Dulmage-Mendelsohn percolation

Random geometry of maximum-matchings, zero modes & Majorana excitations…

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Some generalities

- Quenched disorder matters (often)
- Particles scatter and diffuse (may be anomalously...)
- Matter-waves scatter and localise (sometimes weakly...)

Classical transport of particle-fluid

- Simplest setting: Porous random medium
- Random geometry of medium determines fluid transport
- Paradigm of percolation
- More generally: Diffusion in presence of random potentials

Broadbent & Hammersley, Sinai...

Percolation

- Study end-to-end connectivity of a porous medium
- Can you go from one end to other?
- Answer changes as function of porosity
- Simplest model: Randomly diluted regular lattice (graph)

Broadbent and Hammersley, Percolation processes I, Crystals and Mazes (1957)

Sharp question about the random geometry

Crossing probability

- Consider two dimensional square grid or honeycomb net or three dimensional cubic lattice of linear size L
- Remove fraction n_{vac} of sites and delete links to removed sites.
- $P_w(n_{\text{vac}}, L)$: Probability that one can 'walk' from left end to right end along existing sites and links.
- How does this behave as a function of n_{vac} and L?

Sharp threshold behaviour

Property of thermodynamic limit

- In $d = 2$ and in $d = 3$, $L \to \infty$ limit characterised by sharp threshold behaviour as function of $n_{\rm vac}$
- 'Percolation transition'
- Simplest geometric example of a 'thermodynamic phase transition'
- For $n_{\rm vac} < n_{\rm vac}^{\rm crit}$, $P_w \to 1$ as $L \to \infty$
- For $n_{\rm vac} > n_{\rm vac}^{\rm crit}$, $P_w \to 0$ as $L \to \infty$

Approach to thermodynamic limit

‘Scaling behaviour’

- $L \to \infty$ limit is approached in interesting way
- $P_w(n_{\rm vac}, L) = f(\delta L^{1/\nu})$ where $\delta = n_{\rm vac} - n_{\rm vac}^{\rm crit}$
- $f(x)$ is the ‘universal scaling function’, ν is a ‘scaling exponent’ and $n_{\rm vac}^{\rm crit}$ is the ‘critical’ dilution

Approach to thermodynamic limit

‘Scale invariance’ and ‘universality’

- $f(x)$ and ν believed to be independent of microscopic-scale details. Examples of ‘universal critical properties’. Square lattice and honeycomb net have same $f(x)$ and ν.
- Cubic lattice has different $f(x)$ and ν. Dimensionality dependent.
- Implies different size samples have same P_w for $n_{\text{vac}} = n_{\text{vac}}^{\text{crit}}$
- Scale invariance: Pictures of random geometry look same if we rescale pictures!
- Only true if we ignore lattice scale features, but amazing anyway!

Localization of matter waves

- Anderson localization of electrons in dirty metals
- Localization of quasiparticles in dirty superconductors
- Symmetries of disordered Hamiltonian matter (e.g. in random matrix theory)

Anderson, Ramakrishnan, Abrahams, Thouless, Dyson, Wegner, Mehta...

Simplest lattice model

- $\sum_{j \in i} t_{ij}\psi_j + V_i\psi_i = E\psi_i$ for all i
- E is the energy of the particle represented by wave function ψ_i
- t_{ij} are 'hopping amplitudes' for particle to hop from site i to site j
- V_i are values of external potential at sites i
- Allowed E: Eigenvalues of matrix of $t_{ij} + V_i\delta_{ij}$

Quantum mechanics of bipartite random hopping

- Particle hopping on a bipartite lattice
- Random (real) hopping amplitudes between nearest neighbour sites
- Bipartite (‘chiral’) symmetry: State with energy ϵ has partner at energy $-\epsilon$
- Symmetry broken by random potentials, next-neighbour hopping...

Vacancy disorder

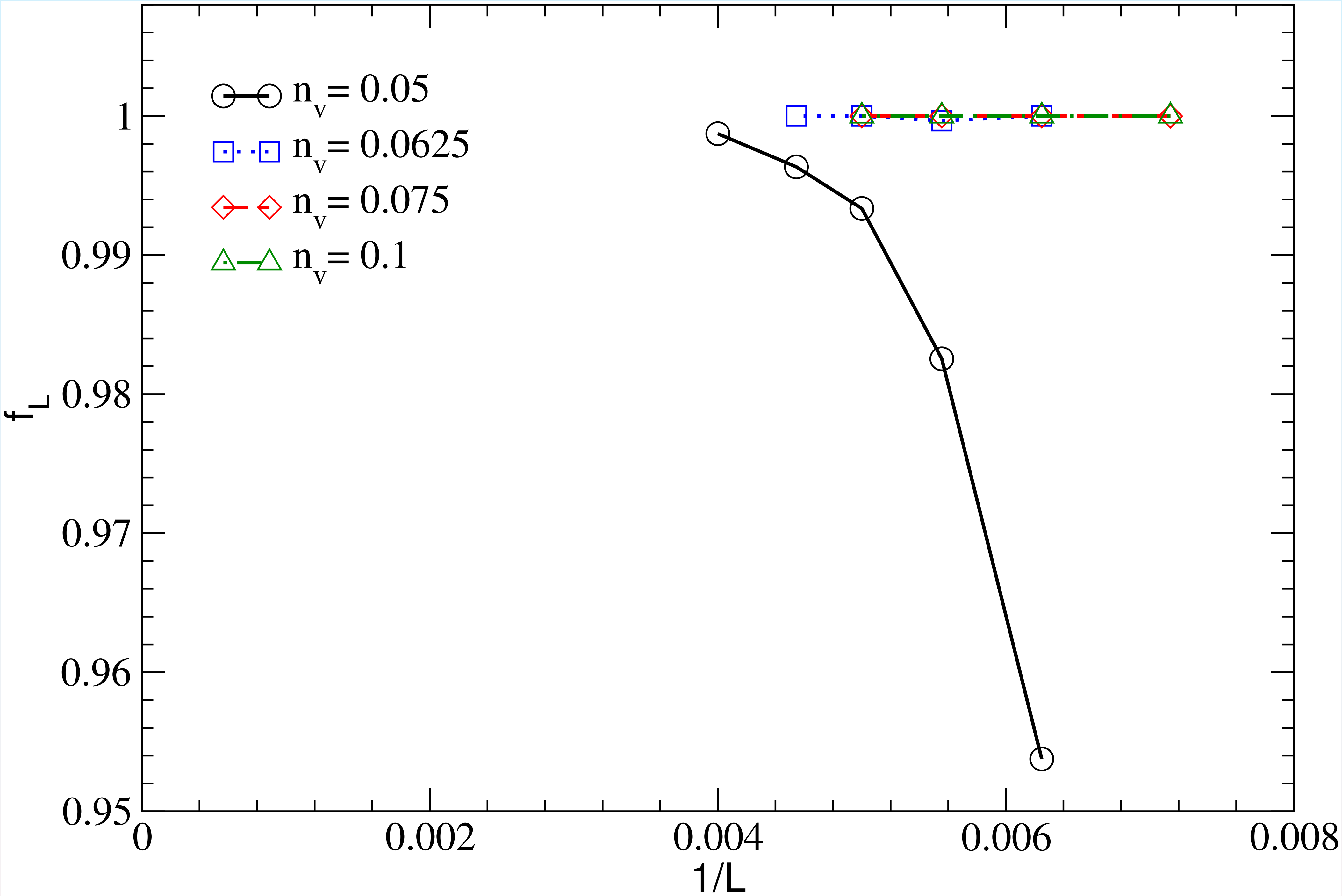
- Vacancy disorder: random dilution of the underlying lattice (missing atoms in crystal structure)
- Also natural if substitutional impurities correspond to missing orbital
- Question: Does vacancy-disorder change the asymptotic low-energy behaviour of $\rho(\epsilon)$?
- Note: No change in symmetries

The question

$\epsilon = 0$ is special

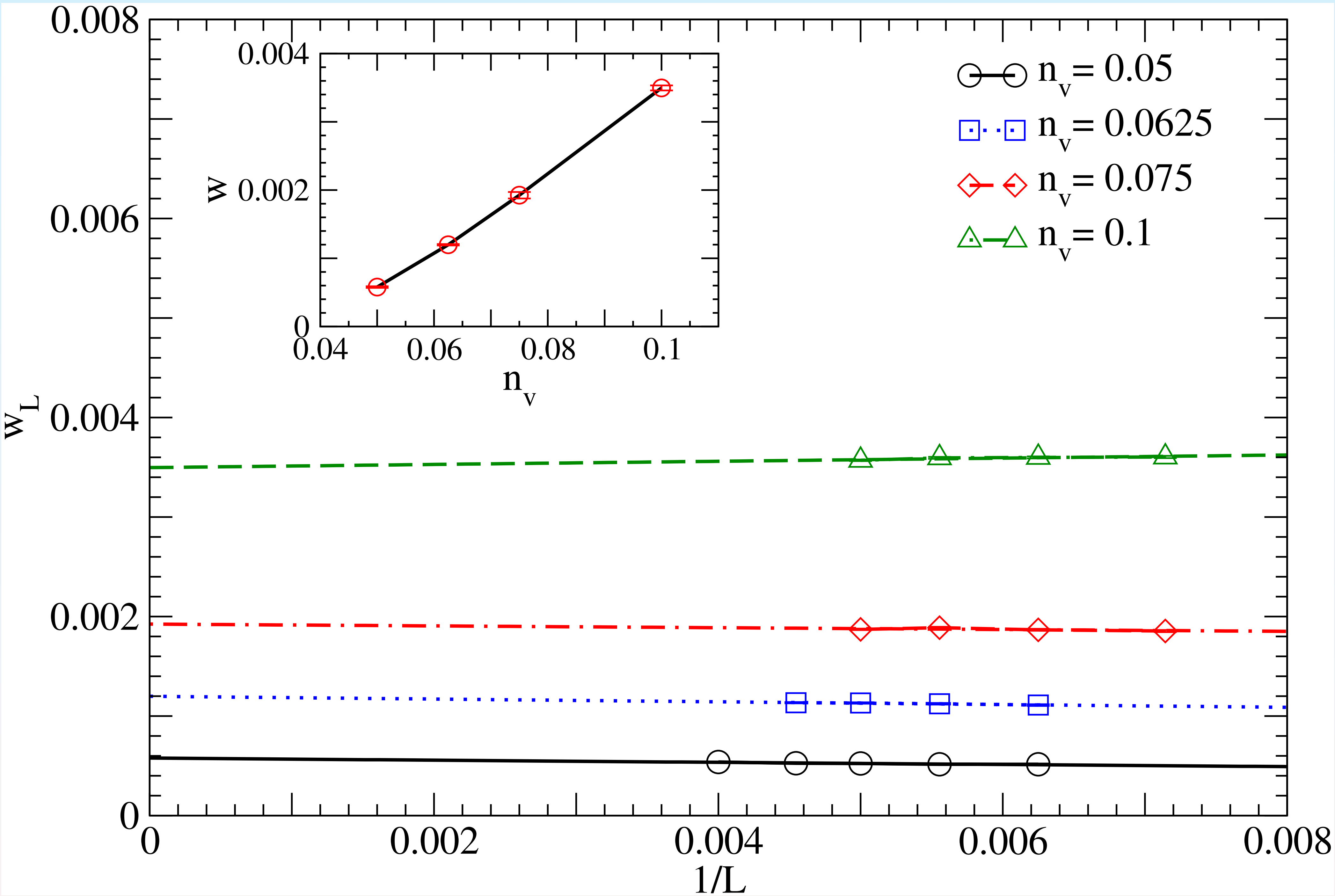
- Does anything interesting happen in the quantum mechanical spectrum of H near $\epsilon = 0$?

