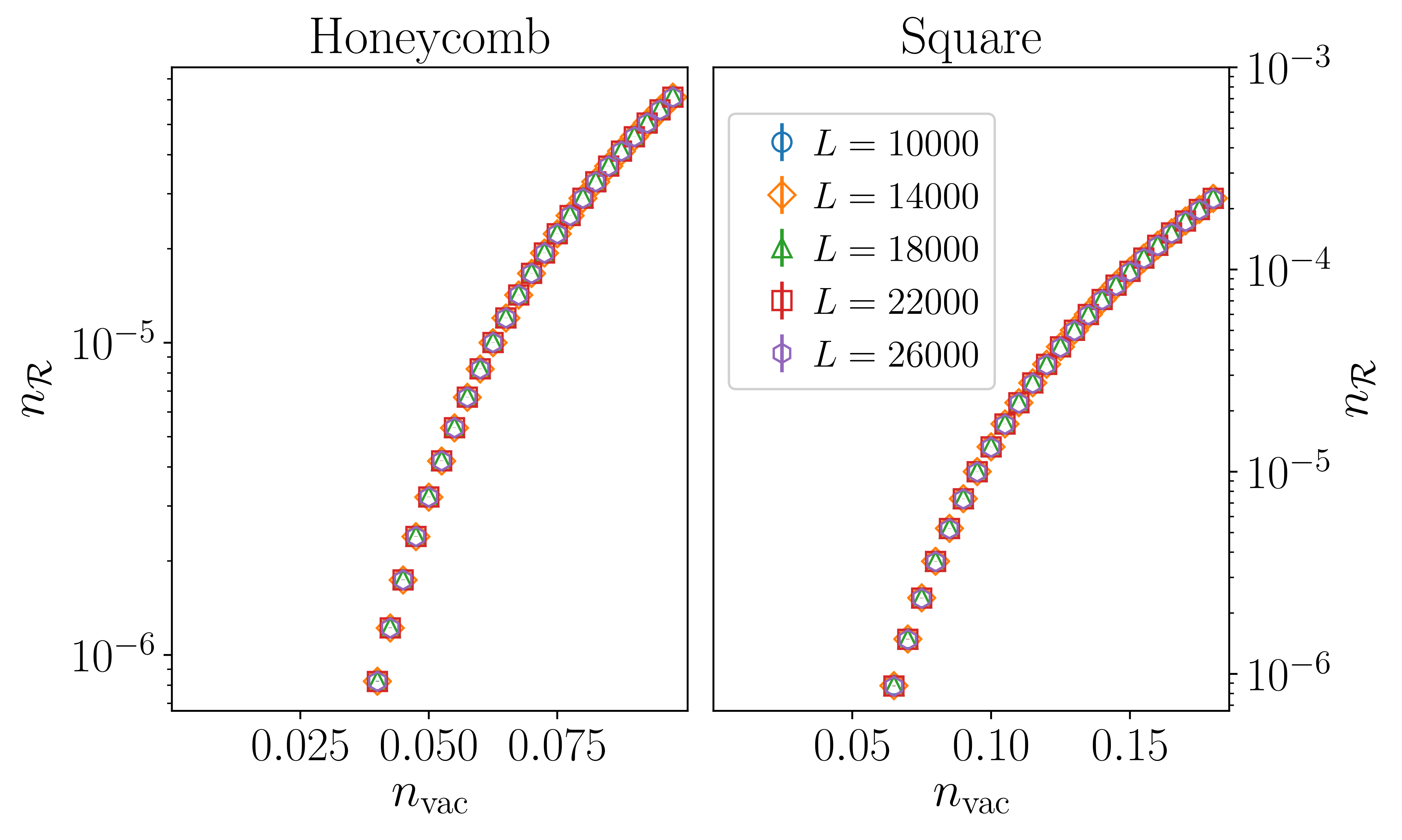
Universality and scaling

- $L \to \infty$ limit is approached in interesting way
- $P_w(n_{\mathrm{vac}}, L) = f(\delta L^{1/\nu})$ where $\delta = n_{\mathrm{vac}} - n_{\mathrm{vac}}^{\mathrm{crit}}$
- $f(x)$ is the universal scaling function, ν is a scaling exponent and $n_{\mathrm{vac}}^{\mathrm{crit}}$ is the critical dilution
- $f(x)$ and ν believed to be universal (independent of microscopic-scale details)
- Square lattice and honeycomb net have same $f(x)$ and ν. Cubic lattice different (dimension dependent)

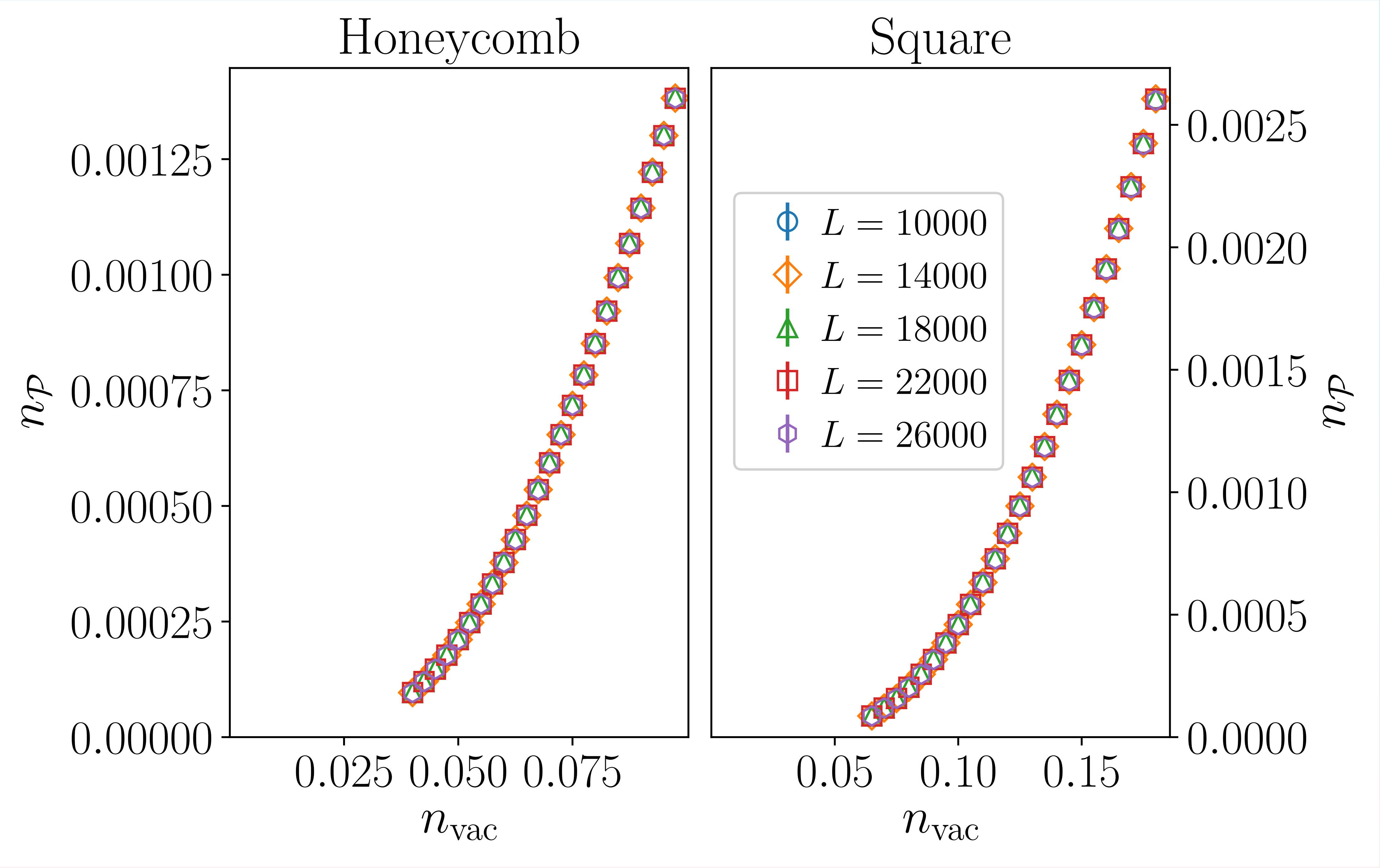
Scale invariance

- Implies different size samples have same P_w for $n_{\mathrm{vac}} = n_{\mathrm{vac}}^{\mathrm{crit}}$
- Scale invariance: Pictures of random geometry look same if we rescale pictures!
- Only true if we ignore lattice scale features, but amazing anyway!

