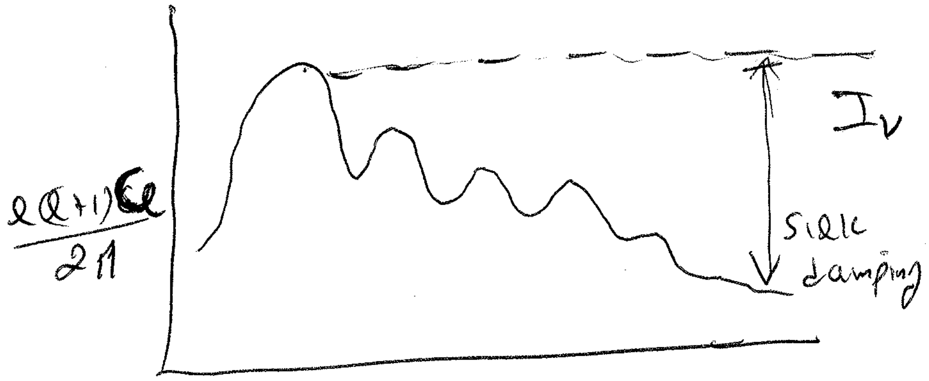
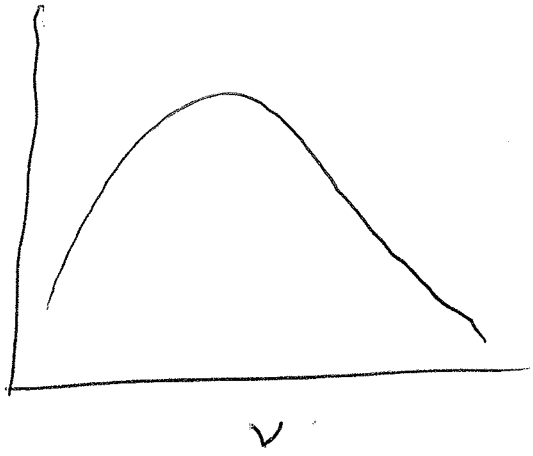
The shape of the CMB Spectrum

It is now often quoted in cosmology seminars, and rightly so, that we are now in the era of precision cosmology. This great sucess of cosmology has been driven in large part by the measurements of the anisotropies ~~de~~ in the cosmic microwave background (CMB) by COBE-DMR, WMAP, SPT, ACT and many other ground-based and balloon-borne experiments. In fact, the first direct evidence that Universe started in a hot big-bang was the discovery of CMB by Penzias and Wilson in 1965

COBE also had another instrument, FIRAS which measured the energy spectrum of CMB and found it to be a blackbody within the sensitivity and errors of the instrument. We will see that this measurement directly implies that the Universe must have been extremely hot and dense at some point in its evolution since these are the conditions necessary for creating a blackbody spectrum. The CMB spectrum, however, is much more than just a confirmation of the standard model of cosmology. The deviations from the perfect blackbody spectrum are possible; in fact inevitable. These tiny deviations are almost as rich in information about the standard cosmological parameters and physics beyond the standard model as the CMB anisotropy power spectrum, There is also a very important link between the two, ~~namely~~ known as Silk damping, which allows us to have a view of inflation spanning 17-eFolds.



CMB anisotropy power spectrum



CMB energy spectrum

$$C_\ell = \langle a_{\ell m} a^*_{\ell m} \rangle$$

$$a_{\ell m} = \int d\hat{n} \, \frac{\Delta T}{T}(\hat{n}) \, Y^*_{\ell m}(\hat{n})$$

$$\frac{\Delta T}{T} = \frac{T(\hat{n}) - \bar{T}}{\bar{T}}, \quad \bar{T} \text{ is average CMB temperature.}$$

$$I_\nu^{Planck} = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_BT} - 1}$$

ergs/s/cm²/steradian/Hz

Some definitions

1. dimensionless frequencies: $x = \frac{h\nu}{k_B T}$, h = Planck's constant

ν = frequency, T = temperature, k_B = Boltzmann's constant.

We will use x defined w.r.t to electron temperature (T_e) or radiation temperature (T_γ) and denote these by x_e and x_γ respectively.

2. Occupation number $n(x) = \frac{c^2}{2h\nu^3} I_\nu$

$$n^{\text{Planck}}_{(x)} = \frac{1}{e^x - 1}$$

$$n^{\text{Bose-Einstein}}_{(x)} = \frac{1}{e^{x+\mu} - 1}, \quad \mu = \frac{-\text{chemical potential}}{T}$$

We will follow the convention in cosmology and sometimes refer to the μ-parameter as just the chemical potential, keeping in mind the above definition.

Any general ~~chemical~~ spectrum can be written in the ~~form~~ Bose-Einstein form by defining a frequency dependent μ-parameter,

$$n(x) = \frac{1}{e^{x+\mu(x)} - 1}.$$

This is of course a Bose-Einstein spectrum only when $\mu(x)$ is independent of x.

3. Reference temperature T.

Once we introduce deviations from a blackbody spectrum, there is ambiguity in defining the original/reference blackbody w.r.t which we are calculating the distortions. Two natural definitions which are useful are using the total energy density or number density of photons and use these to define the reference temperature.

Total energy density of a blackbody $E_{PL} = a_R T^4$

$$a_R = \frac{8\pi^5 k_B^4}{15 c^3 h^3}$$

Total number density $= N_{PL} = b_R T^3$

$$b_R = \frac{16\pi k_B^3 \zeta(3)}{c^3 h^3}, \quad \zeta(3) \text{ is Riemann zeta function.}$$

Thus for an arbitrary spectrum $n(\nu)$ we can define

$$T_E = \left(\frac{E}{a_R}\right)^{1/4}, \quad E = \int \frac{8\pi h \nu^3}{c^3} n(\nu)\, d\nu$$

$$T_N = \left(\frac{N}{b_R}\right)^{1/3}, \quad N = \int \frac{8\pi \nu^2}{c^3} n(\nu)\, d\nu$$

For a blackbody, of course, $T_E = T_N$

Calculations involving bispectrum from second order Boltzmann equation have used T_E as definition of temperature.

For Y-distortion or Sunyaev-Zeldovich effect T_N is used. T_N is the natural definition and is equal to original blackbody spectrum when Compton-scattering in clusters just adds energy to the CMB without changing the photon number.

Aside: Using T_N instead of T_E in the second-order Boltzmann equation separates the 2nd order equation into two decoupled equations: one for pure blackbody part and the other for pure Y-distortion. Both these equations have no frequency dependence.

[Chluba, Khatri and Sunyaev 2012]

The frequency dependence factors out from both equations, just like the first order Boltzmann equation! We can therefore solve the full 2^{nd} order Boltzmann equation keeping the spectral dependence. ~~The tempera~~ ~~Solving for temperature~~

We will use T_N as definition for the reference blackbody and drop the subscript N.

With this definition, the spectral distortion,

$$\Delta n(x) = n(x) - n_{pl}(x), \quad x \equiv \frac{h\nu}{kT_N} \equiv \frac{h\nu}{kT}$$

has the property

$$\int dx\, x^2 \Delta n(x) = 0.$$

4. Y-parameters

Sunyaev-Zeldovich effect

$$y = \int dt \frac{k_B \underbrace{\sigma_T}_{\text{Thomson cross-section}} \underbrace{n_e}_{\text{electron number density}}}{m_e c} (\underbrace{T_e}_{\text{electron temperature}} - \underbrace{T_r}_{\text{radiation temperature}})$$