‘Surprise’: Zero modes

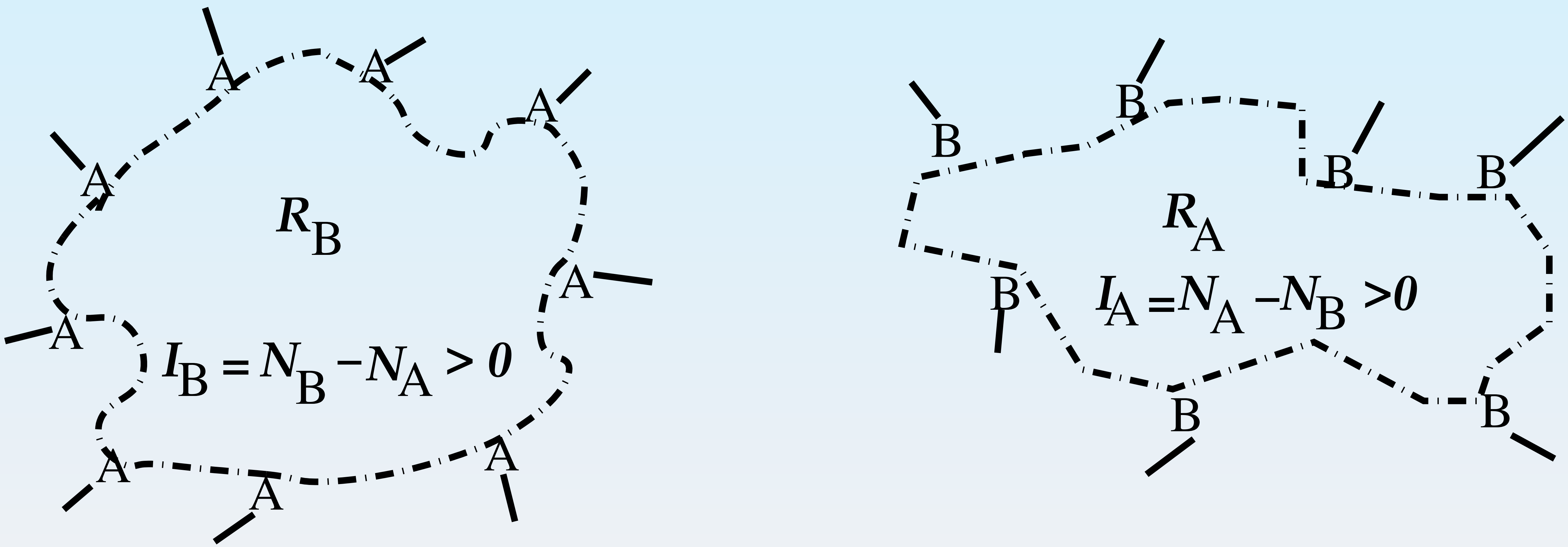


For large enough lattice, almost every sample has at least one zero mode

In fact: Nonzero density of zero modes

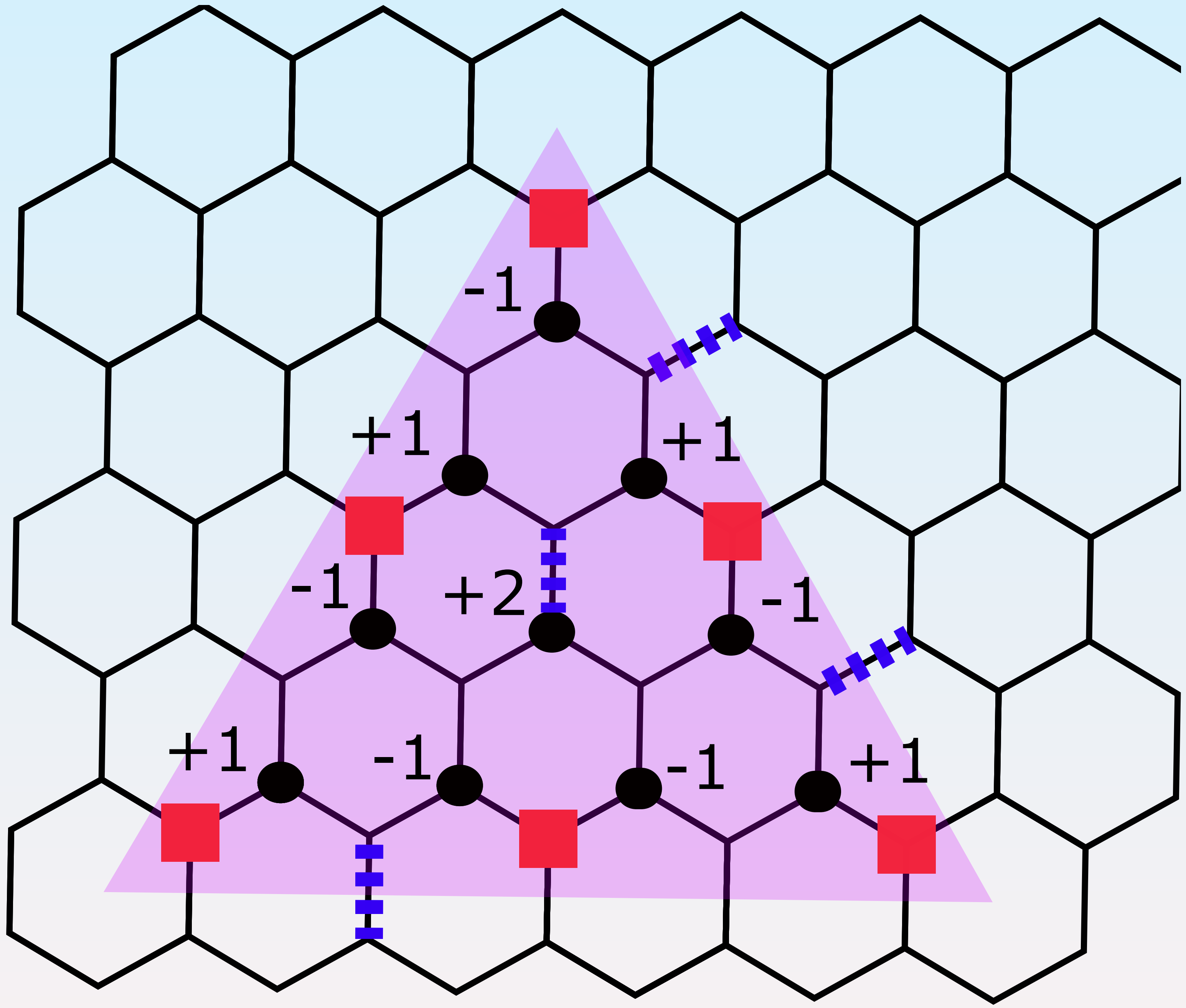


Our idea: R-type regions hosting zero modes



Constraint on zero-energy wavefunction: $\psi_A: \sum_{A \in B_0} t_{AB_0} \psi_A = 0$ $\quad$ $\psi_B: \sum_{B \in A_0} t_{BA_0} \psi_B = 0$

Example of R-type region



Gives a rigorous lower bound on density of zero modes on honeycomb lattice

Major open questions remained

- Actual density of zero modes much larger than lower bound
- What dominates?
- General algorithm for identifying all R-type regions?

Finally: progress...

- Key idea: Disorder-robust zero modes only depend on connectivity, not hopping strengths.
- R-type regions rely on local imbalance between A and B type site densities.
- Suggests thinking in terms of matchings a.k.a dimer covers
- Places that cannot be covered by dimers host wavefunctions?

Counting zero modes from maximum matchings

AIP | The Journal of Chemical Physics

Some Studies in Molecular Orbital Theory I. Resonance Structures and Molecular Orbitals in Unsaturated Hydrocarbons

H. C. Longuet-Higgins

(1950)

Chemist (structure of diborane)

Physicist (advisor of Peter Higgs)

Pioneer in:

- Cognitive science and computer vision
- ‘Machine-intelligence (aka AI!)
- Computer music...

Counting zero modes from maximum matchings

Longuet-Higgins (restated)

- Number of monomers in any maximum matching of bipartite graph gives number of topologically-protected zero modes

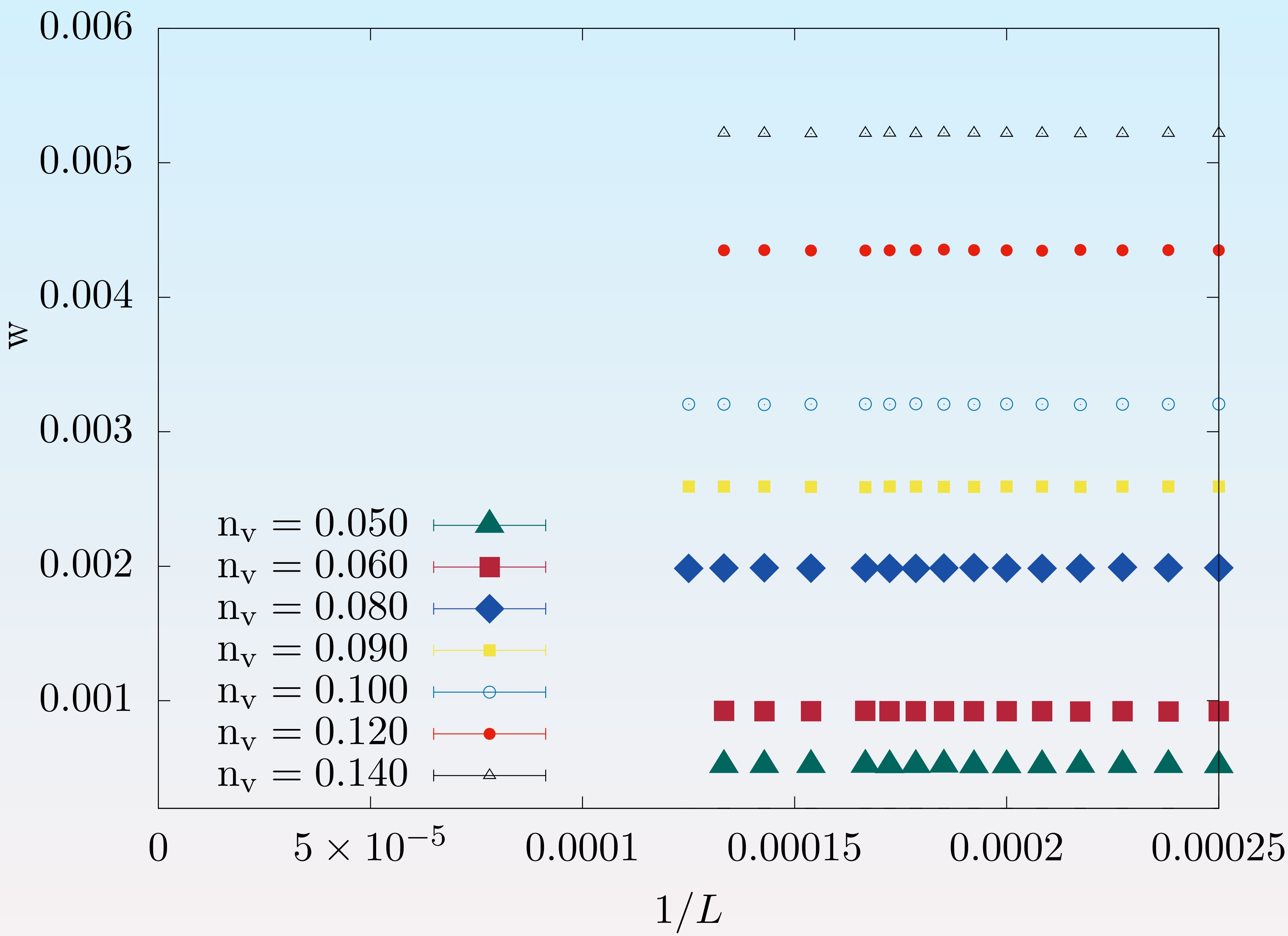
nonzero-defect generalization of Tutte’s Theorem in bipartite case

Gives access to larger sizes

- Somewhat larger-size studies confirm our claim of enormous excess of zero modes above simple lower bounds
- Again: What are these modes?

Numerical results on percolating by Evers group (Weik et al ‘16)

Density of disorder-robust zero modes



Equivalent to counting zero modes of a $10^9 \times 10^9$ matrix (?!)

Wavefunctions?... Longuet-Higgins again:

- Set of all sites that host a monomer in at least one maximum matching form support of all topologically-protected zero mode wavefunctions

Want more: What do the zero modes look like?

Maximally-localised basis for zero-mode subspace?

Basis-invariant consequences for on-shell zero-energy Green function?