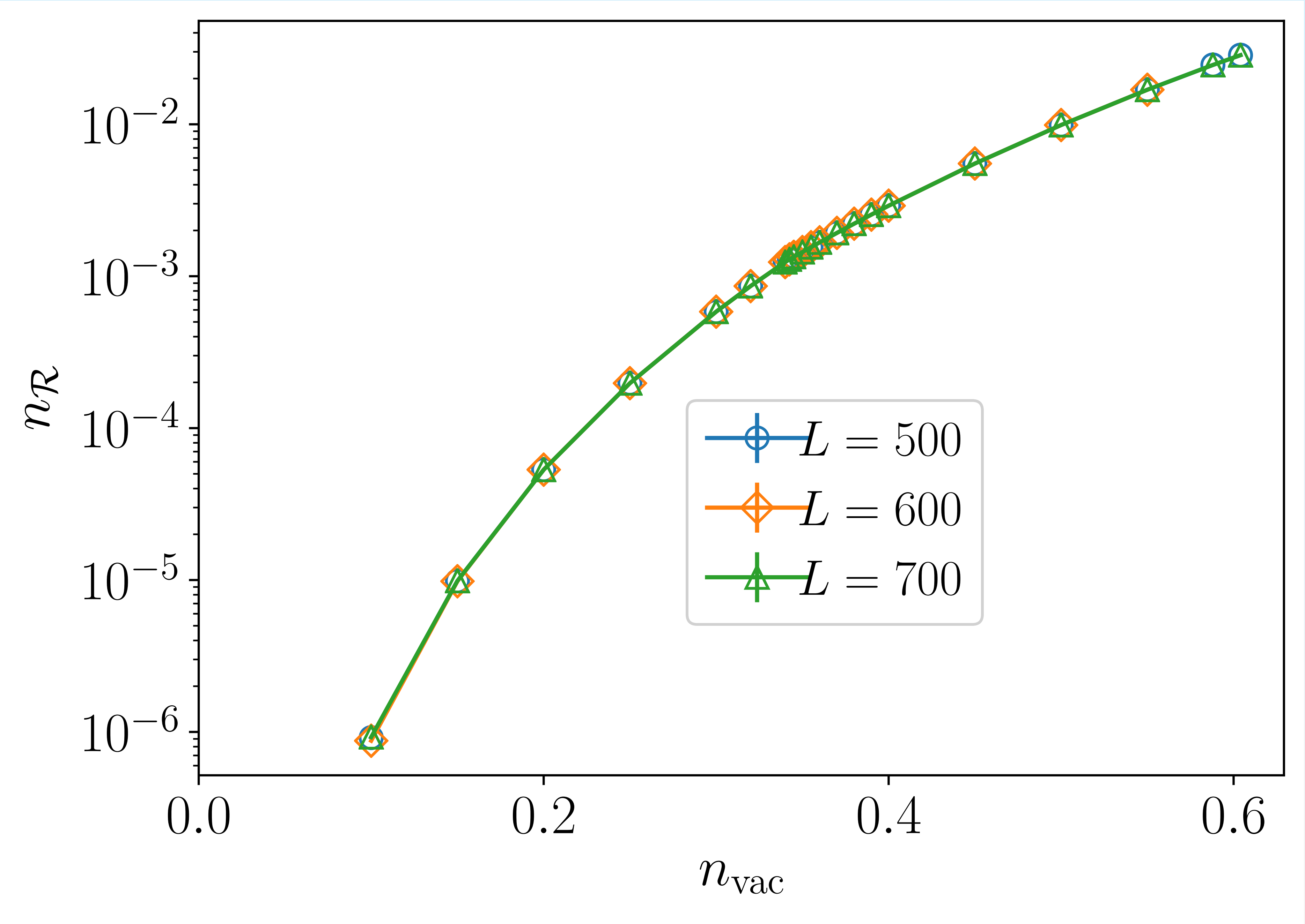
Number density of R-type regions



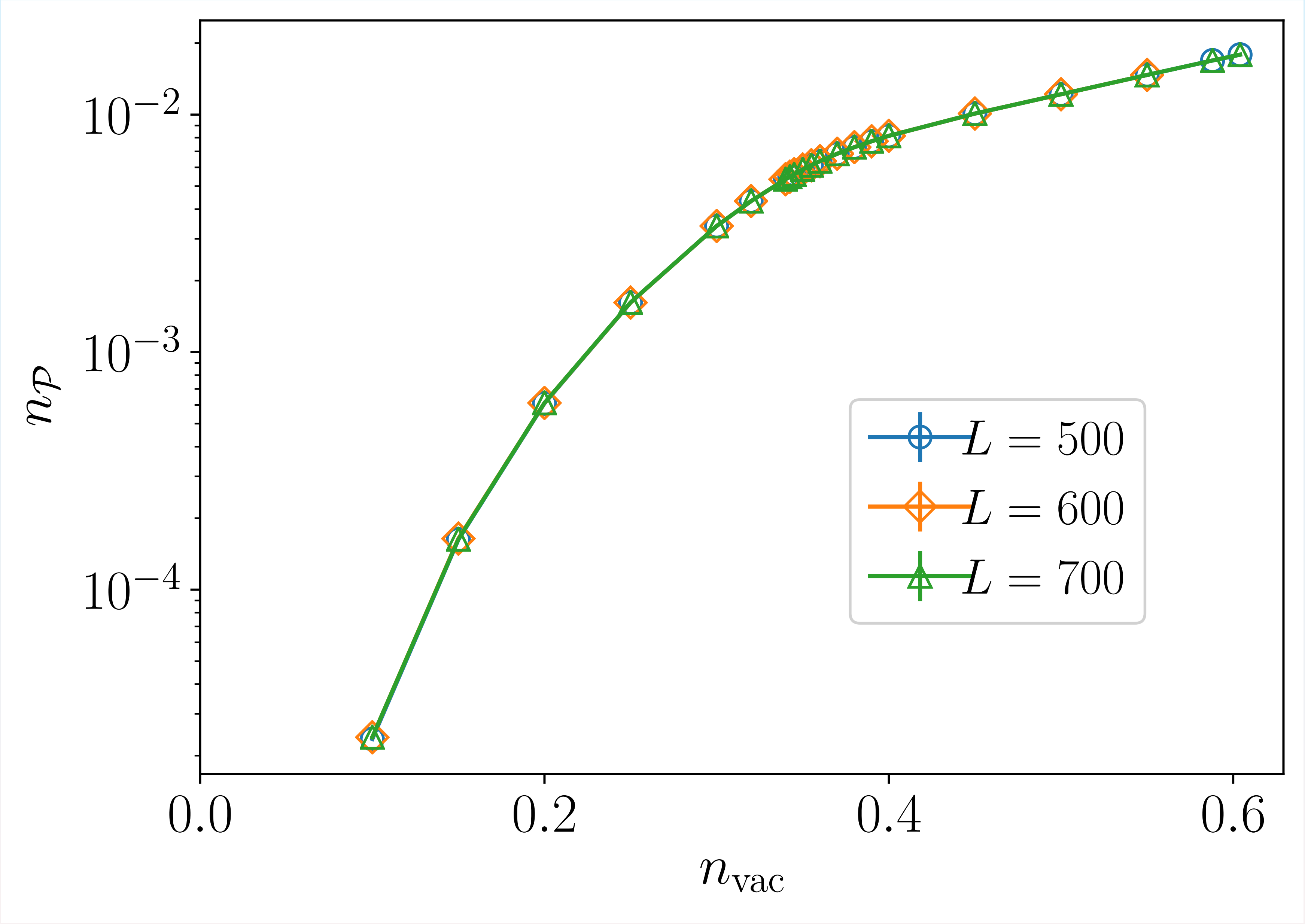
Thermodynamic densities of number of P-type regions