Recoil

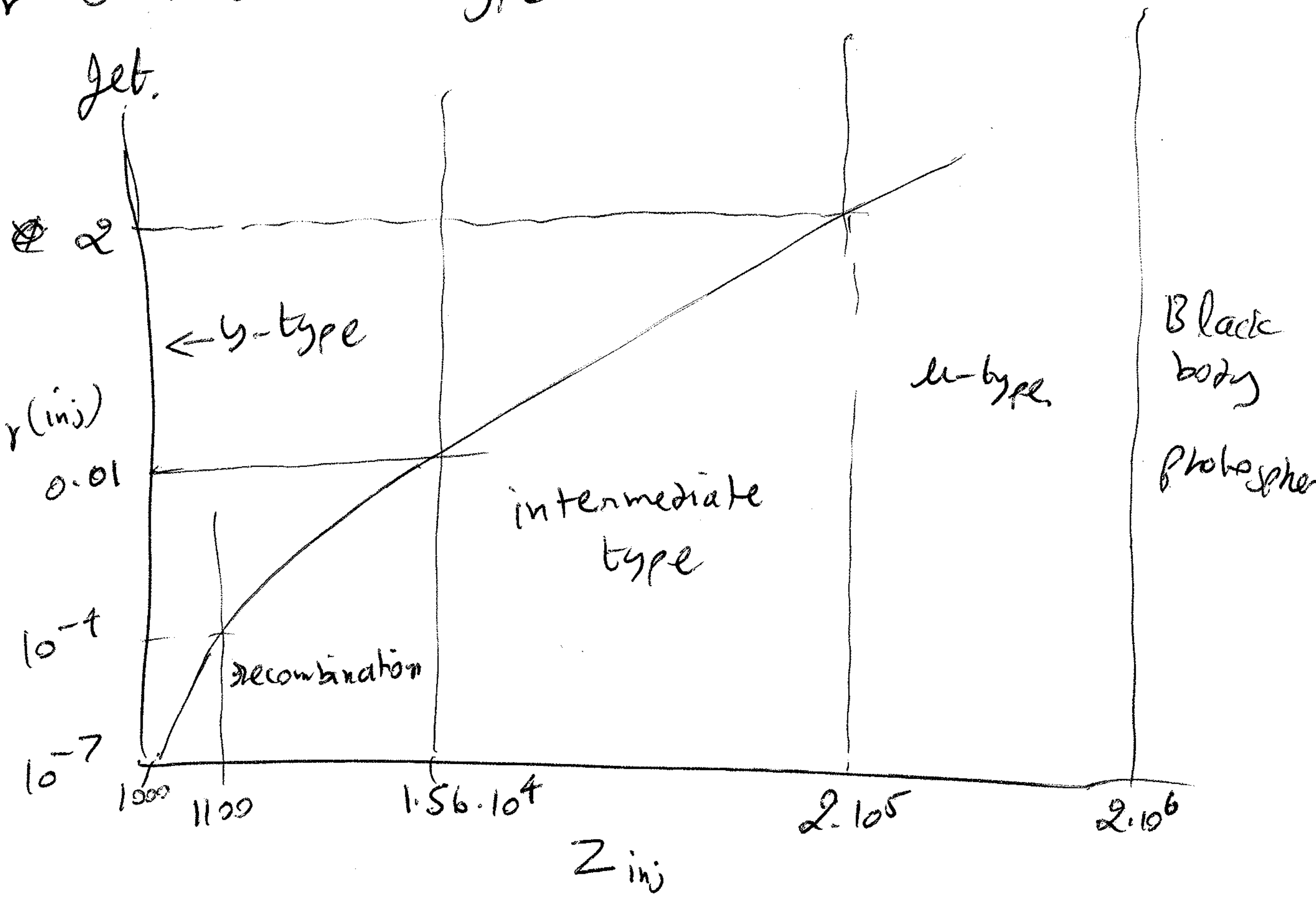
$$y_\gamma = \int dt \frac{k_B \sigma_T n_e}{m_e c} T_r$$

Doppler effect $\quad y_e = \int dt \frac{k_B \sigma_T n_e}{m_e c} T_e$

In early universe, $z \gtrsim 200 \quad y_r \approx y_e$

It is convenient to use y_r as the time coordinate, instead of redshift z or time t.

$y_r^{(\text{or } t_e)}$ decides the type of distortion we will get.

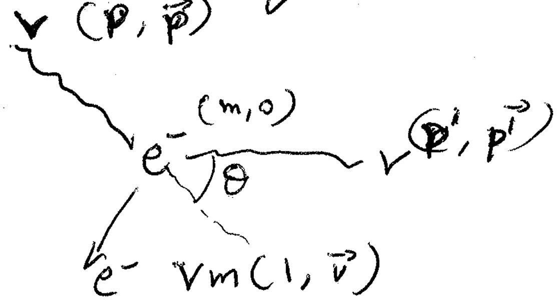


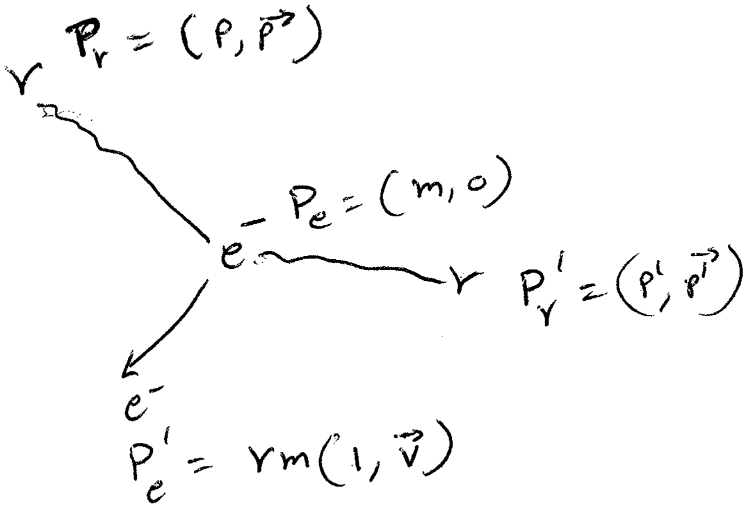
$$y_r(\text{inj}) = \int_{z_{\text{inj}}}^{0} dt\, \frac{k_B \sigma_T n_e}{m_e c} T_r \,, \quad T_r = 2.725(1+z)$$

y-type, μ-type and intermediate-type distortions are all solutions of ~~Kopa~~ Kompaneets equation.

Scattering Processes

1. Compton scattering – Kompaneets (1956).





$$P_\gamma + P_e = P_\gamma' + P_e' \qquad \text{Capital } P_{\gamma,e} \text{ four vectors.}$$

Usual trick – using (see Peskin and Schroeder)

$$P_e^2 = P_e'^2 = -m^2$$
$$P_\gamma^2 = P_\gamma'^2 = 0$$

$$P_e'^2 = -m^2 = (P_\gamma - P_\gamma' + P_e)^2$$
$$= -m^2 + 2 P_e \cdot (P_\gamma - P_\gamma') - 2 P_\gamma \cdot P_\gamma'$$
$$= -m^2 - 2m(p - p') + 2(pp' - \vec{p} \cdot \vec{p}')$$

$$\frac{1}{p'} - \frac{1}{p} = \frac{1}{m}(1 - \cos\theta)$$

$$p' = \frac{p}{1 + \frac{p}{m}(1 - \cos\theta)} \overset{p/m \ll 1}{\simeq} p\left(1 - \frac{p}{m}(1 - \cos\theta)\right)$$

$$\frac{\Delta p}{p} = -\frac{p}{m}(1 - \cos\theta) \equiv -\frac{h\nu}{mc^2}(1 - \cos\theta)$$

= energy transfer to electron in recoil!

~~For a thermal distribution of electrons there will be Doppler effect of order $\left(\frac{v}{c}\right) \sim \left(\frac{T}{m}\right)$~~

For a thermal distribution of electrons (or electrons in motion in general) there will be Doppler shift in the frequency of photons. At first order, the Doppler shift ~~due to motion of ele~~ for an isotropic distribution of electrons will average to zero (over all angles). ~~The~~ A second order there is non zero average transfer of energy from electrons to photons (opposite of recoil effect in electron rest frame)

$$\left(\frac{\Delta E}{E}\right)_{average} \sim \left\langle \frac{v^2}{c^2} \right\rangle \sim \frac{T_e}{m_e}$$

Thus in the early universe we are looking at energy exchange between electrons and photons of order $\frac{P}{m}$, $\frac{T_e}{m} \sim \frac{v^2}{c^2}$ These are thus the 'small parameters' in which we should expand the Boltzmann equation.

$$A \; z \sim 2 \times 10^6 , \; \frac{T_e}{m_e} \simeq 10^{-3} .$$

We will be interested in redshifts

$z <$ few $\times 10^6$. Thus T/m, P/m, $\frac{v^2}{c^2}$ can be used as parameters in which to expand the ~~Compton~~ Boltzmann's equation. The result is Kompaneets equation.

Thus, ~~in~~ the early universe we are looking at energy exchange between electrons and photons of order ~~[illegible]~~ $\frac{h\nu}{mc^2} \sim \frac{k_B T}{mc^2}$

These are the 'small parameters' in which we should expand the Boltzmann equation.

At $z \sim 2\times10^6$, $\frac{T}{m} \simeq 10^{-3}$

[Note: At first order red-shifts and blueshifts in Doppler effect cancel, $\langle \frac{v}{c} \rangle = 0$. Thus the lowest non-zero Doppler effect is at second order in v/c].

~~Thus~~ We will be interested in redshifts ~~z < few~~ $z <$ few times 10^6. Thus $T/m \simeq \frac{p}{m}$ can be used as order parameter to expand the Boltzmann equation. This gives us the Kompaneet's equation. (Kompaneets 1956).

KOMPANEETS Equation

We will follow the method of Dodelson and Bernstein 1990. A different derivation can be found in Rybicki and Lightman's book. Higher order terms are derived ~~in~~ by ~~several authors~~ Hu, Scott and Silk 1994 and Chluba, Khatri and Sunyaev 2012 in context of cosmological perturbation theory.

Kompaneets Equation

We will only give a sketch of ~~the~~ the derivation omitting most of the tedious but straightforward algebra. We will follow the method of Dodelson and Bernstein 1990 which is also used in derivation of ~~Second or~~ Boltzmann equation at second order in perturbation theory, for example Bartolo, Matarrese and Riotto 2007 (arxiv: astro-ph/0703496

Higher order ~~corr~~ terms which mix thermal $\left(\frac{T_e}{m_e}, \frac{p}{m_e}\right)$ and cosmological perturbation orders (i.e. perturbation in curvature ζ) are derived in Hu, Scott and Silk 1994 and Chluba, Khatri and Sunyaev 2012.