Consequences for two-terminal conductance?

Our strategy

Bring into play classic result from graph theory

COVERINGS OF BIPARTITE GRAPHS

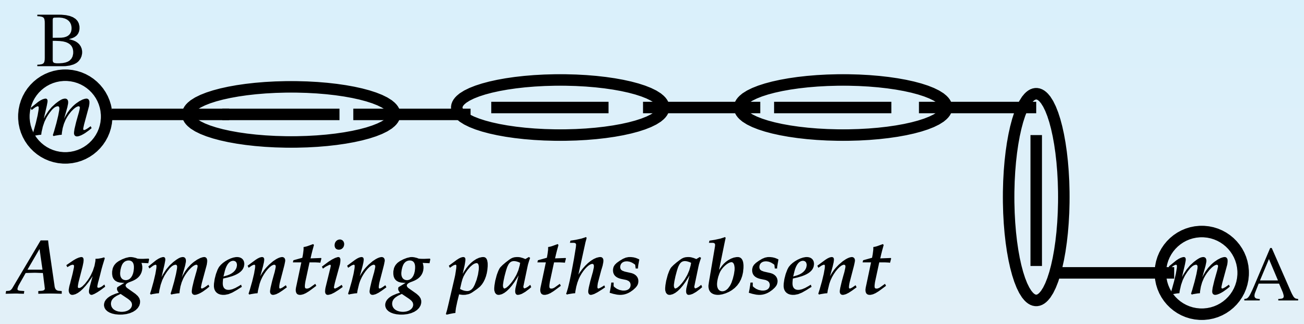
A. L. DULMAGE AND N. S. MENDELSOHN

Can. J. Math. 10: 517, 1958

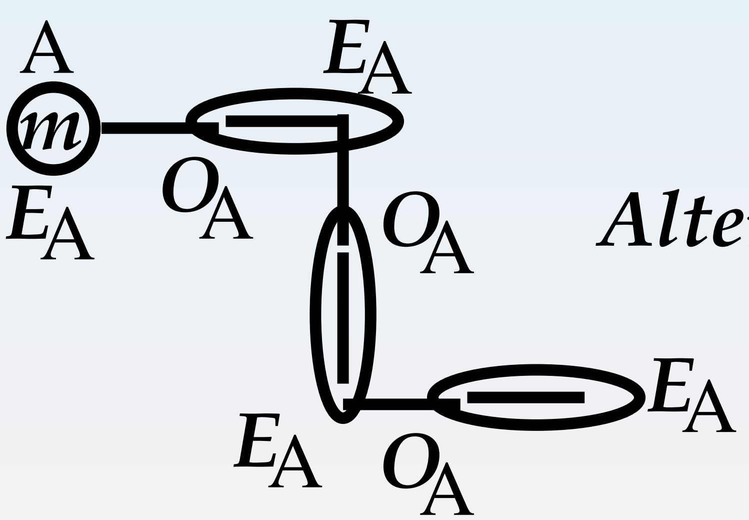
Characterization of regions of lattice that can host monomers

Matchings, augmenting paths, and alternating paths

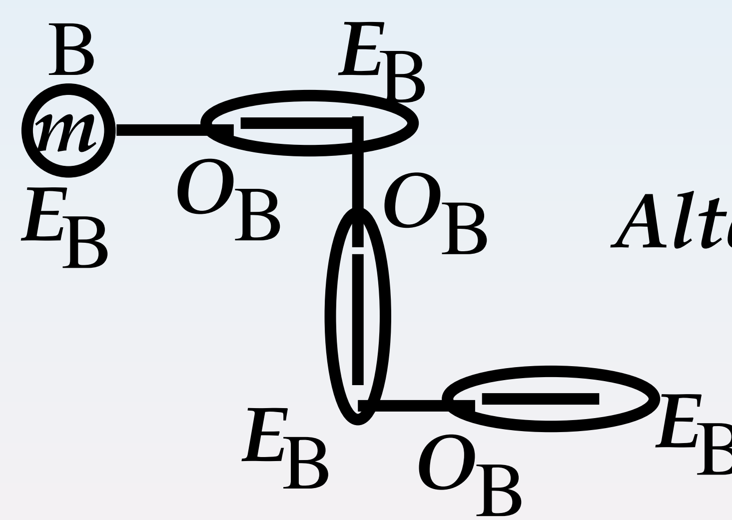
In any maximum matching M:



Augmenting paths absent



Alternating path



Alternating path

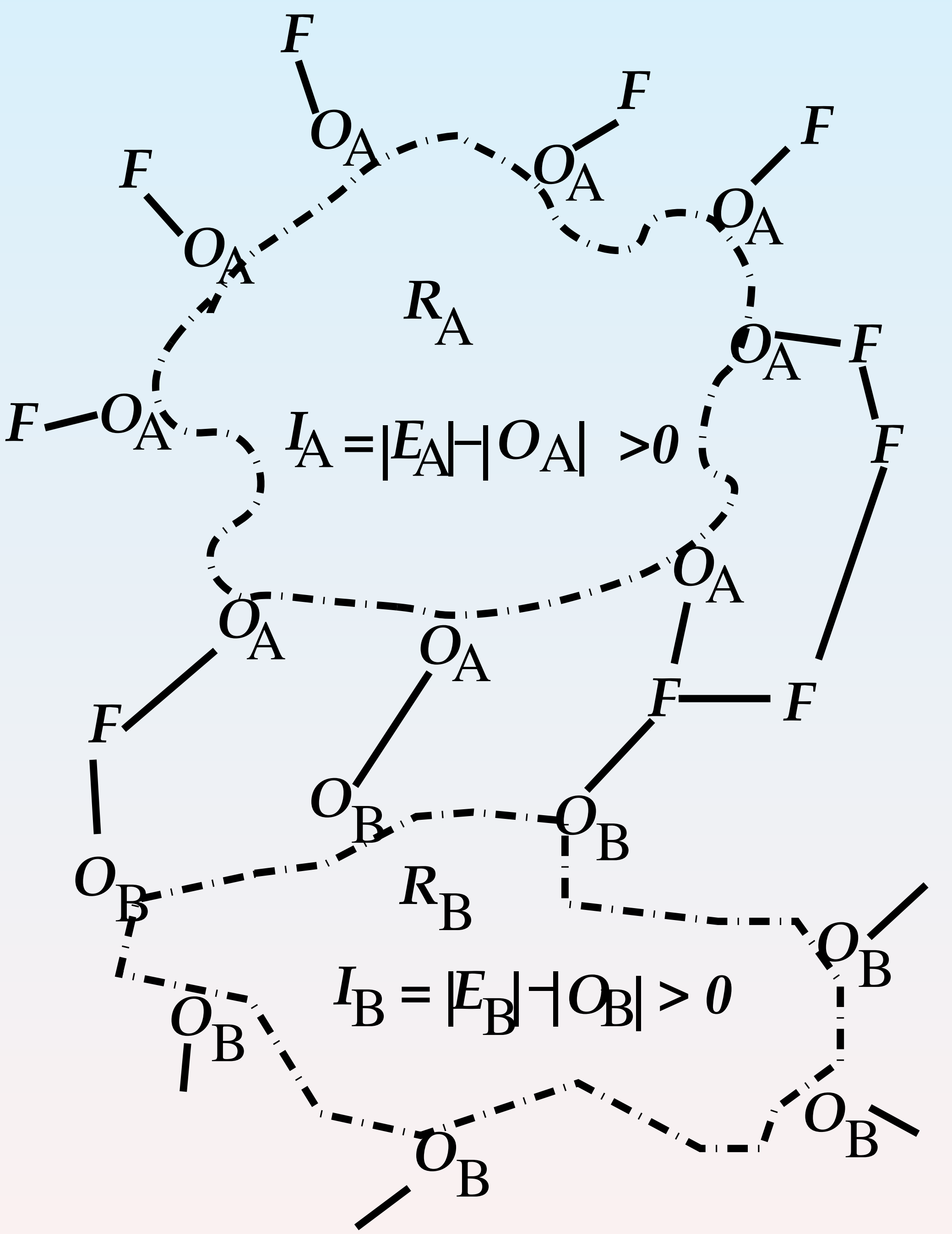
A useful version of the DM decomposition

Even, odd and unreachable sites from any one maximum matching M

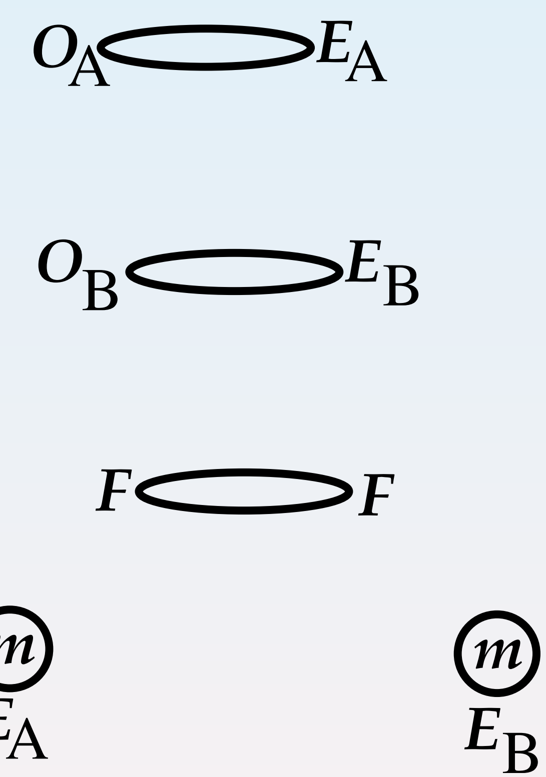
- Even: Reachable by even-length alternating paths from monomers of M (including length zero)
- Odd: Reachable by odd-length alternating paths from monomers of M
- Unreachable: Not...
- Decomposition:
- $C_A : E_A \cup O_A$
- $C_B : E_B \cup O_B$
- $P : U_A \cup U_B$

Key observation: R-type regions from C_A and C_B

Focus on connected components



In any maximum matching:



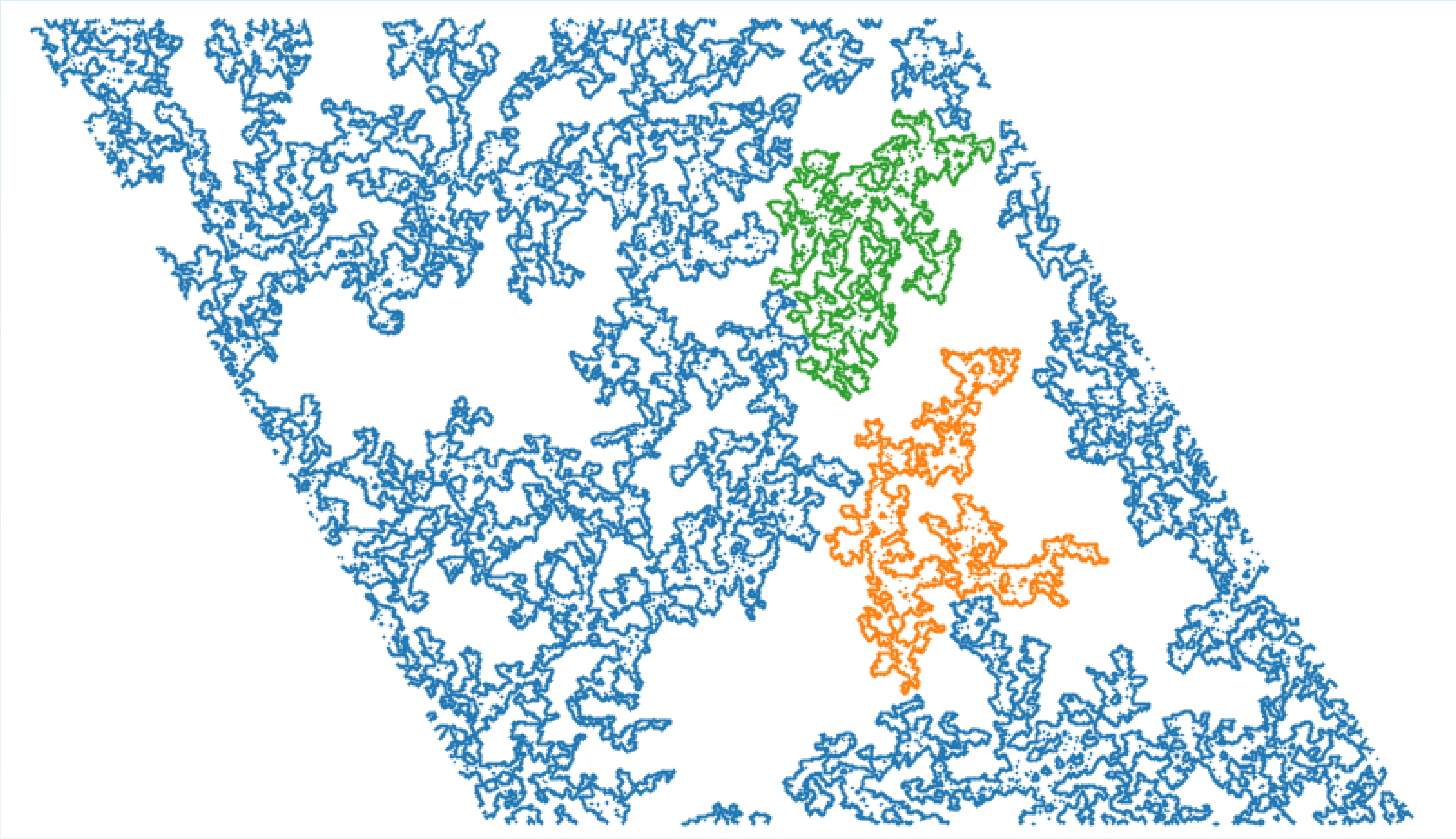
Key observation: Connected components construct R-type regions