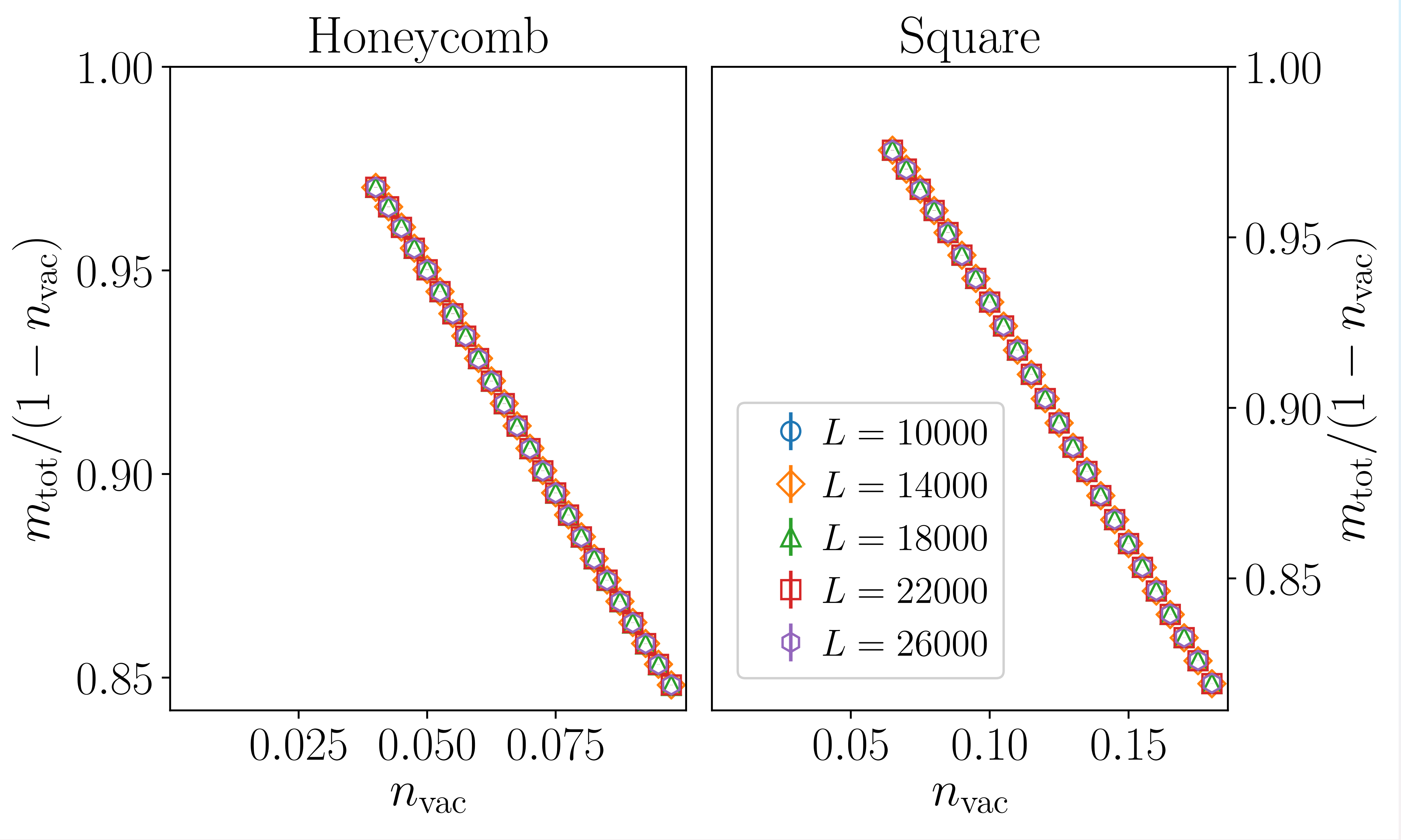
Cubic lattice: Number density of R-type regions



Cubic lattice: Very similar basic picture...