The Boltzmann equation for photons is given by

$$\frac{Dn(\vec{x},\vec{p},t)}{Dt} = C(\vec{x},\vec{p},t)$$

$$= \frac{1}{2p}\int\frac{d^3q}{(2\pi)^3 2E_e(q)}\int\frac{d^3q'}{(2\pi)^3 2E_e(q')}\int\frac{d^3p'}{(2\pi)^3 2E(p')}|M|^2$$

$$(2\pi)^4\,\delta^3(\vec{p}+\vec{q}-\vec{p}'-\vec{q}')\,\delta\left[p-p'+E_e(q)-E_e(q')\right]$$

$$\Big[g(\vec{q}',\vec{x})\,f(\vec{x},\vec{p}')\,(1+f(\vec{x},\vec{p})) -$$

$$g(\vec{x},\vec{q})\,f(\vec{x},\vec{p})\,(1+f(\vec{x},\vec{p}'))\Big].$$

A nice discussion of how various terms arise can be found in Dodelson's Modern Cosmology.

$g(\vec{v})$ is electron distribution function and is given by Maxwellian distribution.

$f(\vec{x},\vec{p})$ is photon distribution which is in general a function of direction and position, for example when doing cosmological perturbation theory. We will assume it to be isotropic and independent of position since we will only deal with spectral distortions of average monopole of CMB. Therefore, $f(\vec{x},\vec{p}) = n(p)$

Also in the absence of metric perturbations we can replace total derivative with partial derivative. $\frac{D}{Dt} \to \frac{\partial}{\partial t}$.

After doing one of the integrals using 3-d Dirac delta function, we get

$$C(\vec{x},\vec{p},t) = \frac{1}{2p}\int \frac{d^3q}{(2\pi)^3 2E_e(q)} \int \frac{d^3p'}{(2\pi)^3 2E(p')} |M^2|$$

$$(2\pi)^4\, \delta\left(E_e(\vec{q}) - E_e(\vec{q}+\vec{p}-\vec{p}\,') + p - p'\right)$$

$$\left[g(\vec{p}+\vec{q}-\vec{p}\,')\, f(\vec{p}\,')\,(1+f(\vec{p})) - g(\vec{q})\, f(\vec{p})\,(1+f(\vec{p}\,')) \right]$$

Electron energy is given by $E_e(q) = (q^2 + m_e^2)^{1/2}$

$$\simeq m\left(1 + \frac{q^2}{2m_e^2}\right)$$

The energy transfer is therefore,

$$E_e(\vec{q}) - E(\vec{q} + \vec{p} - \vec{p}')$$

$$\simeq -\frac{(\vec{p} - \vec{p}')\cdot\vec{q}}{m_e} - \frac{(\vec{p} - \vec{p}')^2}{2m_e} + \text{h.o.}$$

The Dirac delta function can be expanded around $p - p'$.

$$\delta(p - p' + E_e(q) - E_e(q')) = \delta(p - p') + \left[\frac{(\vec{p} - \vec{p}')\cdot\vec{q}}{m} + \frac{(\vec{p} - \vec{p}')^2}{2m}\right]$$

$$\frac{\partial}{\partial p'}\delta(p - p') + \frac{1}{2}\left(\frac{(\vec{p} - \vec{p}')\cdot\vec{q}}{m_e}\right)^2 \frac{\partial^2\delta(p - p')}{\partial p'^2}$$

The electron distribution is Maxwellian,

$$f(\vec{q}) = n_e\left[\frac{(2\pi)}{(m_e T_e)}\right]^{3/2} e^{-\frac{(\vec{q} - m_e\vec{v})^2}{2m_e T_e}}, \quad \vec{v}$$ is peculiar velocity.

denoting $\int \frac{d^3 q}{(2\pi)^3}$ by $\langle\ \rangle$, we have

$$\langle f \rangle = n_e, \quad \langle q_i f \rangle = m_e v_i n_e$$

$$\langle q_i q_j f \rangle = m_e^2 v_i v_j n_e + m_e T_e \delta_{ij} n_e$$

n_e is the restframe electron number ~~nes~~ density.

$$g(\vec{q}+\vec{p}-\vec{p}\,') = g(\vec{q})\Big[1 - \frac{(\vec{p}-\vec{p}\,')\cdot(\vec{q}-m\vec{v})}{m_e T_e}$$

$$- \frac{(\vec{p}-\vec{p}\,')^2}{2 m_e T_e} + \frac{1}{2}\left(\frac{(\vec{p}-\vec{p}\,')(\vec{q}-m\vec{v})}{m_e T_e}\right)^2 + h.o.\Big]$$

$$|M|^2 = 2(4\pi)^2\alpha^2\left[\frac{p'.q}{p.q} + \frac{p.q}{p'.q} + 2m^2\left(\frac{1}{p.q} - \frac{1}{p'.q}\right) + m^4\left(\frac{1}{p.q} - \frac{1}{p'.q}\right)^2\right]$$

(p, q four vectors)

$$= 2\cdot(4\pi)^2\alpha^2\left[1+\cos^2\theta - 2\cos\theta(1-\cos\theta)\left[\frac{\vec{q}\cdot\hat{p}}{m_e} + \frac{q\cdot\vec{p}\,'}{m_e}\right] + h.o\right]$$

Put everything in the collision term, ~~The~~ and do all the integrals!

photon occupation number can also be decomposed into different perturbation orders,

$$f = f^{(0)} + f^{(1)} + f^{(2)} \cdots$$

Gathering together ~~different~~ terms of order $f^{(1)}$ and v (peculiar velocity) (after integrating over d^3q and d^3p')

we get the first order Boltzmann equation. (which describes only scattering but no energy exchange)

~~&~~ Terms of order $f^{(0)}\, p/m$ and $f^{(0)}\,\frac{T_e}{m}$ give the Kompaneets equation which describes the lowest order energy exchange between photons and electrons.

It is convenient to write Kompaneets equation in terms of dimensionless variables, using y_r as time variable.

$$\frac{\partial n}{\partial y_r} = \frac{1}{x^2}\frac{\partial}{\partial x} x^4 \left(n + n^2 + \frac{T_e}{T}\frac{\partial n}{\partial x}\right) \quad (1)$$

Here T is the reference temperature w.r.t which x is defined, $x = \frac{h\nu}{KT}$ and has in general no relation to ~~relat~~ blackbody temperature. It is just a convenient dimensionless variable. If we choose $T = T_0(1+z) = 2.725K(1+z)$ the x becomes independent of expansion of the universe. Eq. (1) is therefore valid in static medium as well as expanding medium.

Solutions: It is easy to verify that the equilibrium solution of Eq (1) is a Bose-Einstein spectrum, with $T = T_e$.

$$n^{eq}(x) = \frac{1}{e^{x+\mu} - 1}, \quad T_e = T.$$ This makes the R.H.S vanish. This solution will be obtained if $y_r \gg 1$.

This is also called μ-distortion.

For $\mu \ll 1$, we can expand $n^{eq}(x)$ in μ,

$$n_1^{eq}(x) = \frac{1}{e^{x+\mu}-1} \simeq \frac{1}{e^x-1} - \frac{\mu e^x}{(e^x-1)^2}$$

The number density in this spectrum is

$$N = \frac{b_R T^3}{I_2} \int dx\, x^2\, n^{eq}(x)$$

$$= b_R T^3 \left(1 - \frac{\mu \pi^2}{3 I_2}\right) \quad \text{—(2)}$$

$$I_2 = \int x^2 n_{pl}(x) dx = 2\zeta(3)\,, \quad n_{pl} = \frac{1}{e^x-1}.$$

But we want to define the reference temperature T_{ref} as the temperature of blackbody with same number density of photons,

$$N = b_R T_{Ref}^3 \quad \text{—(3)}$$

Equating (2) and (3) and expanding in $\frac{T-T_{ref}}{T_{ref}}$

gives, $\frac{T-T_{ref}}{T_{ref}} = 0.456\,\mu$.

Written in terms of T_{ref}, $x_{ref} = \frac{h\nu}{k T_{ref}}$,

$$n^{eq}(x_{ref}) = \frac{1}{e^{x_{ref}}-1} + \underbrace{\frac{\mu e^{x_{ref}}}{(e^{x_{ref}}-1)^2}\left(\frac{x_{ref}}{2.19} - 1\right)}_{\mu\text{-distortion}}.$$

We will drop 'ref' subscript from now on, with the above definition of reference temperature understood.

Solution of Kompaneets equation in the minimal comptonization limit – Sunyaev-Zeldovich effect

~~In~~ Let us solve the Kompaneets equation with $T_e \neq T$, with initial spectrum $n_{pl}(x)$ a blackbody spectrum at temperature T.