- Each R_i^A (R_j^B) hosts I_i^A (I_j^B) topologically-protected zero modes with wavefunctions confined to the region.
- Provides alternate 'local' proof for correspondence between monomers of maximum matchings and zero modes of adjacency matrix
- Gives topologically-robust construction of a maximally-localised basis for zero modes
- Standard proofs (Longuet-Higgins, Lovasz) are 'global'—no information about maximally-localised basis.

Computational strategy

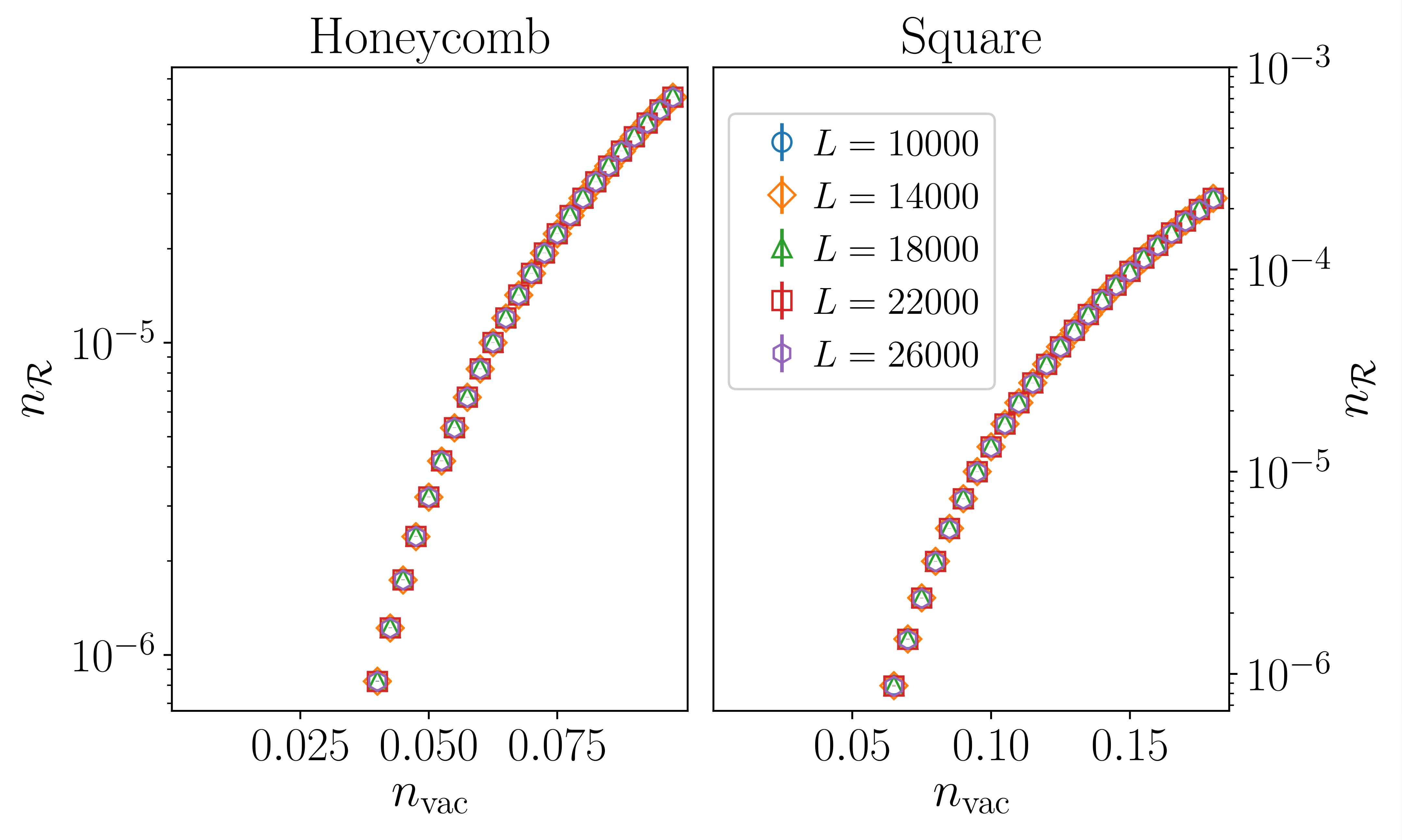
- Compensated disorder ($|A| = |B|$)
- Standard algorithms for finding any one maximum matching
- Alternating path tree from each monomer to obtain DM classification
- Burning algorithm to construct connected components

Basic picture in d=2

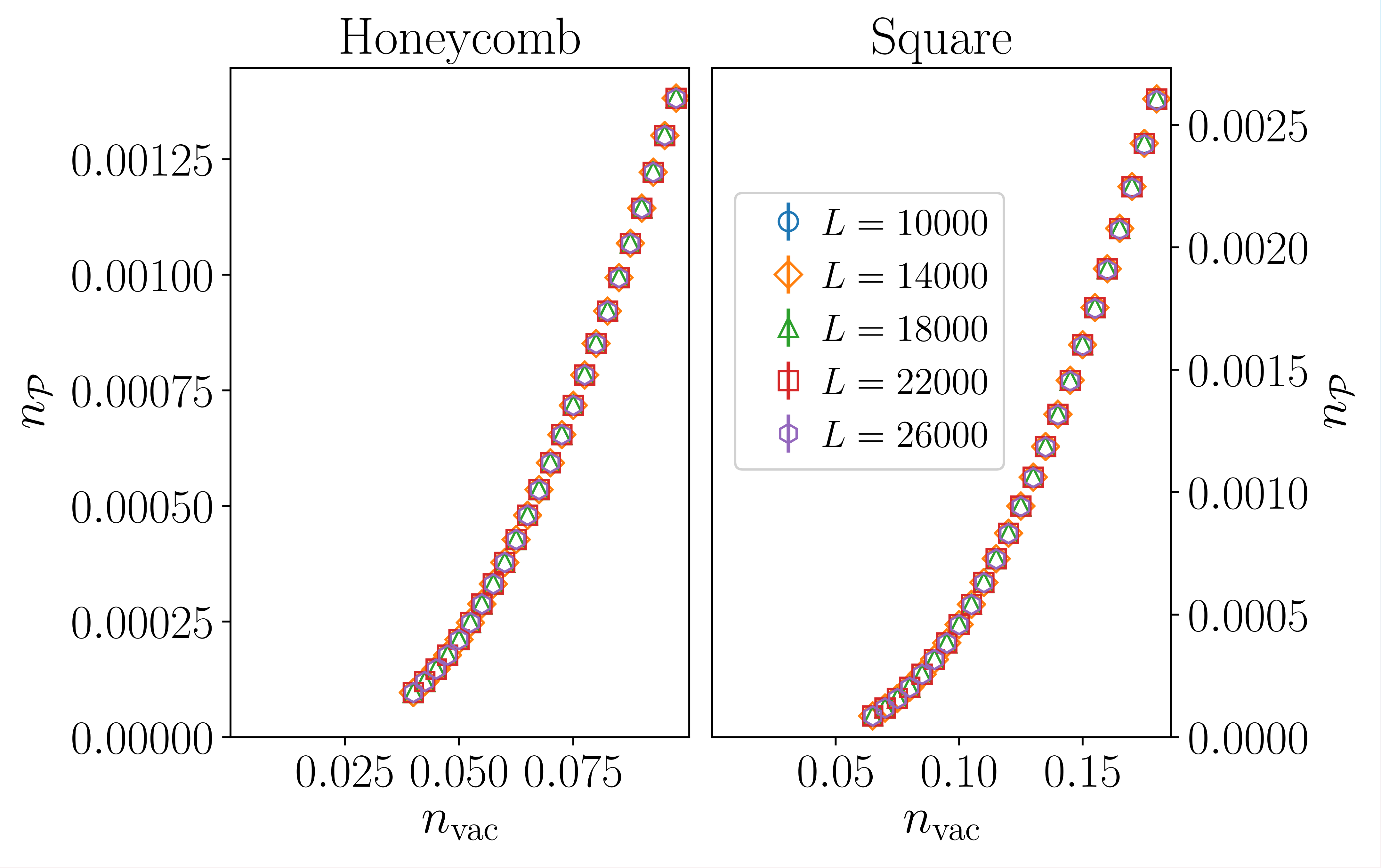


Typical R type regions are BIG at low dilution (of order 5%)

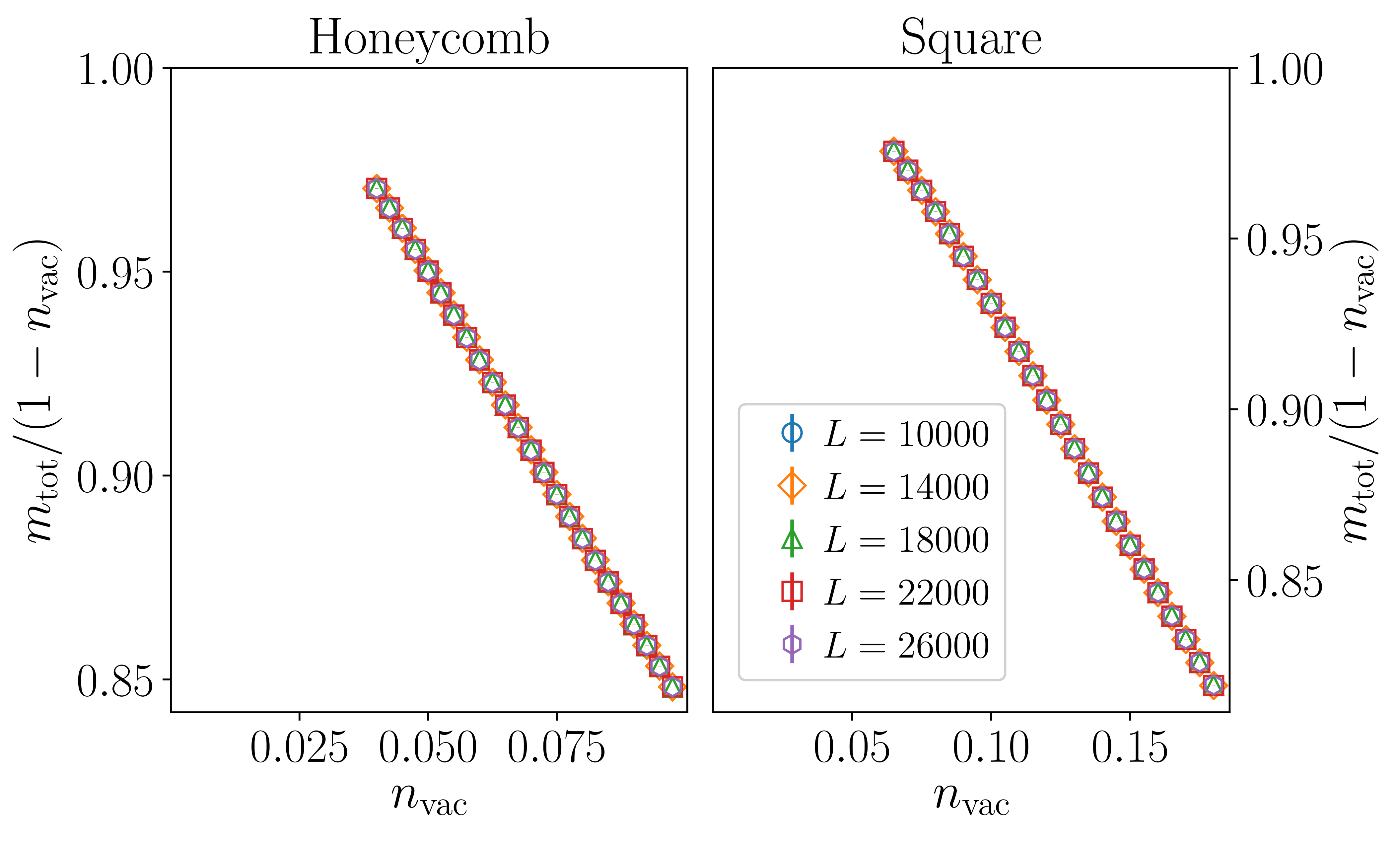
Thermodynamic densities of number of R-type regions



