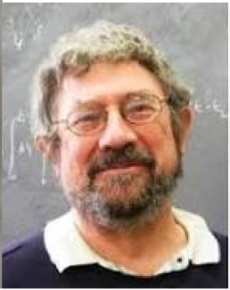

On the work of Thouless, Haldane, and Kosterlitz

“Topological” phases and phase transitions of matter

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Boyle and Newton in the 1600s

- Boyle's gas law:
  $P \propto \frac{1}{V}$ at fixed T
- Newton's law(s) of motion:
  $m\frac{dv}{dt} = F$
  Force of gravity: $F_{12} = G\frac{m_1 m_2}{r_{12}^2}$
  Planetary motion understood...

Beginings of the systematic study of physical properties of materials...
& the science of mechanics...

1750-1850: Bernoulli, Dalton, Avogadro, Carnot, and Kelvin

- Bernoulli & Dalton: Gases made of invisible particles with definite mass (atoms)
- Avogadro: Some gases made of molecules
  Ideal gas equation of state: $PV = Nk_BT$
- Carnot's Therrmodynamics: Connecting heat content to work output (engines)
  absolute temperature, free-energy F, entropy of substances...

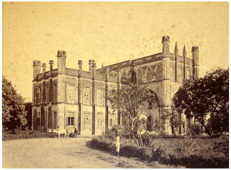
Dynamics of atoms vs thermodynamics of gases

- Trajectories of 10^{23} particles predicted by Newton's laws
  vs
  Thermodynamic concepts heat, temperature, entropy, free-energy?

Where is thermodynamics "hiding" inside the mechanistic view?

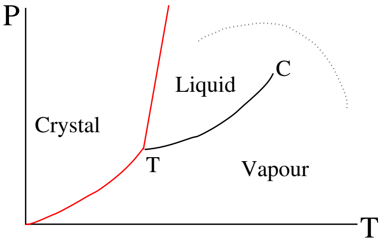
Waterston*, Maxwell, Boltzmann, Gibbs: Statistical mechanics (1850–1920s)

- Statistical description of macroscopic aggregates of atoms
  Each microscopic configuration $C \equiv \{v_i, x_i\}$ arises with probability
  $\text{Probability}(C) = \frac{\exp(-E(C)/k_B T)}{Z}$,
  where $Z = \sum_C \exp(-E(C)/k_B T)$
- Free energy $F = -k_B T \log(Z)$
  (For ideal of atoms : $E = \sum_i \frac{m v_i^2}{2} + \sum_{i,j} V(x_i - x_j)$)



* Grant College Bombay, circa 1860

Ordered phases and phase transitions of matter

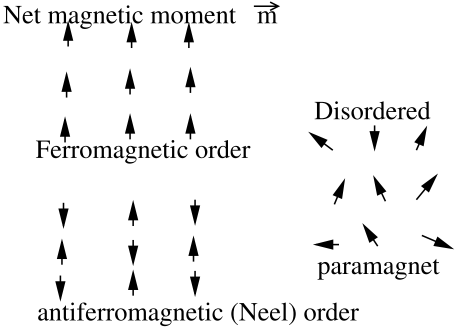


- Phase transformation from fluid to ordered crystalline phases
- Always involves crossing line of singular thermodynamic behaviour

Can statistical mechanics describe such ordering phenomena?

Another example: Ordering of magnetic moments in ionic solids

Intrinsic spin angular momentum of ions quantized in half-integer units ($|\vec{S}| = k/2$). Magnetic moment $\vec{m}_{ion} = 2\mu_B \vec{S}$



Apparent “paradox”: $\text{Prob}(C_{\vec{m}=\uparrow}) = \text{Prob}(C_{\vec{m}=\downarrow})$ by symmetry of E
($E = J \sum_{rr'} \vec{S}_r \cdot \vec{S}_{r'} - A \sum_r (\vec{S} \cdot \hat{n})^2$)
$\langle \vec{m} \rangle = 0$ by Gibbs prescription

Onsager, Anderson: Spontaneously broken symmetry and fluctuation modes

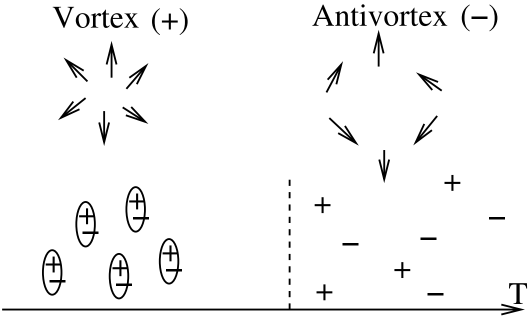
- Resolution of "paradox": Timescale to change direction of $\vec{m}$ diverges with system size.
  "Spontaneous symmetry breaking" and long-range order in "thermodynamic limit"
- Real question: Do small-angle, long-wavelength fluctuations destabilize order?