We can think of solving the Kompaneets equation by taking infinitesimal steps, ~~as~~ $\delta y_{re} \ll 1$

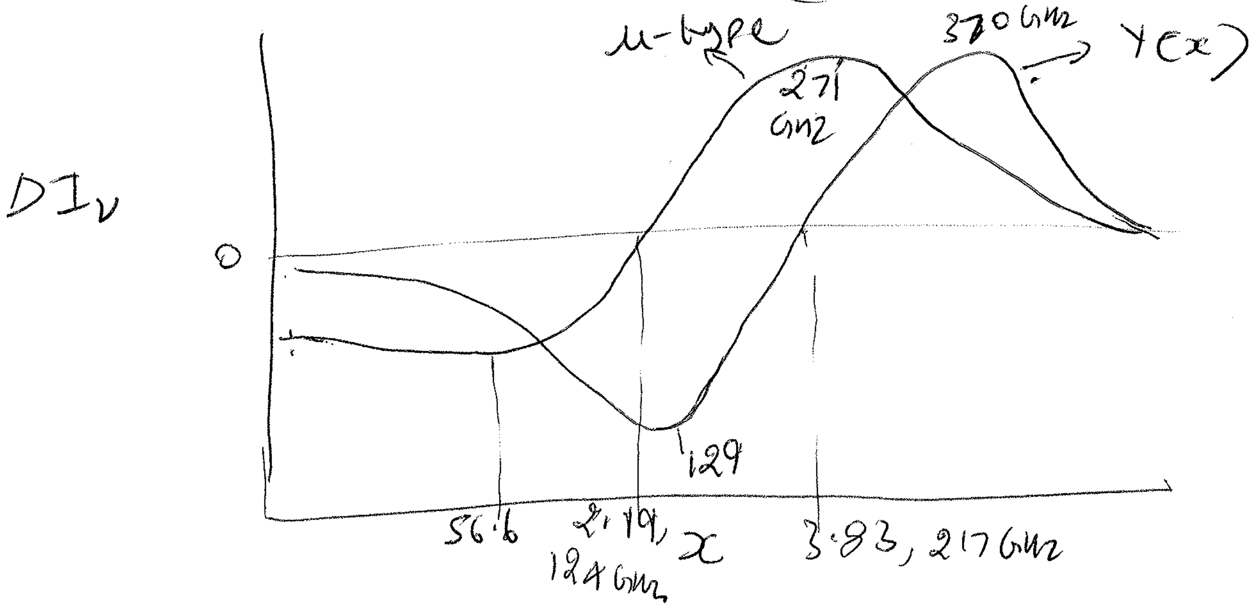
The first step, starting at $y_r = 0$ gives us the y-distortion, by substituting $n_{pl}(x)$ in the R.H.S.

$$\frac{\partial n}{\partial y_r} \simeq \frac{1}{x^2}\frac{\partial}{\partial x} x^4 \left(n_{pl} + n_{pl}^2 + \frac{T_e}{T}\frac{\partial n_{pl}}{\partial x}\right)$$

$$\simeq \left(\frac{T_e}{T} - 1\right)\frac{1}{x^2}\frac{\partial}{\partial x} x^4 \frac{\partial n_{pl}}{\partial x}$$

$$n(x) = \left[\int_0^{\delta y_r} \left(\frac{T_e}{T} - 1\right) dy_r\right] Y(x)$$

$$Y(x) = \frac{1}{x^2}\frac{\partial}{\partial x} x^4 \frac{\partial n_{pl}}{\partial x} = \frac{x e^x}{(e^x - 1)^2}\left[x\left(\frac{e^x + 1}{e^x - 1}\right) - 4\right]$$



Electron temperature

We can also calculate the ~~temp~~ equilibrium temperature that the electrons will have in a radiation field with arbitrary spectrum $n(x)$. In cosmological scenarios, since there are 10^9 photons for every electron, electrons reach equilibrium much faster.

(Zeldovich and Levich 1970, Levich and Sunyaev 1971)

Energy in radiation is proportional to $\int x^3 n(x)\, dx$.

Equilibrium is reached when there is no energy exchange. Multiplying Kompaneets equation with x^3 and integrating over all frequencies, and setting $\int \frac{\partial x^3 n(x)}{\partial y_r} = 0$

we get, after integrating by parts and setting boundary terms to zero, $\left[\begin{matrix} x^4 n(x) \to 0 \\ x^5 n(x) \to 0 \end{matrix} \right\} \begin{matrix} \text{as } x \to 0 \\ \text{and } x \to \infty \end{matrix} \Bigg]$

$$\frac{T_e^{\text{equilibrium}}}{T} = \frac{\int (n + n^2)\, x^4\, dx}{4 \int n x^3\, dx} \qquad (4)$$

for $n(x) = n_{pl}(x) = \frac{1}{e^x - 1}$, $\quad T_e^{\text{equilibrium}} = T$.

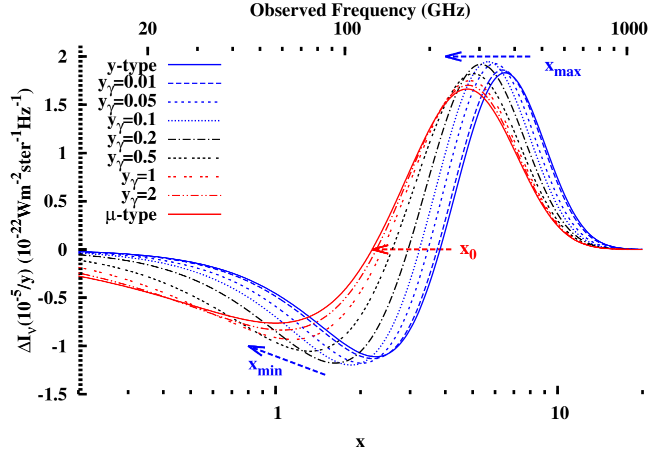
Suppose there is instantaneous energy release in the early universe. The initial spectrum is ~~then~~ given by y-type distortion, $A\,Y(x)$ with amplitude $A = \frac{\Delta E}{E_\gamma}$, where ΔE is the energy release, E_γ is energy density (density) of radiation at that time. All the energy ends up in photons since baryons have negligible heat capacity. This initial ~~...~~ y-type spectrum will evolve through Compton scattering until it reaches equilibrium Bose-Einstein distribution. We can follow this evolution by solving Kompaneets equation with initial spectrum $n(x, y_\gamma=0) = A\,Y(x)$, $A \ll 1$ is the amplitude

The result is shown in next figure.

For realistic cases of continuous energy release we can construct the resulting spectrum as superposition of y, μ-type and intermediate type spectra. (Khatri and Sunyaev 2012)

Y-type distortion,

$$n_{y\text{-type}} = 0.25 \int_{z(y_\gamma=0.01)}^{z=0} \frac{dQ}{dz}\, Y(x)$$



Observed Frequency (GHz)
20
100
1000
y-type
y_γ=0.01
y_γ=0.05
y_γ=0.1
y_γ=0.2
y_γ=0.5
y_γ=1
y_γ=2
μ-type
x_max
x_0
x_min
ΔI_ν(10^-5/y) (10^-22 Wm^-2 ster^-1 Hz^-1)
2
1.5
1
0.5
0
-0.5
-1
-1.5
1
10
x

$$n_{i\text{-type}} = \frac{1}{\theta_{num}} \sum_i \frac{d\theta}{dy_r}(y_r^i)\, \delta y_r^i \; n(y_r^i)$$

$d\theta = \dfrac{\Delta E}{E_r}$ is the fractional energy injected in interval δy_r^i at redshift $z(y_r = y_r^i)$

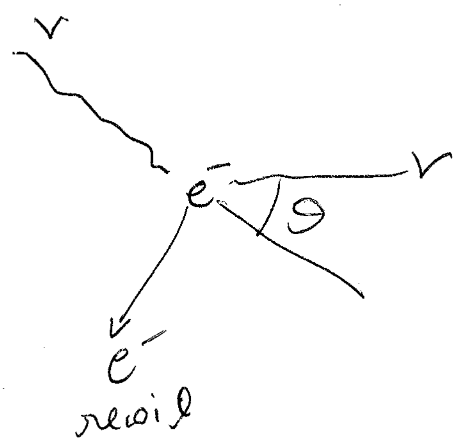
$n(y_r^i)$ is the solution of Kompaneets equation at $y_r = y_r^i$

$$n_{\mu\text{-type}} = 1.4\, n_\mu \int^{z(y_r=2)} \frac{d\theta}{dz} e^{-\tau}\, dz.$$

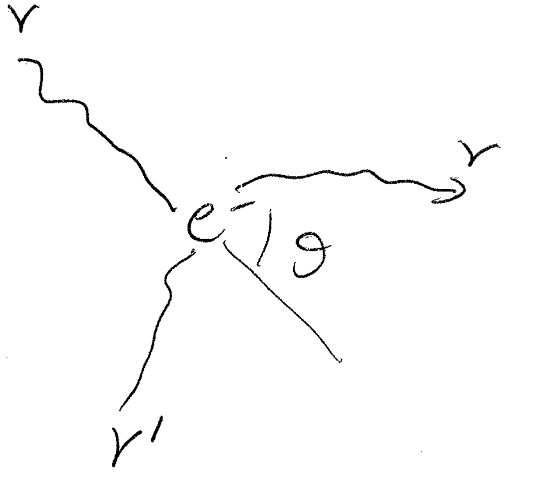
where τ = blackbody optical depth. and takes into account suppression of μ-type distortions at high redshifts due to the action of double Compton scattering and bremsstrahlung.

We will find analytic solutions for $\tau(z)$ next.