Thermodynamic densities of number of P-type regions

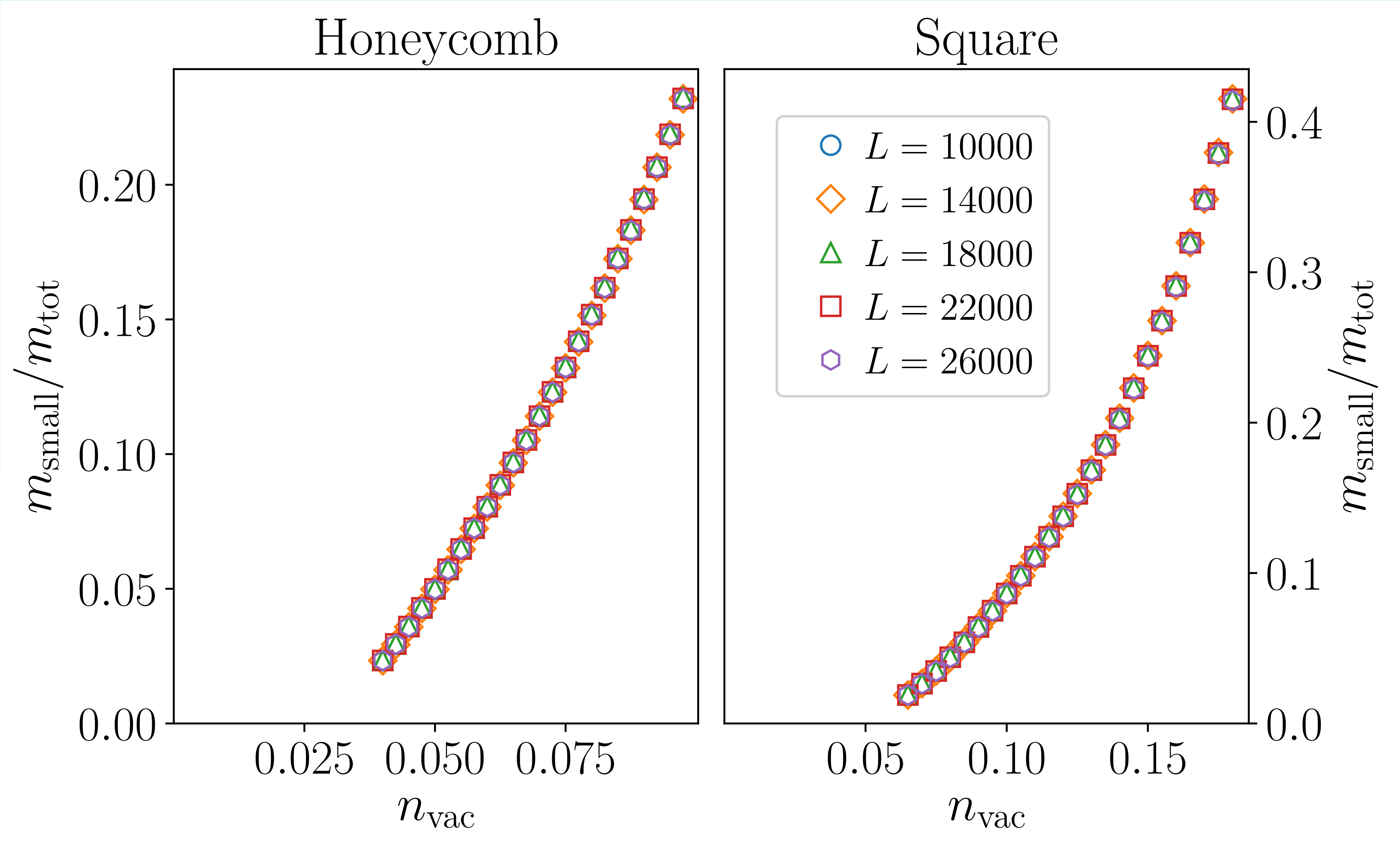


Thermodynamic densities of total mass in R-type regions



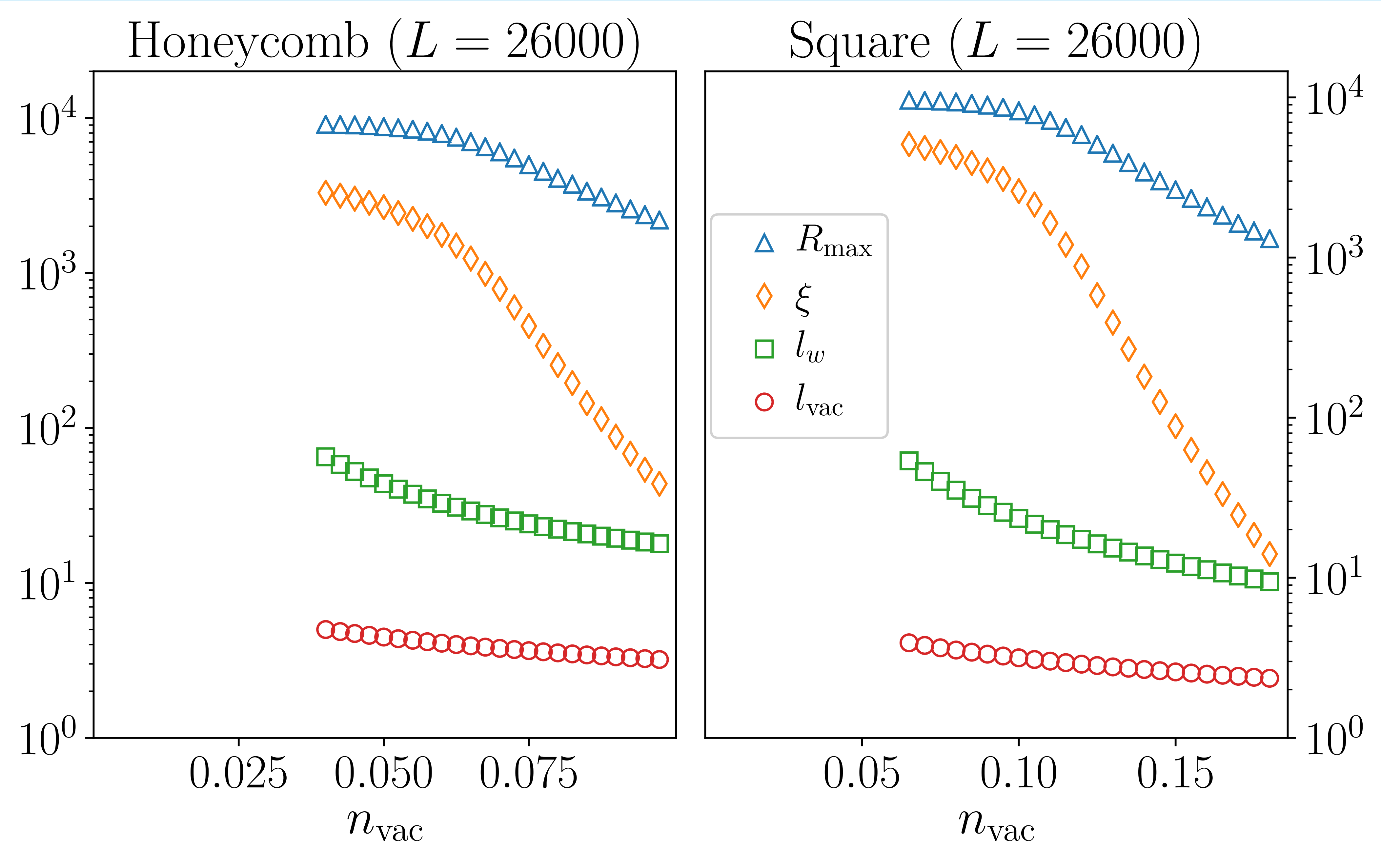
R-type regions take over the lattice in low-dilution limit!

Dominated by large-scale structures



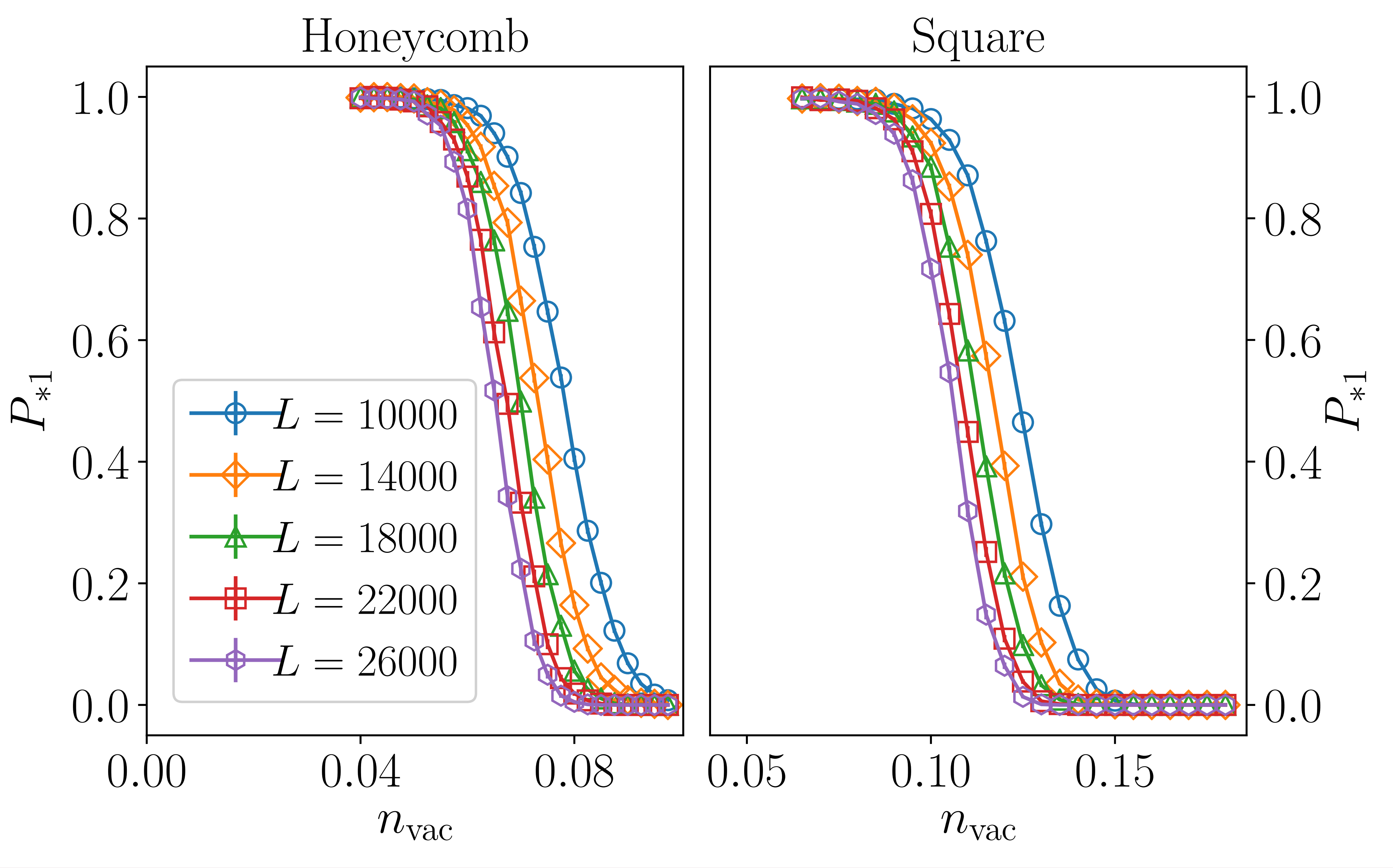
‘Small’ defined to have 10000 vacancies(!)