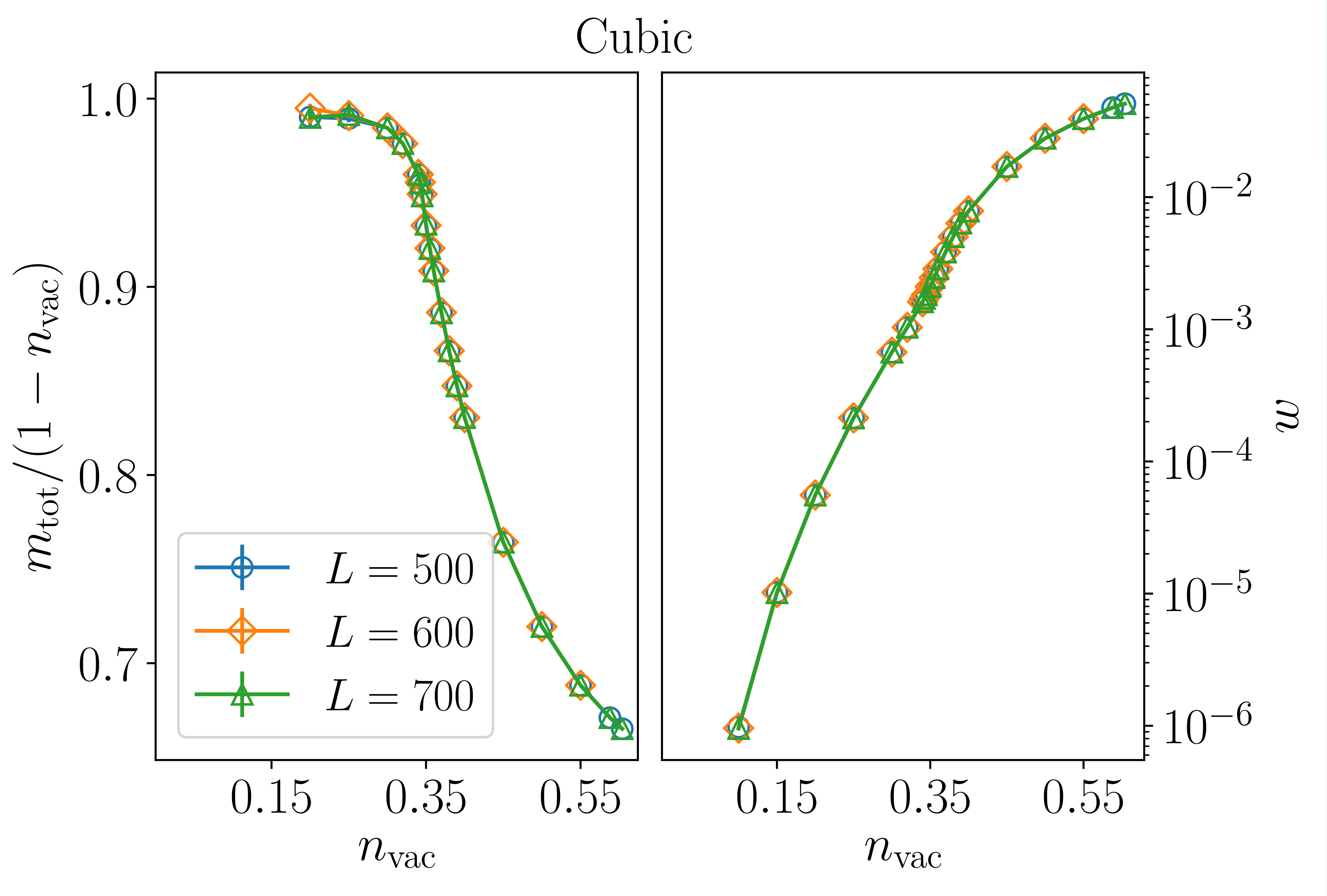


Total mass density of all R-type regions

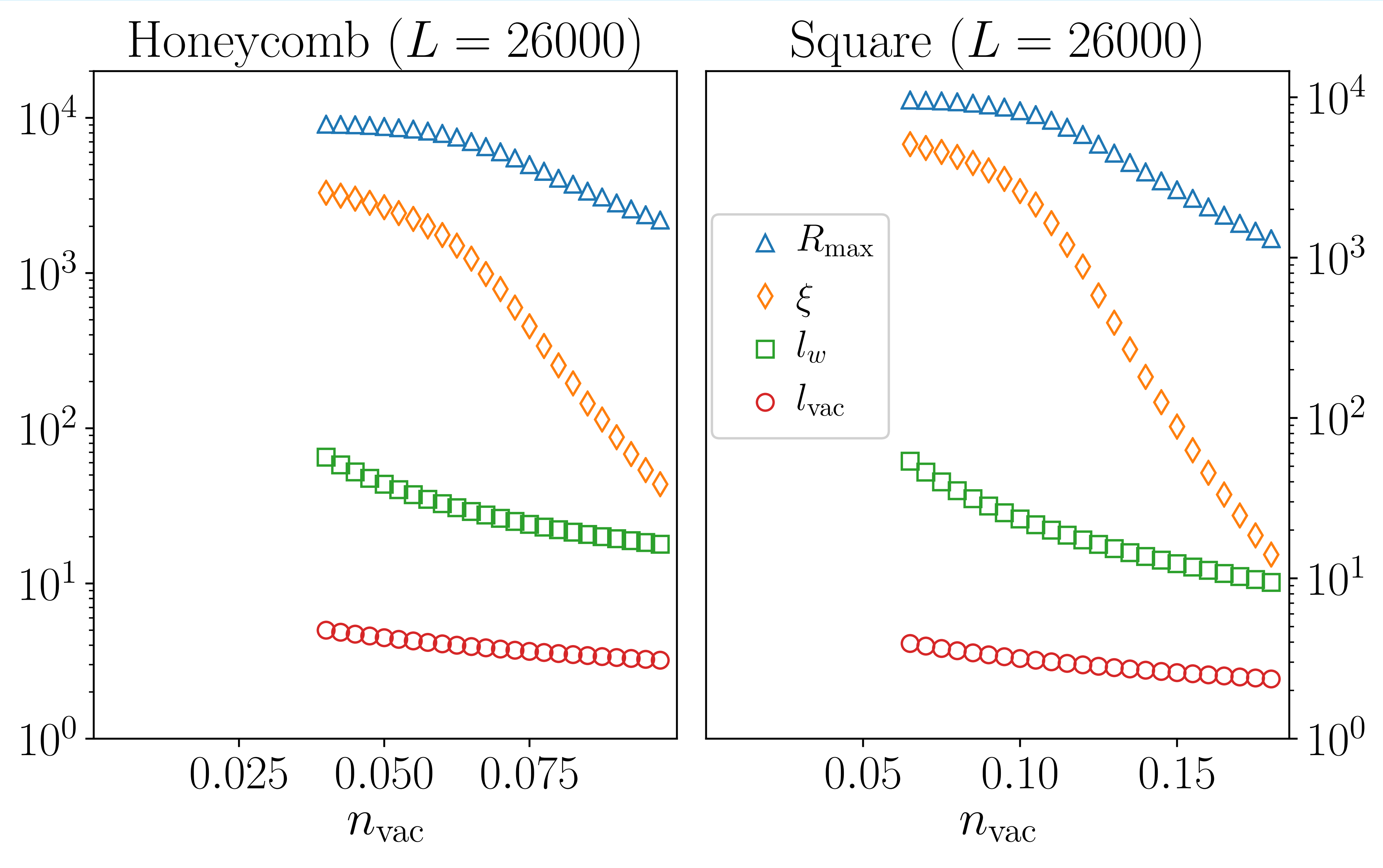


R-type regions take over the lattice in low-dilution limit!