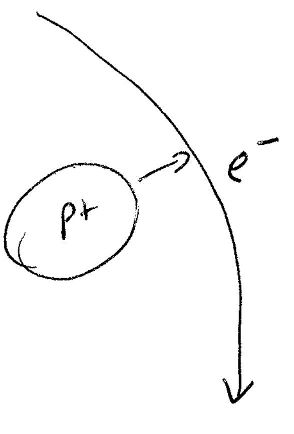
we will use $x_e = \frac{h\nu}{kT_e}$ as frequency variable defined w.r.t. the electron temperature T_e.



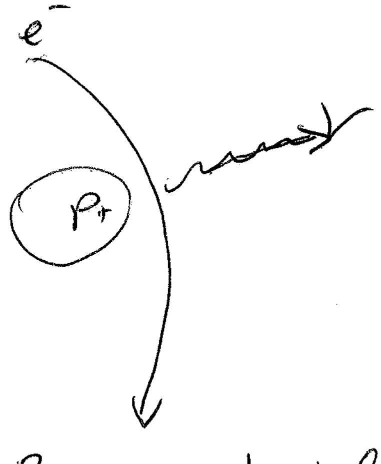
Compton scattering



Double compton Scattering



Coulomb Scattering



Bremsstrahlung.

The kinetic equation including Compton, double Compton and bremsstrahlung is:

$$\frac{\partial n}{\partial t} = K_C \frac{1}{x_e^2} \frac{\partial}{\partial x_e} x_e^4 \left[\frac{\partial n}{\partial x_e} + n + n^2 \right] + \left(K_{dC} + K_{br} \right) \frac{e^{-x_e}}{x_e^3} \left[1 - n\left(e^{x_e} - 1\right) \right]$$

$$+ \; x_e \frac{\partial n}{\partial x_e} \frac{\partial}{\partial t}\left[\ln\left(\frac{T_e}{T_{cmb}(1+z)}\right)\right], \quad T_{cmb} = 2.725\,K.$$

S

The last term takes into account the change in electron temperature as the spectrum of photons ~~changes~~ evolves, but factoring out the trivial evolution due to the expansion of the universe. [The partial time derivative is at constant x_e now, $\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t}\Big|_{x_e}$

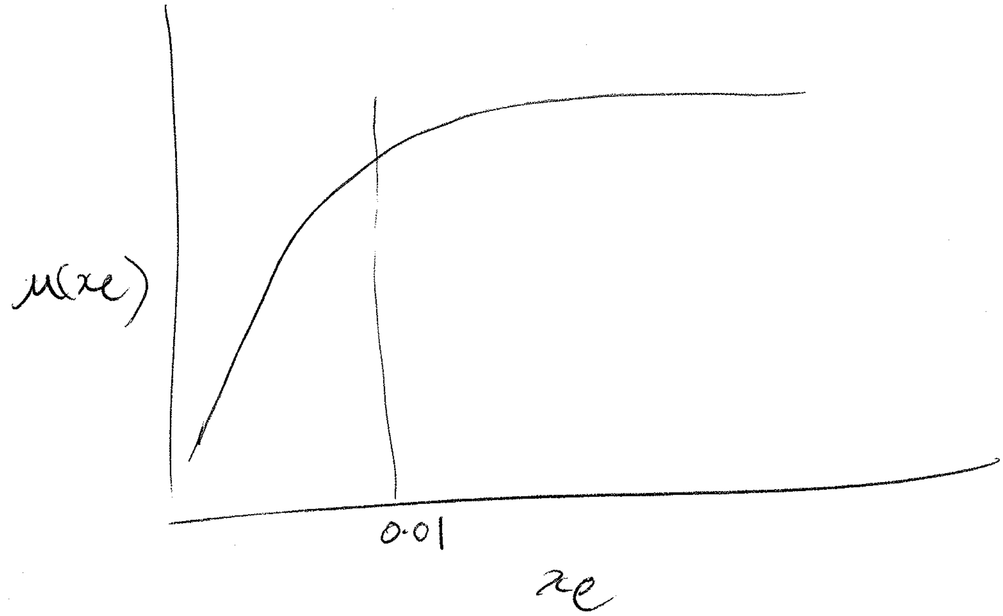
An approximate solution to this problem was found by ~~[illegible]~~ Sunyaev and Zeldovich 1970.

What makes this problem tractable is the fact that the solution at frequencies $x_e \gtrsim 0.1$ is described by a Bose-Einstein spectrum. ~~Since Comptonization at the~~ Thus we only need to find the two parameters of the spectrum, μ and temperature T.

For this we need two constraints, energy injection rate and photon production rate due to double Compton and bremsstrahlung.

Bremsstrahlung and double Compton are very efficient at frequencies $x_e \lesssim 0.01$ and drive the spectrum to blackbody. At $x_e \gtrsim 0.01$ Compton scattering dominates and tries to maintain Bose-Einstein spectrum by redistributing the photons produced at low frequencies. Their combined action drives the μ-parameter to zero.

$$n(x_e) = \frac{1}{e^{x_e + \mu(x_e)} - 1}$$



For $\mu << 1$, we have for total energy density and ~~ee~~ number density,

$$E = \frac{a_R T_e^4}{I_3} \int dx_e\, x_e^3\, n(x_e) \simeq a_R T_e^4 \left(1 - \mu \frac{6\zeta(3)}{I_3}\right)$$

$$N = \frac{b_R T_e^3}{I_2} \int dx_e\, x_e^3\, n(x_e) \simeq b_R T_e^4 \left(1 - \mu \frac{\pi^2}{3 I_2}\right)$$

$$I_3 = \int x^3 n_{pl}(x)\,dx = \frac{\pi^4}{15}, \quad I_2 = \int x^2 n_{pl}(x)\,dx = 2\zeta(3)$$

Taking time derivatives, with E_r, N_r energy and number density of blackbodies with Temperature $T_r = 2.725(1+z)$

$$\frac{d}{dt} \ln\left(\frac{E}{E_r}\right) \equiv \frac{\dot{\varepsilon}}{\varepsilon} = 4 \frac{d}{dt} \ln\left(\frac{T_e}{T_r}\right) - 6\zeta(3) \frac{d\mu}{dt}$$

$$\frac{d}{dt} \ln\left(\frac{N}{N_r}\right) \equiv \frac{\dot{N}}{N} = 3 \frac{d}{dt} \ln\left(\frac{T_e}{T_r}\right) - \frac{\pi^2}{3 I_2} \frac{d\mu}{dt}$$

Multiplying Eq 5 by x_e^2 and integrating over x_e and using it in time derivative of (6) gives us the production rate of photons,

$$\frac{d}{dt}\ln\left(\frac{N}{N_r}\right) = \frac{1}{I_2}\int_0^\infty dx_e\,(K_{dC}+K_{br})\frac{e^{-x_e}}{x_e}\left(1-n(e^{x_e}-1)\right)$$

$$\simeq \frac{K_{dc}+K_{br}}{I_2}\int dx_e \frac{\mu(x_e)}{x_e(e^{x_e}-1)}$$

~~If we can find~~

We can find approximate solution for $\mu(x_e)$ if we neglect the time derivatives in (5). Keeping terms linear in $\mu(x_e)$ gives us the ODE,

$$0 = -K_C\frac{1}{x_e^2}\frac{d}{dx_e}x_e^2\frac{d\mu}{dx_e} + (K_{dC}+K_{br})\frac{\mu}{x_e^4}$$

The solution with boundary condition $\mu(0)=0$ is

$$\mu(x_e) = \mu_c\, e^{-x_c/x_e}$$

$$x_c \simeq \left(\frac{K_{dC}(x_c)+K_{br}(x_c)}{K_C}\right)^{1/2}$$

$$\simeq 0.01$$

we can now evaluate the production rate of photons,

$$\frac{d}{dt}\ln\left(\frac{N}{N_\gamma}\right) \simeq \frac{K_{dC}+K_{br}}{I_2}\int dx_e \, \frac{\mu_c e^{-x_c/x_e}}{x_e^2}$$

$$\simeq \frac{\mu_c}{I_2}\left[(K_{dC}+K_{br})K_C\right]^{1/2}$$

And from (6)

$$\frac{d\mu}{dz} = \frac{C}{(1+z)H}\left[\mu\left[(K_{dC}+K_{br})K_C\right]^{1/2} - B\frac{\dot{\mathcal{E}}}{\mathcal{E}} + \frac{4B}{3}\frac{\dot{N}}{N}\Big|_{x\,In}\right]$$

$C = 0.7768$, $B = 1.803$

This can be integrated to give

$$\mu(0) = \mu(z_i)e^{-\tau(z_i)} + CB\int_{z_{min}}^{z_i}\frac{dz}{(1+z)H}\left(\frac{\dot{\mathcal{E}}}{\mathcal{E}} - \frac{4}{3}\frac{\dot{N}}{N}\Big|_{x}\right)e^{-\tau(z)}$$

$z_{min} = 2\times10^5$ (where $y_\gamma = 2$)

and $$\tau(z) = \int_0^z dz' \, C\frac{\left[(K_{dC}+K_{br})K_C\right]^{1/2}}{(1+z)H}$$

$$\simeq \left[\left(\frac{1+z}{1+z_{dC}}\right)^5 + \left(\frac{1+z}{1+z_{br}}\right)^{5/2}\right]^{1/2}$$

$$+ \varepsilon\ln\left[\left(\frac{1+z}{1+z_\varepsilon}\right)^{5/4} + \sqrt{1+\left(\frac{1+z}{1+z_\varepsilon}\right)^{5/2}}\right]$$

$z_{dC} \simeq 1.96\times10^6$ $\quad z_\varepsilon \simeq 3.67\times10^5$

$z_{br} \simeq 1.05\times10^7$ $\quad \varepsilon \simeq 0.0151$

We can improve on this solution by using it to approximate the time derivative in (5).

$$\frac{\partial n}{\partial t} \simeq \frac{-1}{x_e^2}\frac{\partial \mu}{\partial t} \simeq \frac{C\mu\left[(K_{dc}+K_{br})K_c\right]^{1/2}}{x_e^2}$$

Using this in the kinetic equation we get improved solution for $\mu(x_e)$

$$\mu(x_e) = A\mu_c\sqrt{\frac{2}{\pi}}\sqrt{\frac{x_c}{x_e}}\,K_{0.5\sqrt{1-4C\alpha_c}}(x_c/x_e)$$

K is modified Bessel function of second kind.