Length-scales

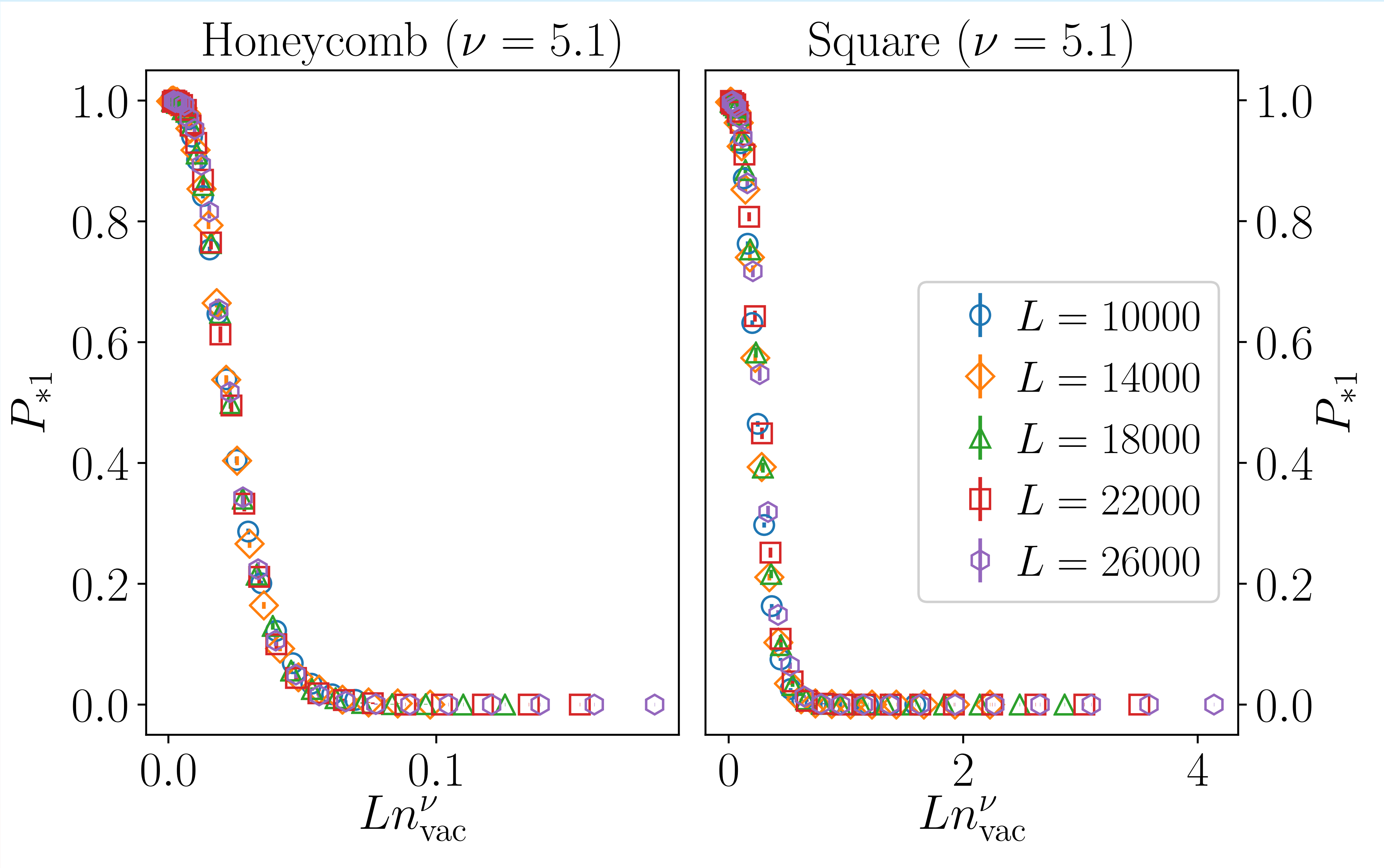


Size of largest R-type region is limited by system size at low dilution

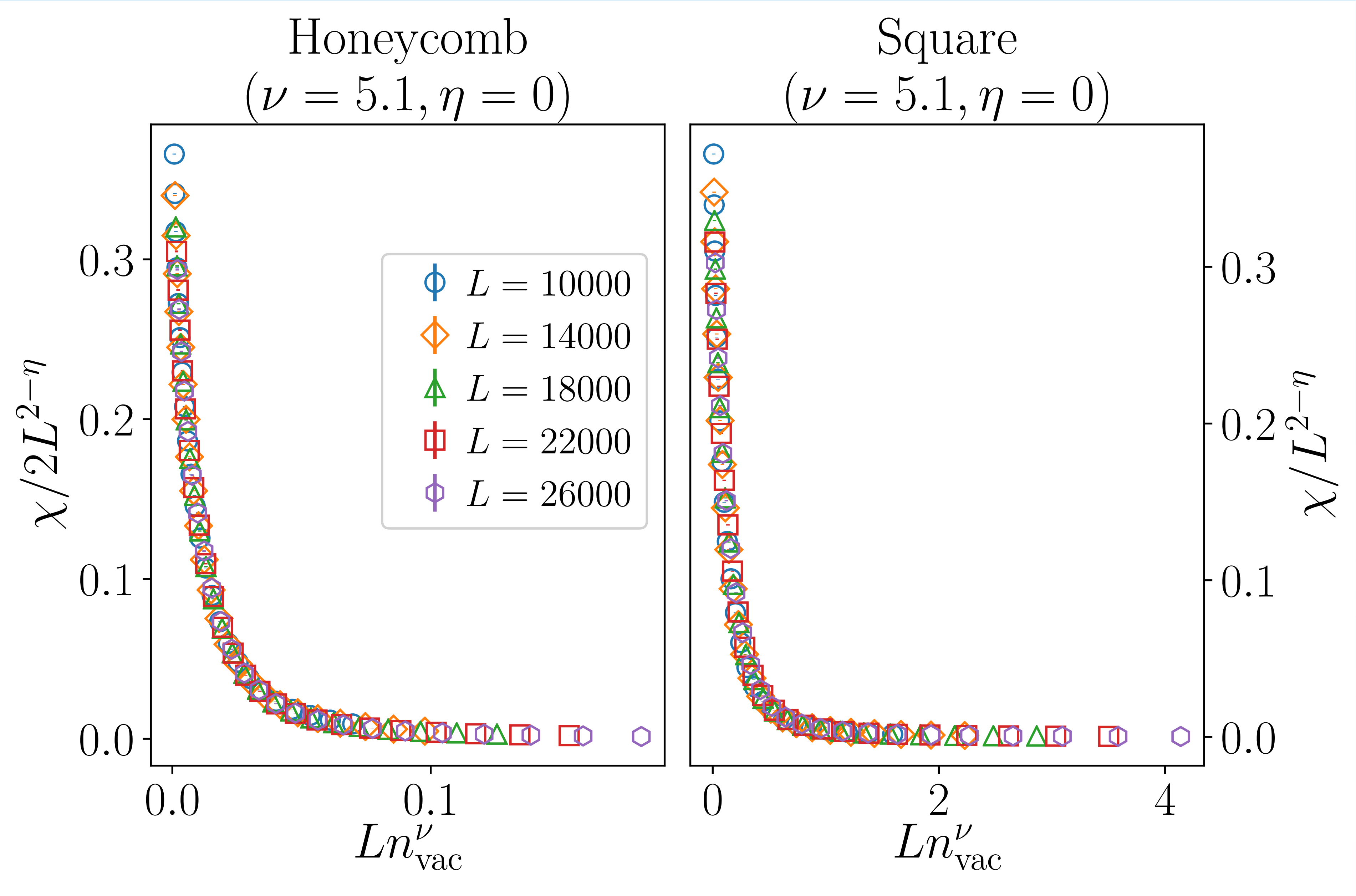
Incipient percolation: wrapping probabilities



Is there universal scaling?

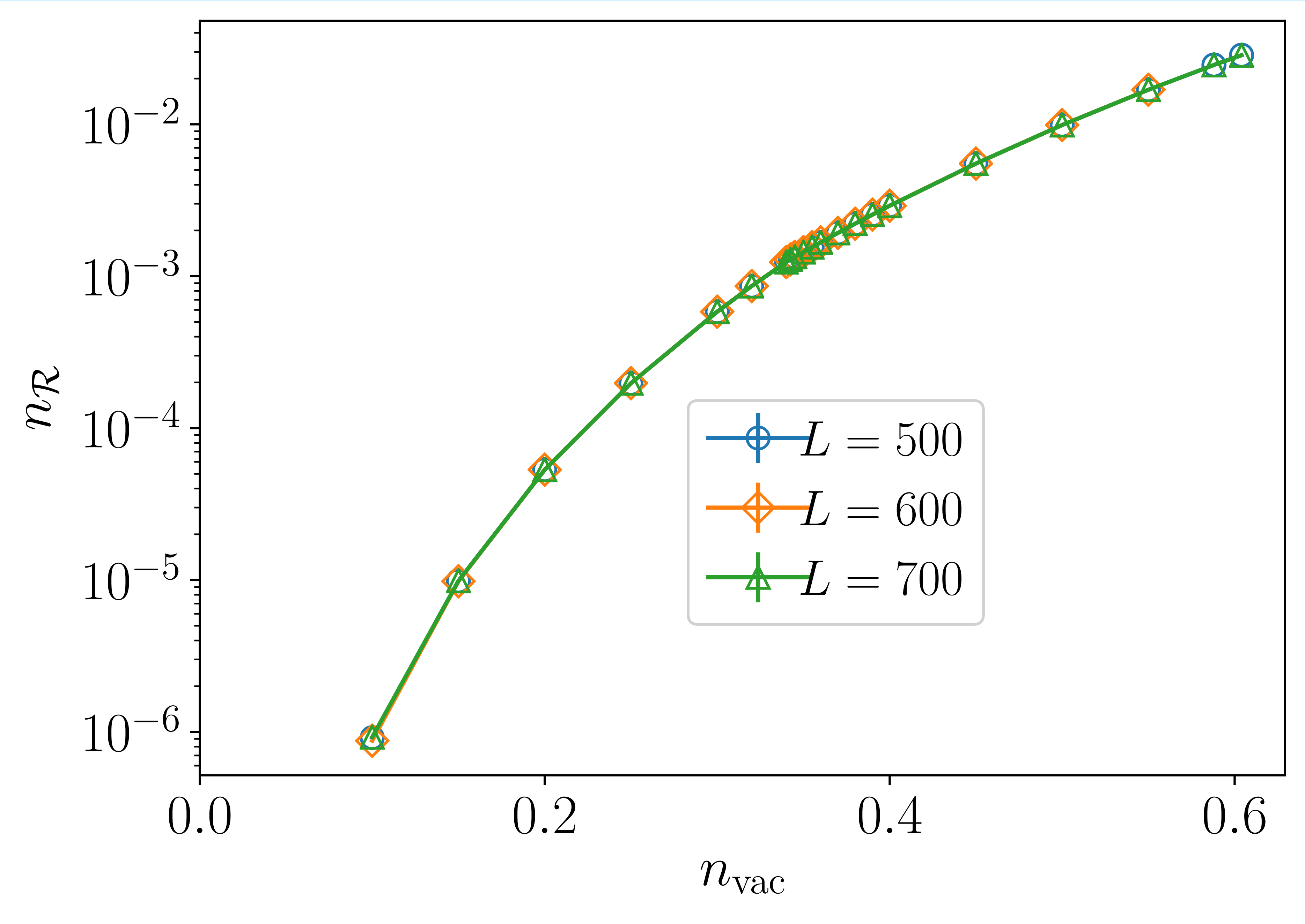


Anomalous exponent?