Peierls, Mermin-Wagner: Role of spatial dimension and magnetic anisotropy

- Peierls: In easy-axis limit ($A > 0$), long-range order possible in planar structures ($d = 2$), and three-dimensional materials, but not in chain-like structures ($d = 1$)
- Mermin-Wagner: Without anisotropy ($A = 0$), or with easy-plane anisotropy ($A < 0$), long-range order possible in $d = 3$, but not $d = 2$ or $d = 1$

Long-range magnetic order only possible in sheet-like materials if magnetic moments can only point up or down along fixed axis (eliminates small-angle fluctuations)

Kosterlitz-Thouless: Vortex-binding transition in $d = 2$ easy-plane case



$\langle \vec{m}(r) \cdot \vec{m}(r') \rangle \sim \exp(-|r - r'|/\xi)$ for $T > T_{KT}$ (paramagnet)
$\langle \vec{m}(r) \cdot \vec{m}(r') \rangle = \frac{C}{|r-r'|^{\eta}}, \eta \in (0, \frac{1}{4})$ for $T < T_{KT}$ (Power-law ordered)

Distinction is "topological"

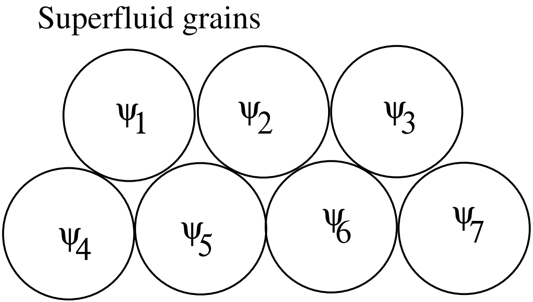
- KT transition is transition between "ionized plasma" of single vortices and neutral gas of vortex-pairs
- Distinction has a "topological" flavour to it:
  Can count number of isolated vortices (signed sum) in region by simply looking at winding of moments along the boundary

Connection to superfluid/superconducting thin films?

Enter: Quantum mechanics

- Linear superposition of alternatives: $a|\uparrow\rangle + b|\downarrow\rangle$
- Trajectories replaced by sum over paths, weighted by $exp(iS)$ for each path $S = \int dt(\frac{m}{2}(\frac{dx}{dt})^2 - V(x))$
- Particles behave like waves in some ways

From Superfluids/superconductors to magnets with easy-plane anisotropy



Each grain r is in state $|\psi_r\rangle \sim \sum_{n=0}^{\infty} \frac{\psi_r^n}{\sqrt{n!}} |n\rangle$

$|n\rangle$: State with exactly n particles in grain

$E = -J \sum_{rr'} (\psi_r^* \psi_{r'} + \psi_{r'}^* \psi_r)$

$\psi = |\psi| e^{i\theta} \to \vec{m}$ with easy-plane anisotropy

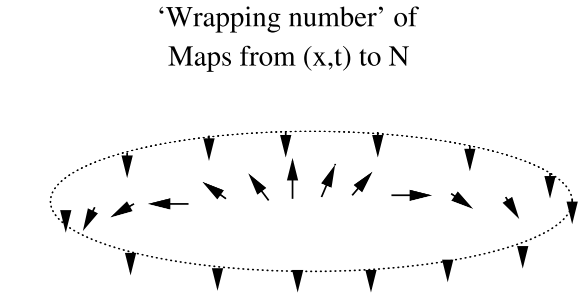
Consequence: Thin-film superfluidity

- $\langle \psi^*(r)\psi(r') \rangle \sim 1/|r - r'|^\eta$
- Superfluid fraction $\rho_s \propto \eta$
- $\rho_s/(k_B T_{KT}) = 2m^2/\pi\hbar^2$
  Observable universal jump in superfluid fraction (Nelson-Kosterlitz)

Haldane: Quantum mechanics of linear chains of magnetic moments:

- $E = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$ with $J > 0$
  Recall: Intrinsic spin angular momentum of ions quantized in half-integer units ($|S| = k/2$). Magnetic moment $\vec{m}_{ion} = 2\mu_B \vec{S}$
- Quantum-mechanics: Trajectories replaced by sum over histories
- Best formulated in terms of Neel vector $\vec{N}_i \propto \vec{S}_i - \vec{S}_{i+1}$ with $|\vec{N}_i| = 1$
  $\vec{N}$ varies slowly with i: Continuum approximation valid $\vec{N}_i \to \vec{N}(x)$
- Weight of trajectory is $\exp(iS)$ with
  $S = \int dt \int dx \left[(\frac{d\vec{N}}{dt})^2 - (\frac{d\vec{N}}{dx})^2 \right] + S_B$

S_B: Depends only on "topology" of $\vec{N}(x,t)$



$S_B = \pi k Q[\vec{N}]$, where $Q = \frac{1}{4\pi}\int dx dt \vec{N}\cdot(\frac{d\vec{N}}{dx}\times\frac{d\vec{N}}{dt})$ counts "wrapping number" of configuration

Consequences

- Chains with $S = 1$ (e.g. Ni^{2+})have exponentially small low-temperature susceptibility and specific heat
- Chains with $S = 1/2$ (e.g. Cu^{2+}) have much higher susceptibility and specific heat
  In fact, power-law ordered
- "Topologically protected" boundary spin $S = 1/2$ moments in integer spin chains cut by non-magnetic impurities: Seen directly in NMR experiments.
  "Topological protection" Presence of free spin $S = 1/2$ at chain-ends is totally independent of details of crystal structure and energetics

Thouless-Nightingale-Kohmoto-Den Nijs, Haldane: Winding of wavefunctions

- Electron waves in "diffraction grating" created by regular crystalline arrangement of ionic cores
- Described by wavefunctions arranged in groups called bands
- Winding number of band "wavefunction" u of electrons determines Hall-voltage
- Result: Hall conductivity $\sigma_H = ne^2/h$
  $n = \frac{1}{2\pi}\int d^2kB(\vec{k})$ where $B = \frac{dA_y}{dk_x} - \frac{dA_x}{dk_y}$ with $A_j = i\langle u_k|\frac{d}{dk_j}|u_k\rangle$
  Not pictorially obvious: But counts "winding number" n of wavefunction

Impact

- Inspiration for sustained efforts to expand our classification of phases of matter, beyond “obvious” distinctions of spontaneously broken symmetry and long-range order
- Much known by now about topologically distinct phases of matter and transitions between them
- Most remarkably: Purely theoretical ideas playing key role in understanding new materials
  e.g. “topological insulators” with “protected” surface states