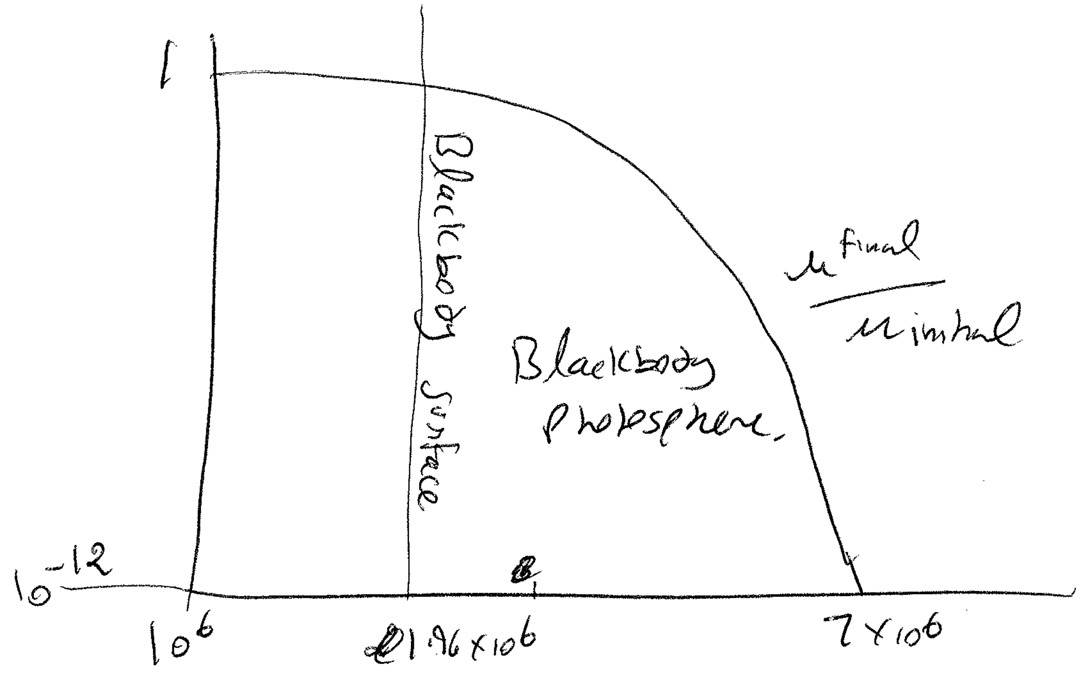
Choosing normalization, so that $\mu(x_e) \simeq \mu_c$ at $x_e \simeq 0.5$, $A = 1.007 + 3.5x_c$.

Repeating the above steps with this solution gives improved blackbody optical depth

$$T(z) = 1.007\,T(z)_{\text{original}} + \left(\frac{1+z}{1+z_{dc}'}\right)^3 + \left(\frac{1+z}{1+z_{br}'}\right)^{1/2}$$

$$z_{dc}' \simeq 7.11\times10^6$$
$$z_{br}' \simeq 5.41\times10^{11}.$$

The improved solution has accuracy of ~1%.



1
10^{-12}
Blackbody Surface
Blackbody Photosphere.
$\frac{u^{final}}{u^{initial}}$
10^6
1.96×10^6
7×10^6

With in the standard model, we have only two dominant sources of spectral distortions of y, μ and intermediate-type before recombination: Adiabatic cooling of baryons which gives "negative" distortions and Silk damping. The spectral distortions from these two sources are inevitable

Adiabatic cooling of baryons

Baryons try to cool ~~due~~ adiabatically (index 5/3) with the expansion of the universe as $T_e \propto (1+z)^2$. But Compton scattering tries to maintain $T_e \simeq T_\gamma \propto (1+z)$ (adiabatic index 4/3)

Thus there is transfer of energy from photons to baryons. For small ~~distortions~~ energy transfer, we get spectral distortions in photons which are ~~eq. ex~~ opposite to what we get in case of heating.

The evolution equation for electron temperature can be written as (Kompt, Zeldovich, Sunyaev 1968 Peebles 1968)

$$\frac{dT_e}{dz} = \frac{2T_e}{1+z} - \frac{2}{3kn_{eb}} S_{compton} \qquad (8)$$

where $n_{eb} = \frac{\rho_b}{\mu_{mol}}$ = total number density of free ~~p~~ baryonic particles, ρ_b is baryon mass density and μ_{mol} is the mean molecular weight.

$$S_{compton} = \frac{4kE_\gamma n_e \sigma_T (T_\gamma - T_e)}{m_e c H (1+z)}$$ is the energy transfer rate per unit volume from radiation

to baryons, by Compton scattering.

Since $T_e \simeq T_r \propto (1+z)$, $\frac{dT_e}{dz} \approx \frac{T_r}{1+z}$

Substituting in (8) we get for $S_{compton}$

$$S_{compton} = \frac{3}{2} k_B n_{eb} T_r \frac{1}{(1+z)} = \frac{E_B}{1+z}$$

$$\left.\frac{d\theta}{dz}\right|_{compton} = \frac{E_B/E_r}{(1+z)}$$

~~The fact that $\mu \geq 0$, means that cooling of~~
~~protons [illegible] is [illegible]~~

This cooling of protons (or Bose gas) starting with chemical potential of 0 (blackbody) is in fact Bose-Einstein condensation. In practice no condensate forms since the protons moving towards ground state ($x=0$) get absorbed by bremsstrahlung, when the reach around $x \sim 0.01$.

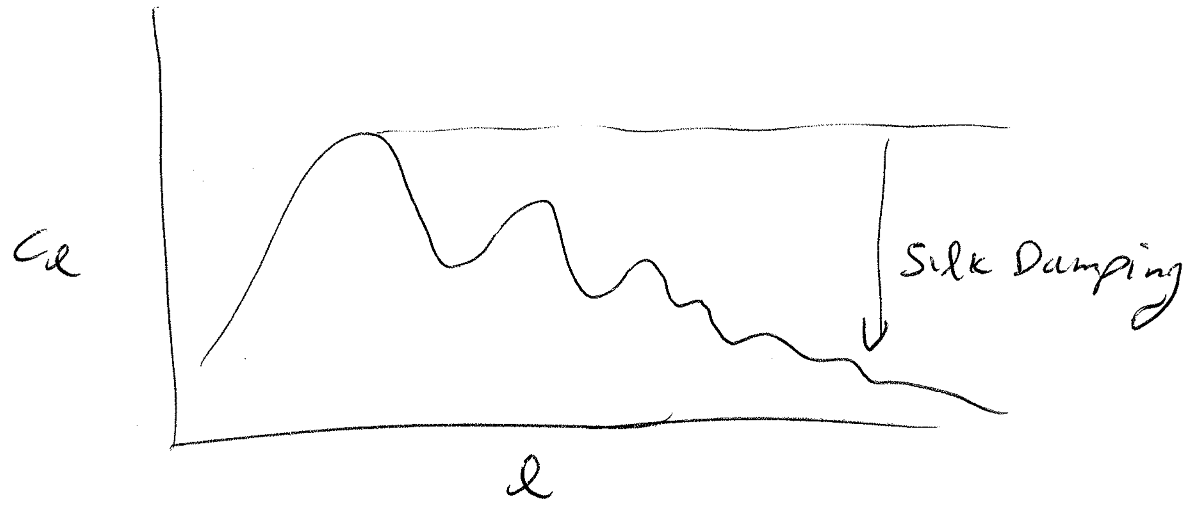
Note The condition for Bose-Einstein condensation of a Bose gas is that the chemical potential should go to 0. The blackbody CMB is thus already critical and any cooling means Bose-Einstein condensation. Non-relativistic particles, like Helium, have a Maxwellian distribution with a huge chemical potential. Therefore in lab we need to cool the particles to drive chemical potential to zero.