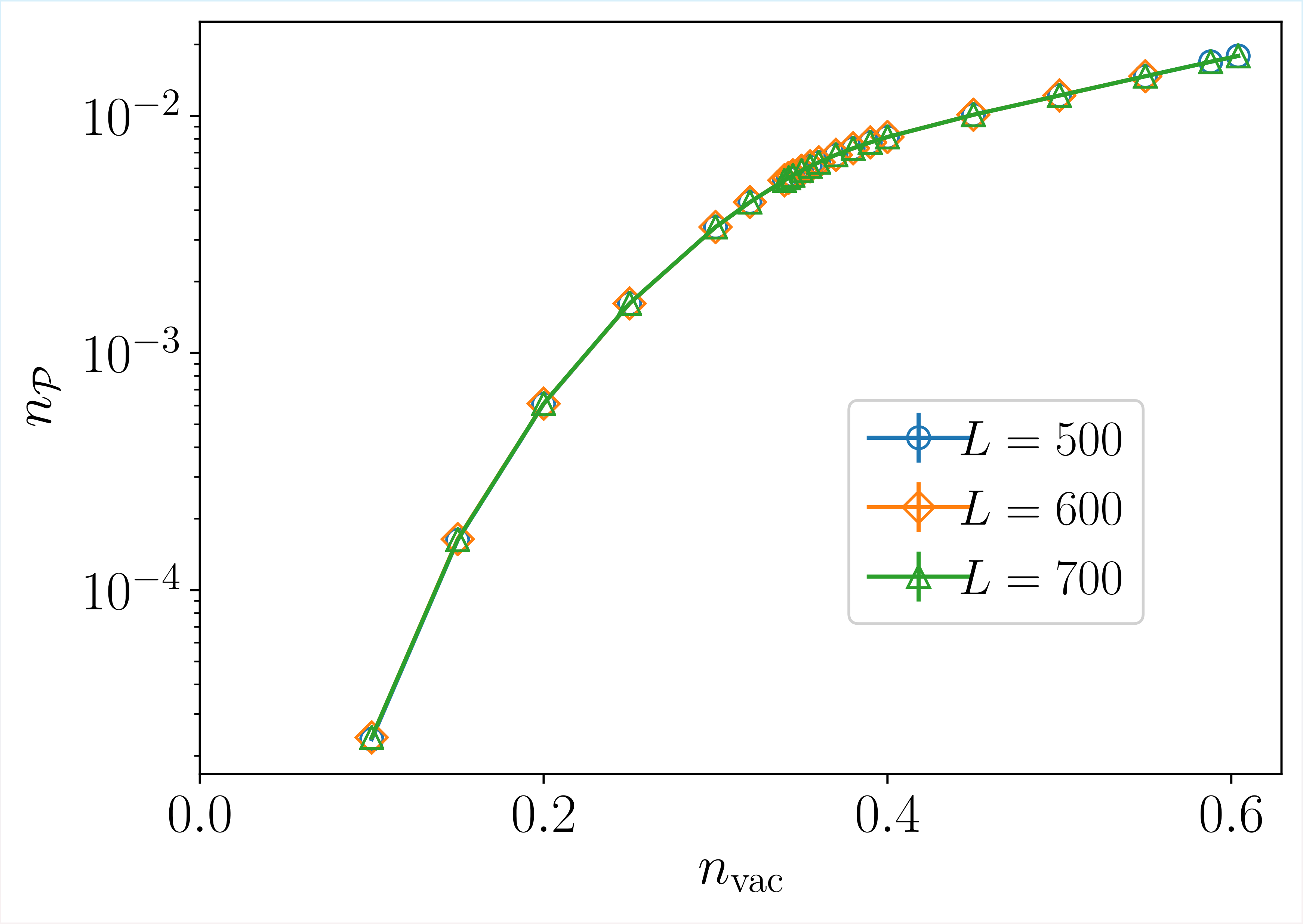


η very close to zero or zero…

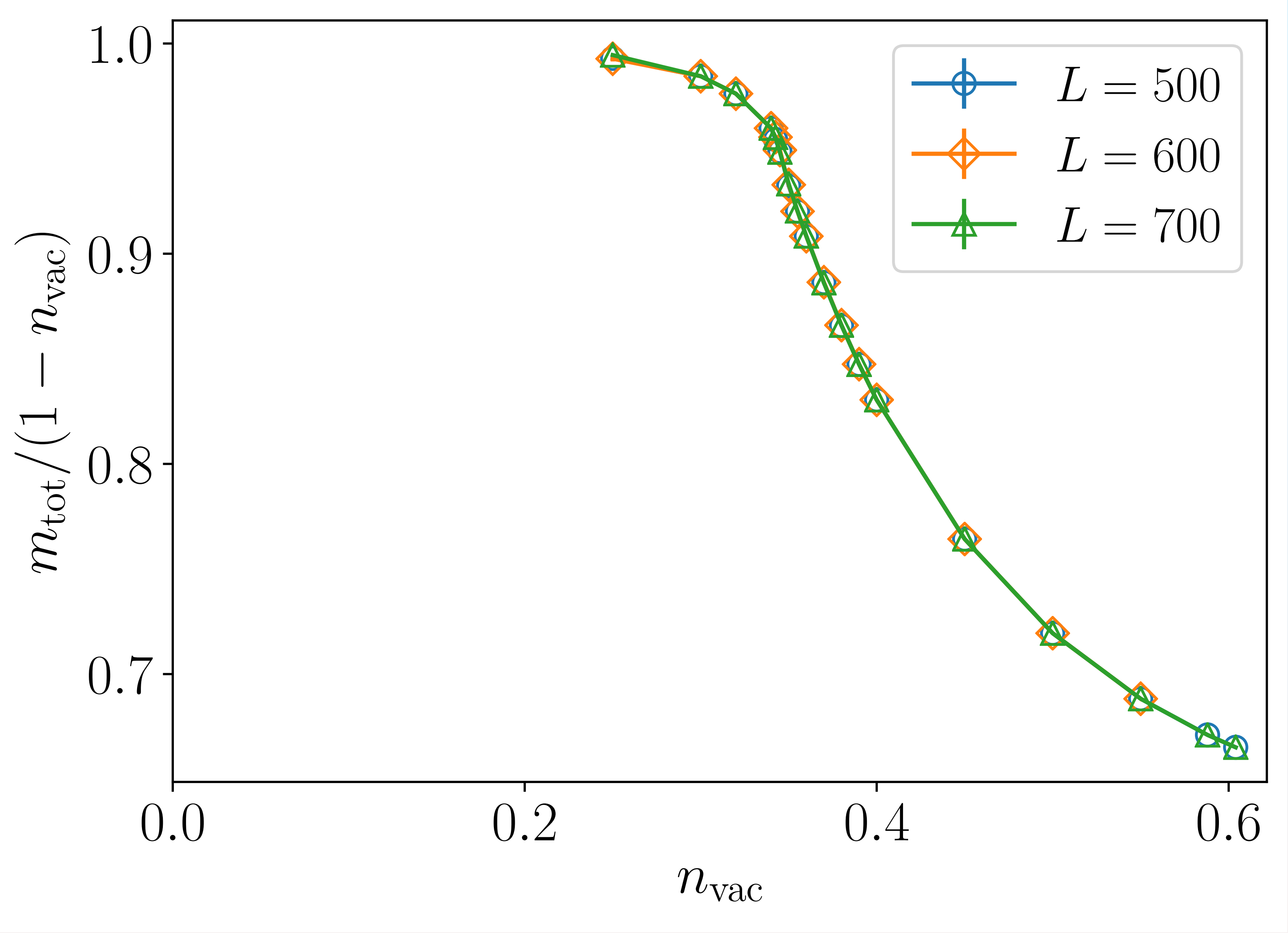
What about three dimensional cubic lattice?



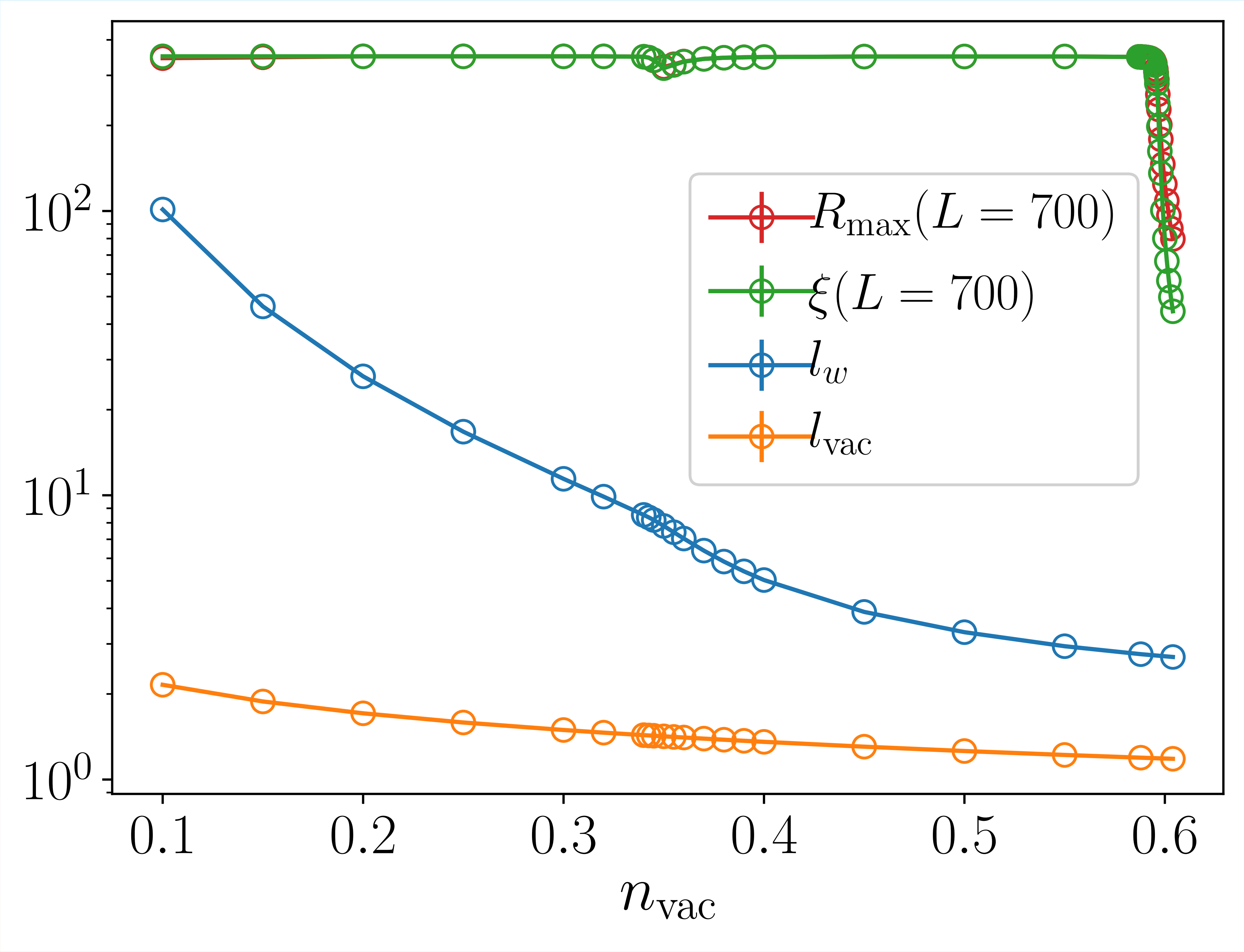
Very similar basic picture...