Cubic lattice: R-type regions take over lattice at low dilution

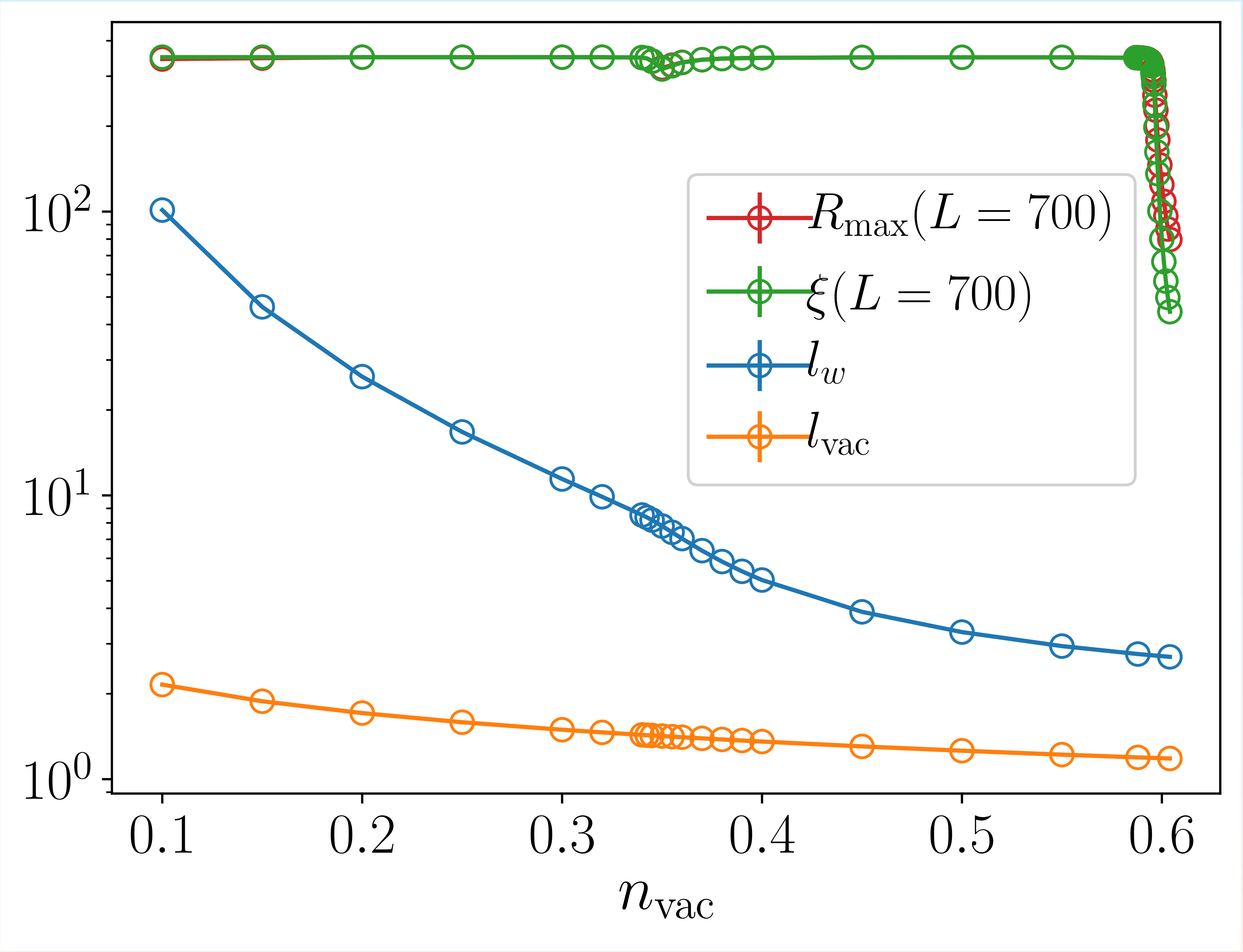


Comparison of length-scales

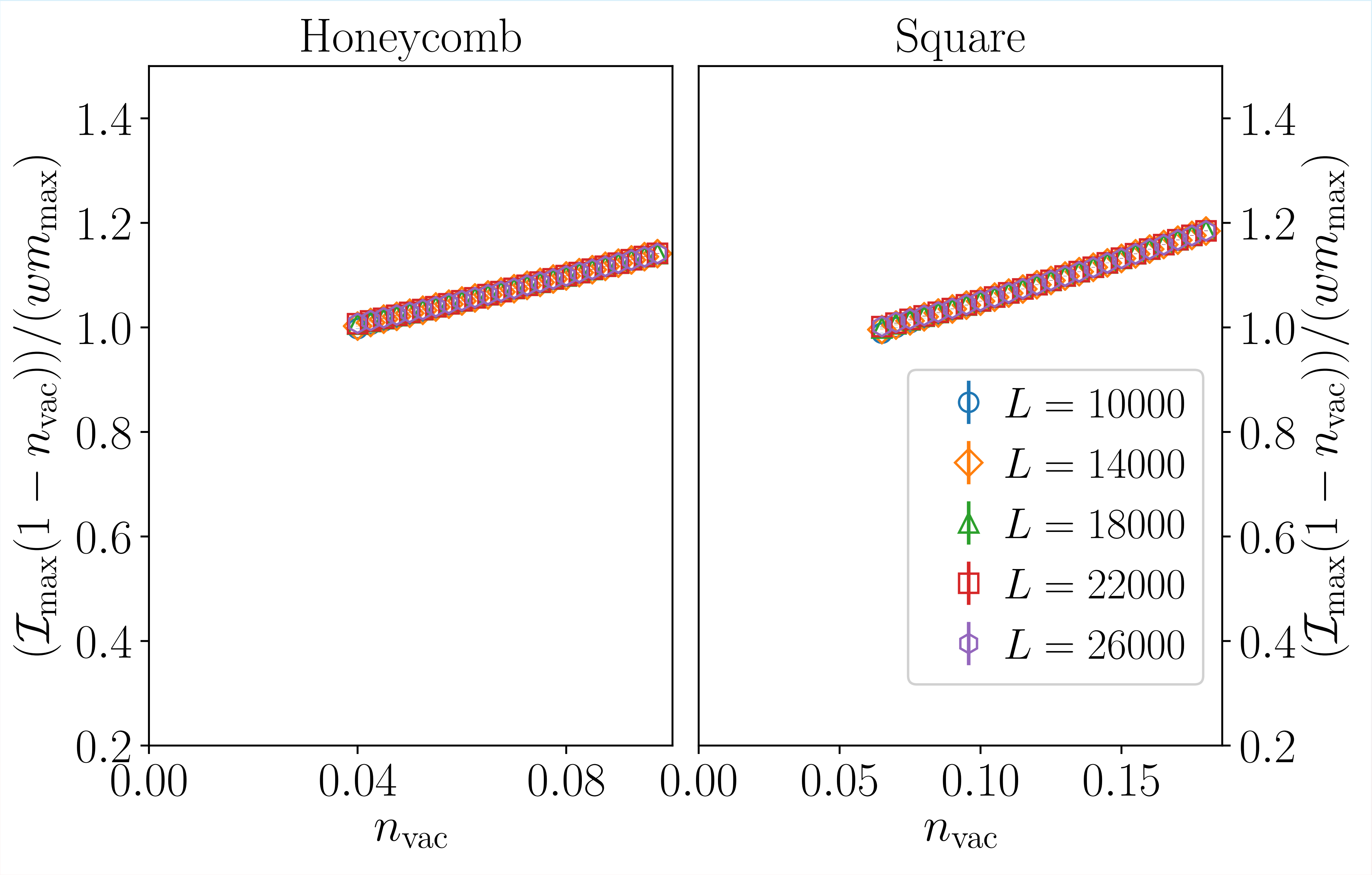


Size of largest R-type region is nearly system-size limited at low dilution

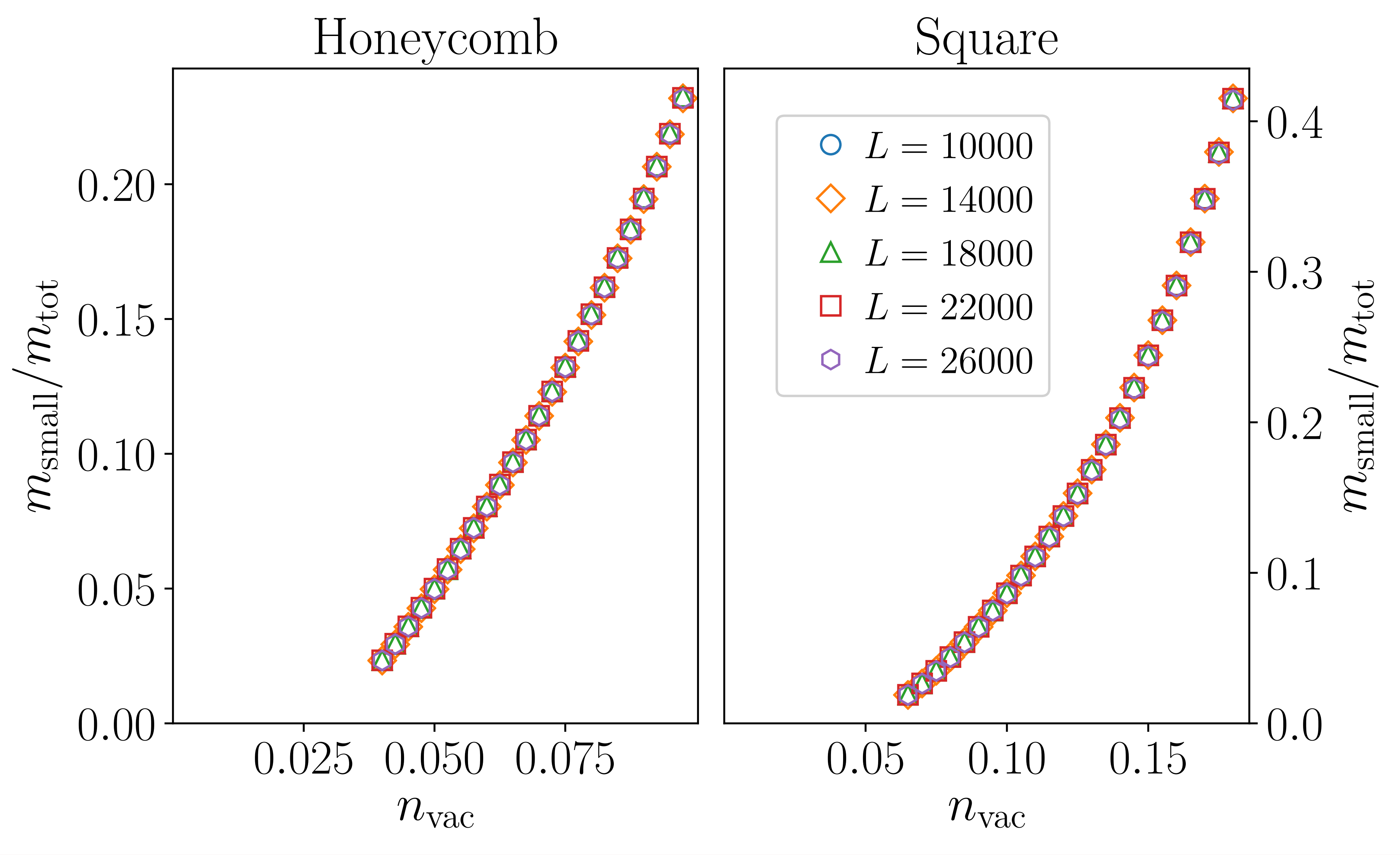
Emergence of a large length scale



Mode density in large R-type regions looks ‘typical’