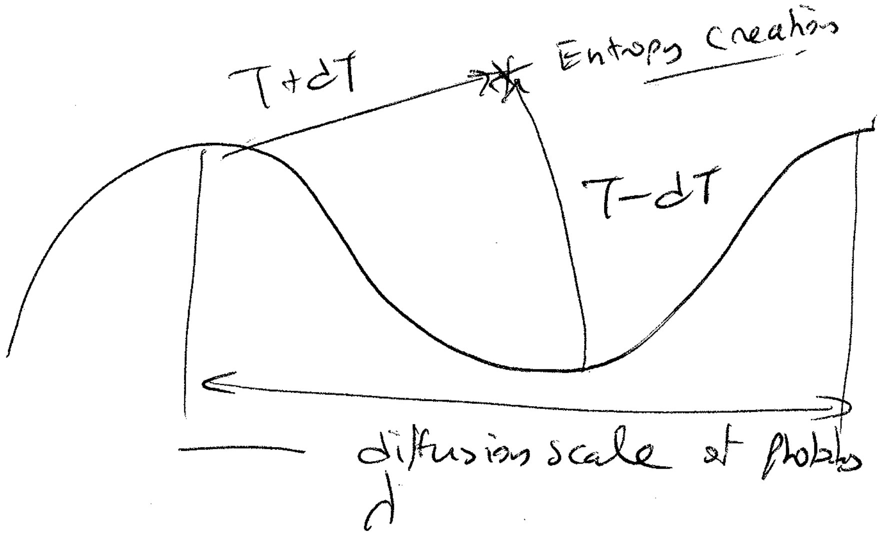
Silk damping.

Adiabatic cooling depends on standard cosmological parameters which are well measured by CMB anisotropies with much better precision than we can hope from measuring the spectral distortions from cooling. Therefore cooling is fixed and there is no uncertainty in it.

From the point of view of learning something new about fundamental physics and initial conditions, therefore, Silk damping is the only source of μ, and i-type distortions. In particular there is no other degeneracy within standard model!

The simplest way which gives the exact result without going through rigors of 2^{nd} order perturbation theory on relativistic fluid mechanics is ~~....~~ by thinking about the mixing of blackbodies.





Photons diffuse through primordial plasma and erase perturbations on scales of order diffusion length. This diffusion mixes and averages blackbodies of different temperatures. The result is a Y-type distortion as can be seen by Taylor expanding the photon occupation number and averaging,

$$\langle n_{pl}(T+\Delta T)\rangle = \left\langle \frac{1}{e^{\frac{\nu}{T+\Delta T}}-1} \right\rangle$$

$$\simeq \left\langle n_{pl}(T) + \ln\left[1+\frac{\Delta T}{T}\right]\frac{\partial n_{pl}}{\partial \ln T} + \frac{1}{2}\left[\ln\left(1+\frac{\Delta T}{T}\right)\right]^2 \frac{\partial^2 n_{pl}}{\partial (\ln T)^2} \right\rangle$$

$$= n_{pl}(T) + \left(\left\langle\frac{\Delta T}{T}\right\rangle + \left\langle\frac{\Delta T}{T}\right\rangle^2\right) T\frac{\partial n_{pl}}{\partial T} + \frac{1}{2}\left\langle\left(\frac{\Delta T}{T}\right)^2\right\rangle$$

$$T^4\frac{\partial}{\partial T}\frac{1}{T^2}\frac{\partial n_{pl}}{\partial T}$$

$$= n_{pl}\left(T + \left\langle\left(\frac{\Delta T}{T}\right)^2\right\rangle\right) + \frac{1}{2}Y(x)\left\langle\left(\frac{\Delta T}{T}\right)^2\right\rangle$$

The result is new planck spectrum with temperature

$T + \left\langle\left(\frac{\Delta T}{T}\right)^2\right\rangle$ and a y-type distortion

of amplitude $\frac{1}{2}\left\langle\left(\frac{\Delta T}{T}\right)^2\right\rangle$.

The total energy added to the original spectrum $n_{pl}(T)$ ϵ is,

$$\epsilon \left\langle a_R (T+\Delta T)^4 \right\rangle - a_R T^4$$

$$= a_R T^4 \cdot 6 \left(\frac{\Delta T}{T}\right)^2 .$$

out of which the $\frac{1}{3}$ energy goes into y-type distortion of amplitude $\frac{1}{2}\left(\frac{\Delta T}{T}\right)^2$.

Applying this to Silk damping, the total acoustic energy is

$$\frac{\Delta E}{E_\gamma} = 6\left\langle\left(\frac{\Delta T}{T}\right)^2\right\rangle$$

$$= 6\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int \frac{d^3k'}{(2\pi)^3} \left[\Theta(\vec{k}',\hat{n})\Theta(\vec{k}-\vec{k}',\hat{n})\right]$$

$$= 6\int \frac{k^2 dk}{2\pi^2} P_i(k) \left[\sum_{\ell=0}^{\infty}(2\ell+1)\Theta_\ell^2(k)\right]$$

where $\Theta(\vec{k})$ is Fourier transform of $\frac{\Delta T}{T}(\vec{x})$

and Θ_ℓ are the transfer functions of multipole moments ℓ,

$$\Theta(\hat{n}.\hat{k}, k) = \sum_\ell (-i)^\ell (2\ell+1)\, P_\ell(\hat{n}.\hat{k})\, \Theta_\ell(k)$$

$\hat{n}$ is of course the line of sight direction.

At high redshifts we have tight coupling and $\ell \geq 2$ multipole moments are suppressed.

Therefore in tight coupling limit we have

$$\left.\frac{\Delta E}{E_\gamma}\right|_{acoustic} = 6 \int \frac{k^2 dk}{2\pi^2} P_i(k) \left[\Theta_0^2 + 3\Theta_1^2\right]$$

We can use ~~tight~~ analytic tight coupling solutions of Hu and Sugiyama 1995 to evaluate the above expression,

$$\Theta_0(\eta) \simeq 3\Theta_0(0) \cos(k r_s)\, e^{-k^2/k_D^2}$$

$$\Theta_1(\eta) \simeq 3\frac{\Theta_0(0)}{\sqrt{3}} \sin(k r_s)\, e^{-k^2/k_D^2}$$

r_s is the sound horizon, $r_s(\eta) \equiv \int_0^\eta d\eta' c_s(\eta')$

η is conformal time.

An important point to note is that $(\Theta_0^2 + 3\Theta_1^2)$ is independent of time in the absence e^{-k^2/k_D^2} damping factor. This is expected for standing sound waves, in which thermal (Θ_0)

and kinetic (Θ_1) energy oscillates with time but total energy ~~does~~ ($\Theta_0^2 + 3\Theta_1^2$) doesn't oscillate and is conserved.

K_D is damping wavenumber given by

$$\frac{1}{K_D^2} = \int_z^\infty dz \frac{c(1+z)}{6H(1+R)\, n_e \sigma_T} \left(\frac{R^2}{1+R} + \frac{16}{15} \right)$$

$R = \frac{3\rho_b}{4\rho_\gamma}$, ρ_b is baryon energy density, ρ_γ is photon energy density

Energy injection into μ (i-type) distortions is

$\frac{1}{3}\frac{d}{dt}\frac{\Delta E}{E_\gamma}\Big|_{acoustic}$. It can be put in gauge invariant form as follows.

$$\frac{1}{3}\frac{d}{dt}\frac{\Delta E}{E_\gamma}\Big|_{acoustic} = \left\langle -2\frac{d}{dt}\left(\frac{\Delta T}{T}\right)^2 \right\rangle$$

$$= -4\left\langle \frac{\Delta T}{T}\frac{d}{dt}\frac{\Delta T}{T} \right\rangle$$ use 1st order Boltzmann equation and ignore metric perturbations

$$4 n_e \sigma_T \int \frac{k^2 dk}{2\pi^2} P_i(k) \Big[\Theta_1 (3\Theta_1 - v) + \frac{9}{2}\Theta_2^2 - \frac{1}{2}\Theta_2 (\Theta_2^P + \Theta_0^P) + \sum_{l \geq 3} (2l+1)\Theta_l^2 \Big]$$

impose gauge invariance on velocity term

$$\longrightarrow 4 n_e \sigma_T \int \frac{k^2 dk}{2\pi^2} P_i(k) \Big[\frac{(3\theta_1 - v)^2}{3} + \frac{9}{2}\theta_2^2 - \frac{1}{2}\theta_2(\theta_2^P + \theta_0^P) + \sum_{l \geq 3} (2l+1)\,\theta_l^2 \Big]$$

This is the exact expression which we can get ~~a~~ from 2^{nd} perturbation theory also in a more rigorous manner. In particular this expression is valid at all times and not just in the tight coupling regime. We added a $\frac{v^2}{3}$ term to make the expression gauge invariant, v is electron peculiar velocity. This ~~is~~ ~~the~~ term is the equivalent of $\frac{T_e}{m}$ term in the Kompaneets equation and like that term gives y-type distortion.

The ~~dip~~ $(3\theta_1 - v)^2$ term mixes the dipole in electron rest frame and ~~is the~~ is the result of thermal conductivity. The quadrupole / ~~polarization~~ polarization ~~Polar~~ term ~~is~~ can be similarly identified as due to shear viscosity.