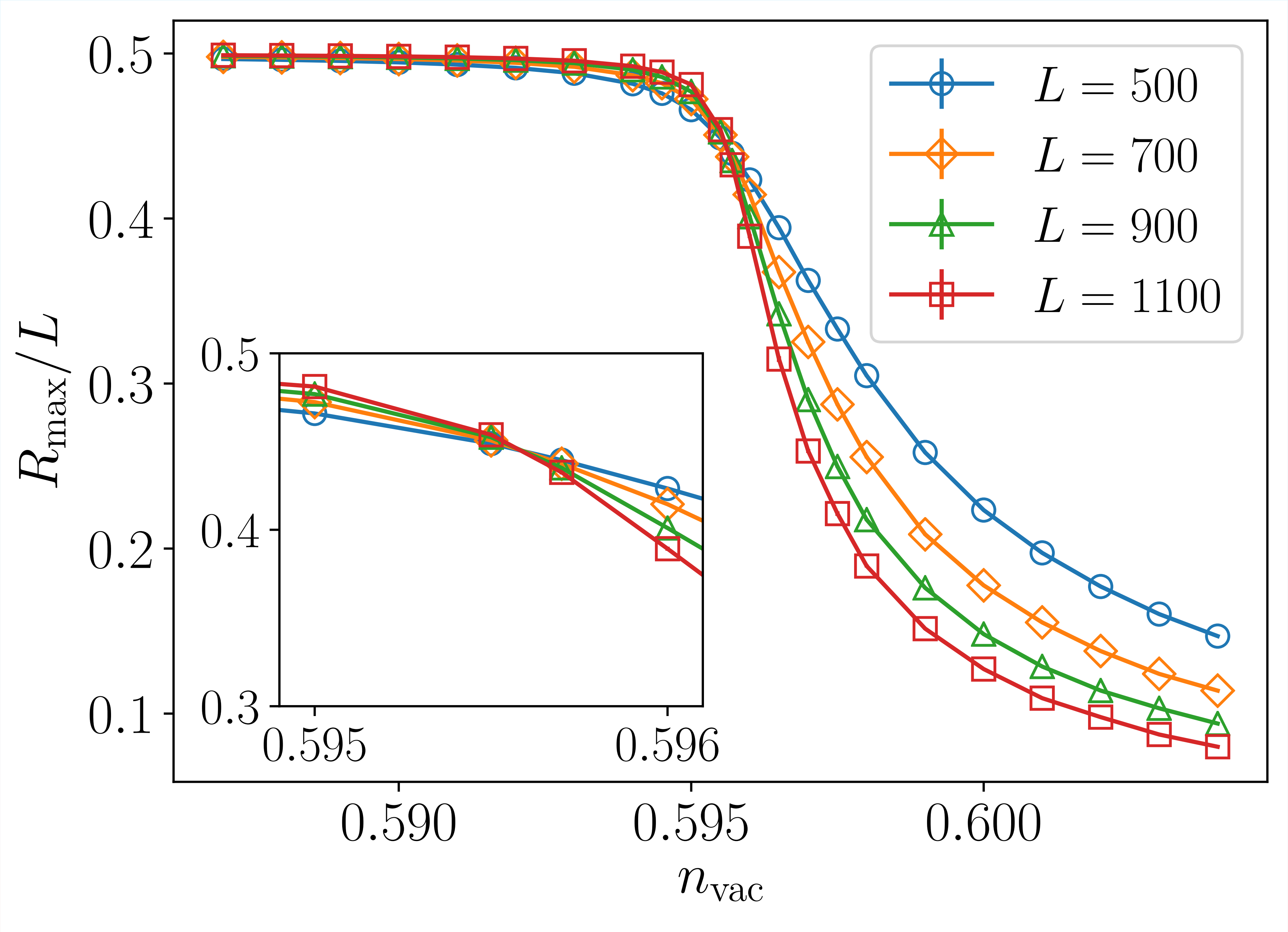
R-type regions invade lattice



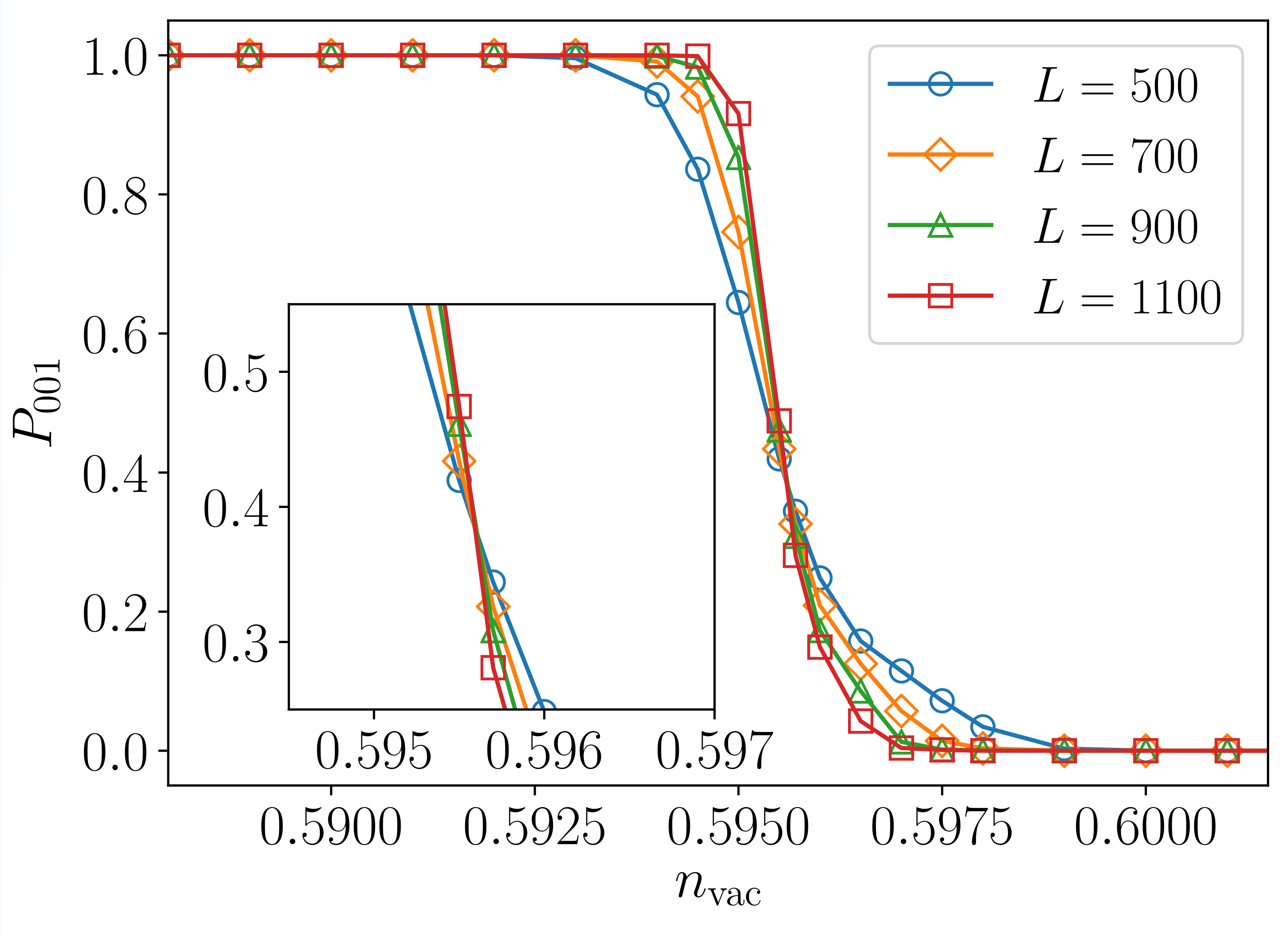
Dominated by large length scales



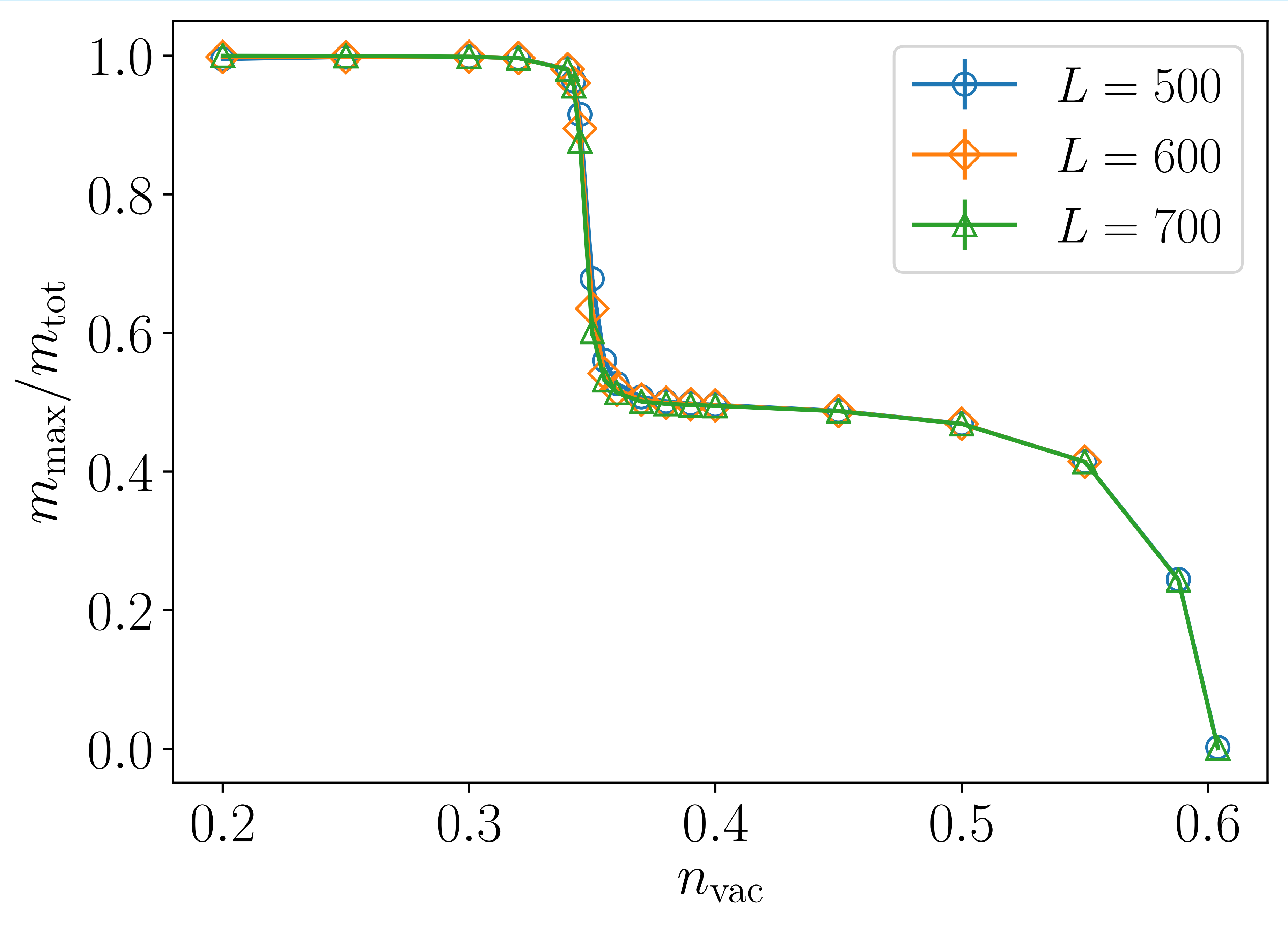
Percolation transition



Percolation transition



A second transition (!)



Spontaneous sublattice-symmetry breaking transition within percolated phase

Summary a la Wodehouse

- Patient perseverance produces percolative paradigms!

Consequences

- Infinitesimal dilution localises monomers of the maximally-packed dimer model in two dimensions
- Infinitesimal dilution causes sublattice symmetry-breaking in the monomer gas in three dimensions
- There is no quantum percolation transition in two dimensions (long story, starting ‘70s)
- Precise determination of quantum percolation threshold in three dimensions and evidence for second transition
- Majorana zero modes hosted by R-type regions with odd imbalance undergo a percolation transition

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