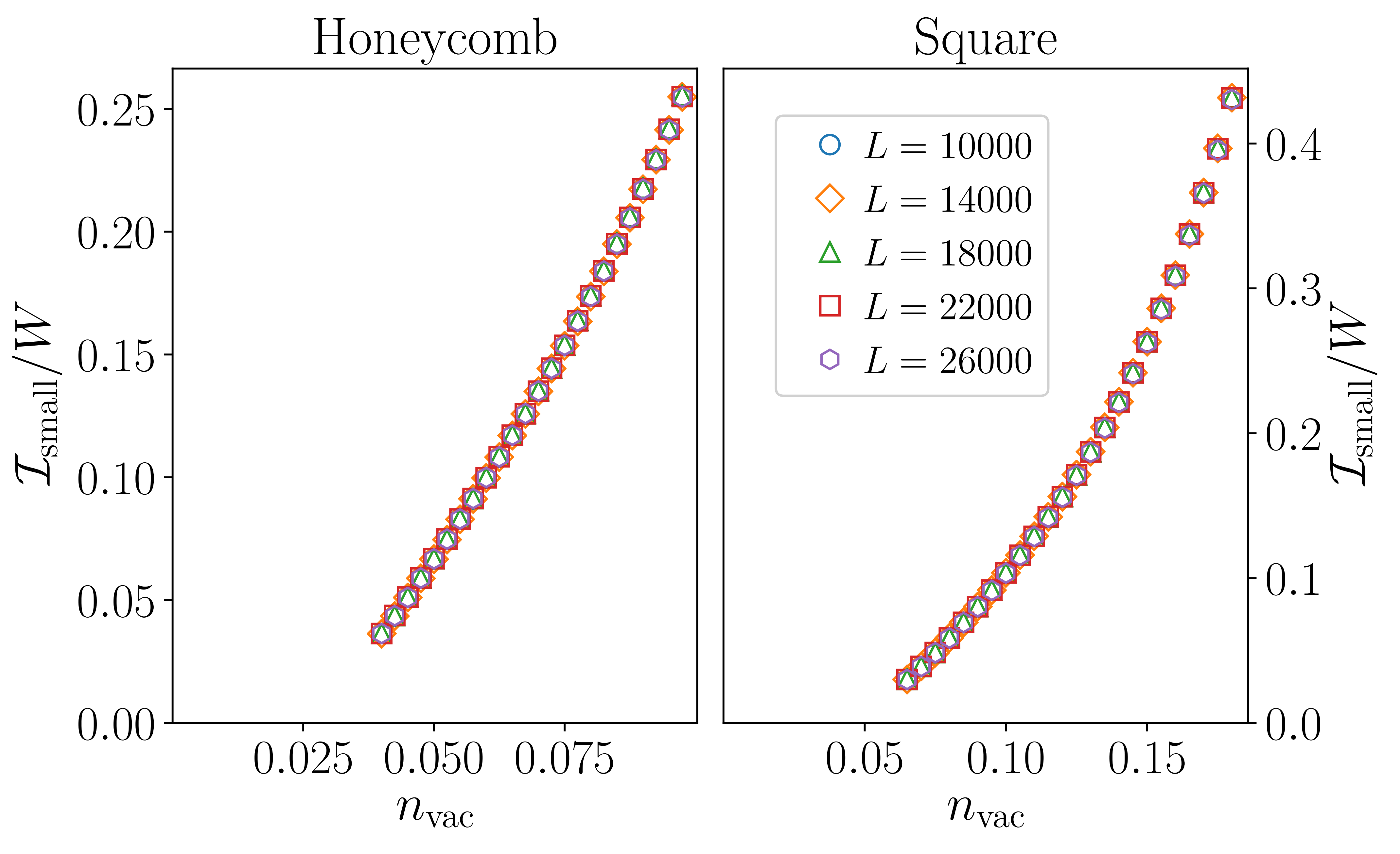


Random geometry dominated by large-scale structures



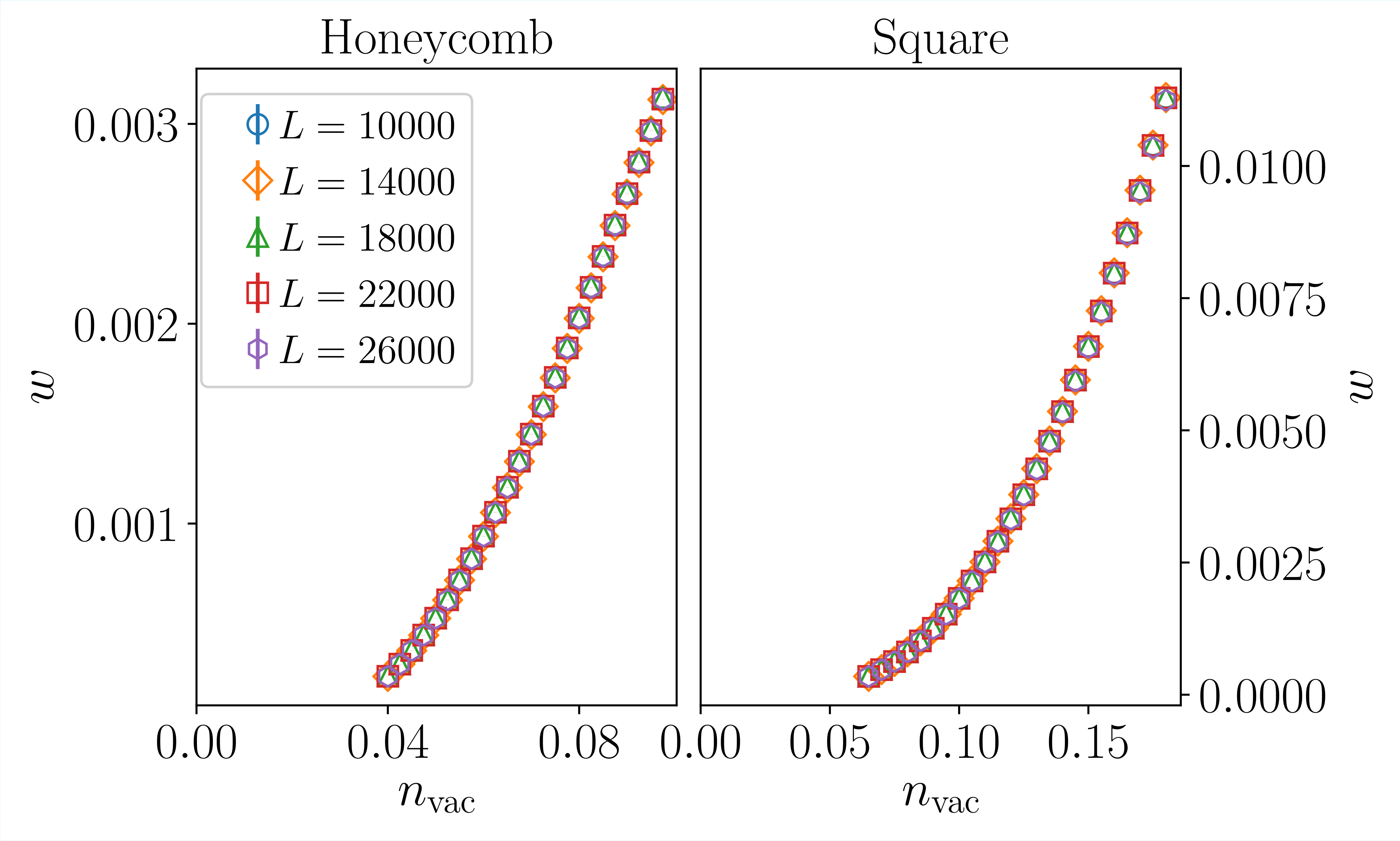
'Small' defined to have $\lesssim 10^4$ vacancies(!)