The final result in tight coupling limit when the initial power spectrum as spectral index n_s can be evaluated analytically,

$$\frac{d\Theta/dz}{\Theta} = -9.75\,\frac{A_\zeta}{k_0^{\,n_s-1}}\,2^{-(3+n_s)/2}\,\Gamma\!\left(\frac{n_s+1}{2}\right)$$

$$A_D^{(1-n_s)/2}\,(1+z)^{(3n_s-5)/2}$$

A_ζ is the amplitude of primordial curvature perturbation defined as in WMAP-7 papers.

$$A_D = \frac{8c}{135 H_0\,\Omega_r^{1/2}\,n_{e0}\,\sigma_T} = 5.92\times 10^{10}\ \mathrm{Mpc}^2$$

radiation energy density. in units of critical density.

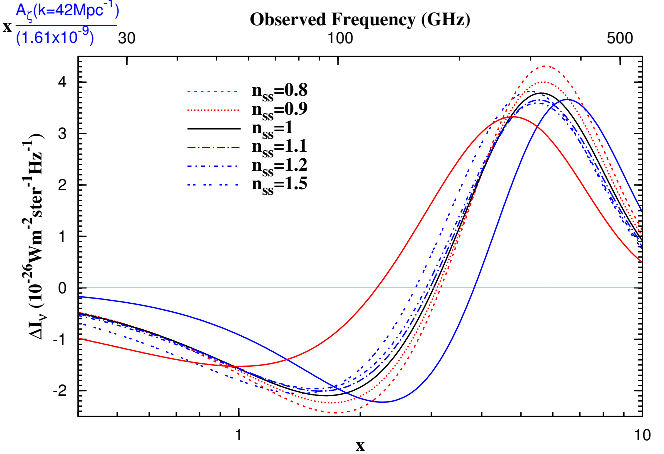
electron number density today of fully ionized primordial plasma

As can be seen, $d\Theta/dz \propto (1+z)^{(3n_s-5)/2}$.

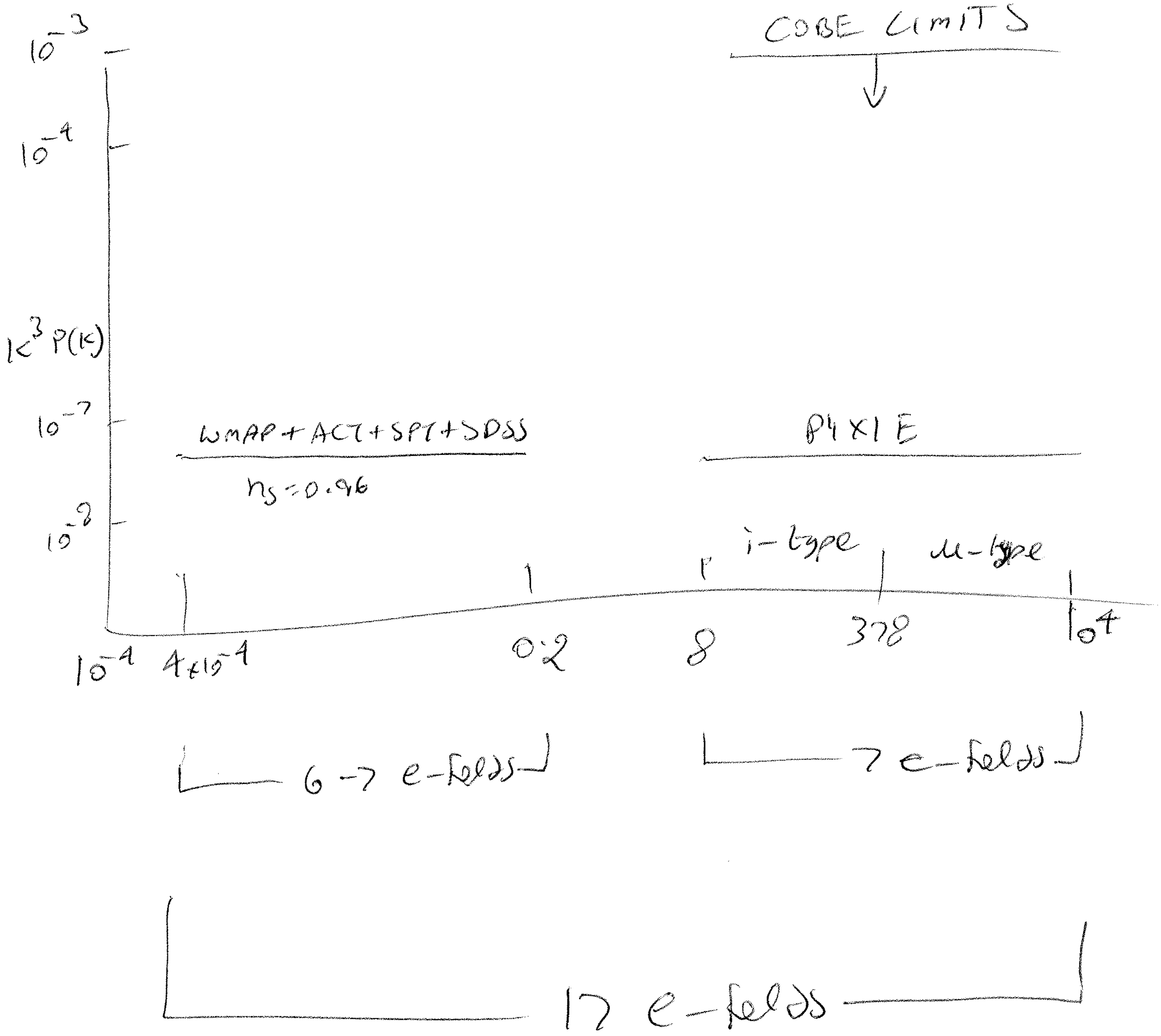
This dependence will give different shape for i-type spectrum for different n_s.

We can therefore measure n_s on small scales ($k_0 \simeq 45\ \mathrm{Mpc}^{-1}$) independently of anisotropies on large scale.

PIXIE (Kogut et al. 2011) can detect Silk damping for WMAP-7 power spectrum, $n_s \simeq 0.97$!



x $\frac{A_\zeta(k=42\text{Mpc}^{-1})}{(1.61\times10^{-9})}$
Observed Frequency (GHz)
30
100
500
n_{ss}=0.8
n_{ss}=0.9
n_{ss}=1
n_{ss}=1.1
n_{ss}=1.2
n_{ss}=1.5
ΔI_ν (10^{-26}Wm^{-2}ster^{-1}Hz^{-1})
4
3
2
1
0
-1
-2
1
10
x


COBE LIMITS
10^{-3}
10^{-4}
$k^3 P(k)$
10^{-7}
10^{-8}
WMAP + ACT + SPT + SDSS
$n_s = 0.96$
PIXIE
i-type
μ-type
10^{-4}
4×10^{-4}
0.2
8
378
10^4
6 → 7 e-folds
7 e-folds
17 e-folds

~~Old~~ Sources of μ, and i-type distortions beyond the standard model

If we go beyond standard model, there are several new sources of energy injection. ~~a~~ This introduces degeneracies with Silk damping. However ~~can be~~ these sources have different characteristic dependence ~~on~~ of energy injection rate on redshift. It is difficult but not impossible to distinguish between different sources using i-type distortions. ~~It also~~

1. <u>decay of particles</u> Chluba and Sunyaev 2012, Khatri and Sunyaev 2012

$$\frac{dQ}{dz} \propto \frac{e^{-\left(\frac{1+z_{decay}}{1+z}\right)^2}}{(1+z)^4}$$

2. <u>Cosmic Strings</u> (superconducting)
Tashiro, Sabancilar, Vachaspati 2012

$$\frac{dQ}{dz} \propto \text{Constant},$$ Constrain current, string tension

3. <u>Primordial magnetic fields.</u> Jedamzik, Katalinic and Olinto 2000

$$\frac{dQ}{dz} \propto (1+z)^{(3n+7)/2}$$ n = spectral index of magnetic field power spectrum

4. Evaporating primordial black holes.

Tashiro and Sugiyama 2008, Carr et al. 2010

$\frac{dQ}{dz}$ depends on mass function

5. Quantum wave function collapse

Lochan, Das, Bassi 2012

$$\frac{dQ}{dz} \propto (1+z)^{-4}$$

Other types of distortions in CMB

1. Cosmological recombination spectrum gives a measurement of primordial helium.

Kurt, Zeldovich, Sunyaev, Peebles, Dubrovich, Chluba, Rubino-Martin - -

Rubino-Martin, Chluba and ~~Sunyaev~~ Sunyaev 2008.

2. Resonant scattering of CMB on C, N, O and other ions during and after reionization makes the optical depth to last scattering surface τ_{LSS} frequency dependent. By ~~ob~~ comparing CMB anisotropy ~~power~~ spectrum at different frequencies, we can measure metal abundance during reionization!

Basu, Hernandez-Monteagudo, Sunyaev 2004.

3. y-distortion from hot electrons during reionization (~~as well as~~ contamination from peculiar velocities, $\frac{v^2}{3}$ effect?)
Can give a measurement of electron temperature

4. Primordial non-gaussianity on small scales gives fluctuations in μ.
Pajer and Zaldarriaga 2012
Ganc and Komatsu 2012

Proposed Experiment PIXIE holds lots of promise. CMB spectrum is very rich in information about the early Universe, late-time Universe and fundamental physics.

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and promises a view of inflation spanning 17-efolds!

Mathematica code and i-type distortions available at
mpa/home/~khatri/idistort.html