Zero mode density dominated by large-scale structures

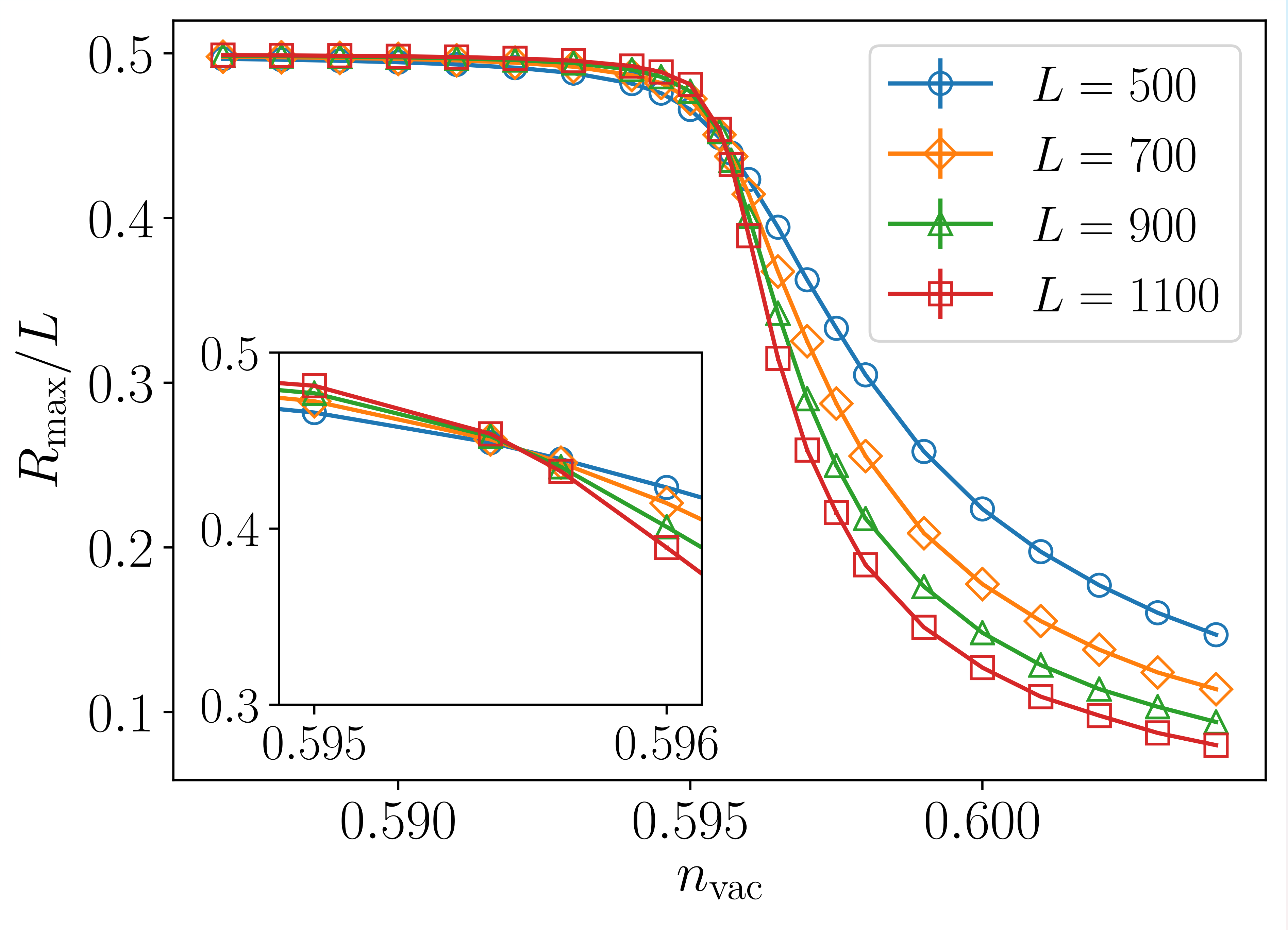


'Small' defined to have $\lesssim 10^4$ vacancies(!)

Zero-mode density: featureless

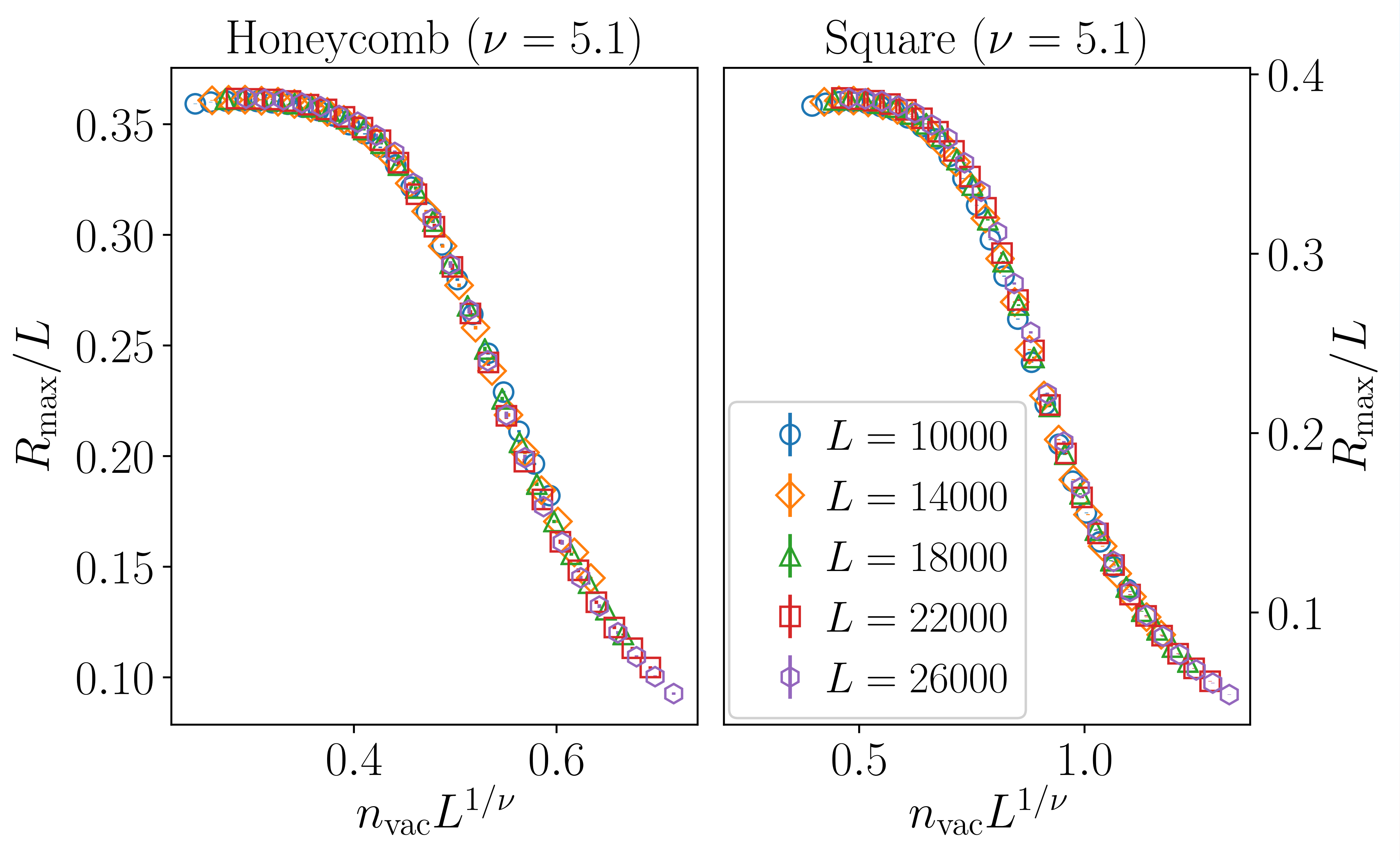


Percolation transition

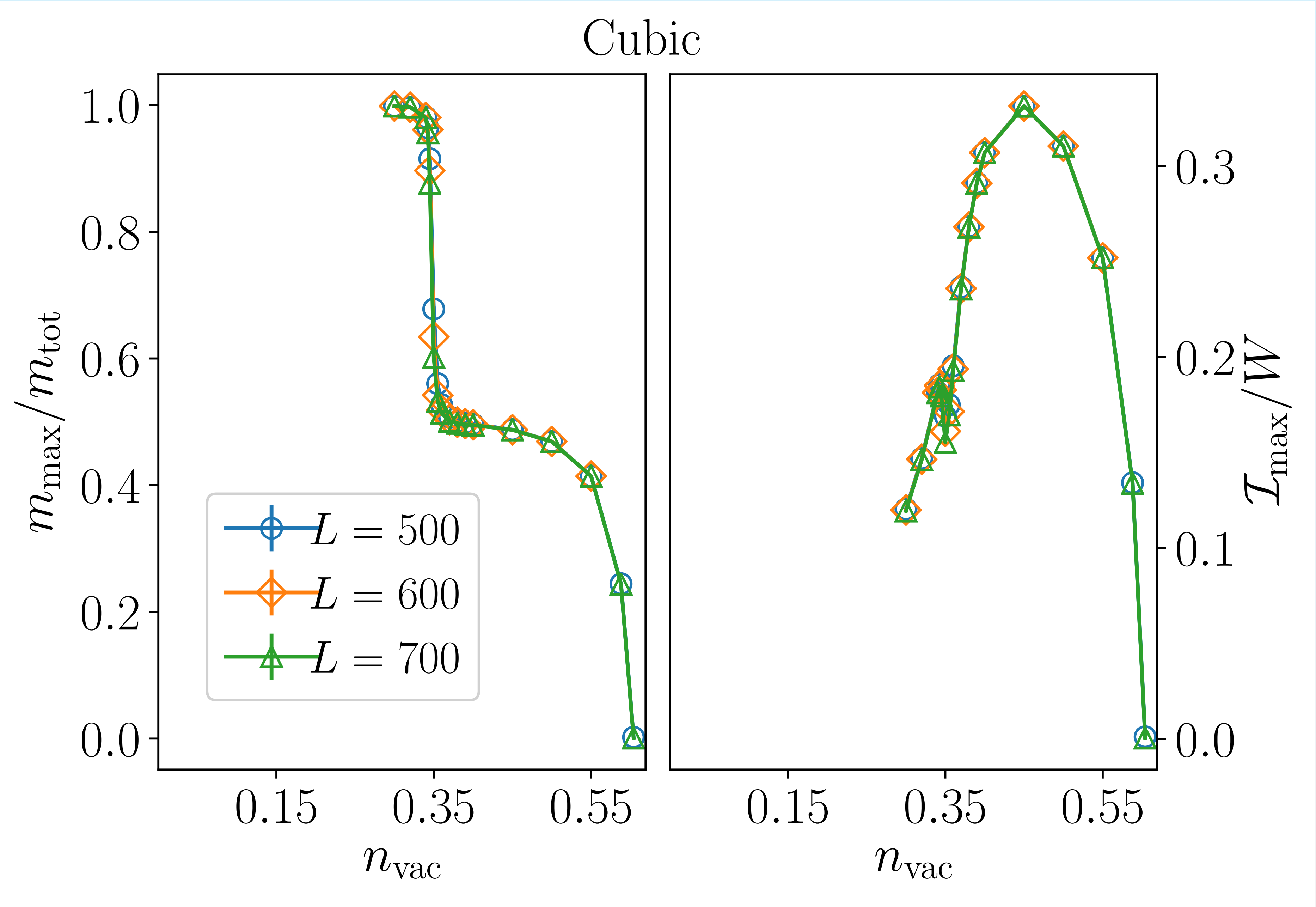


Unconnected with geometric percolation transition of lattice itself

Universal scaling at zero-dilution critical point



A second transition?



Consequences

- Infinitesimal dilution localises monomers of the maximally-packed dimer model in two dimensions
- There is no bipartite quantum percolation transition in two dimensions (long story, starting ‘70s)
- Precise determination of quantum percolation threshold in three dimensions.
- Infinitesimal dilution causes sublattice symmetry breaking in the monomer gas in three dimensions. Consequences for electronic system(?)
- In corresponding Majorana network: Majorana zero modes hosted by R-type regions with odd imbalance undergo a percolation transition
- Perhaps most directly interesting: Low energy triplet excitations in diluted quantum antiferromagnets in the extremely low-dilution regime.