Assignment-1: Classical Mechanics

Solutions

1. Consider the action

$$S = \int \left(\sum_n \tfrac{1}{2} m_n \dot{x}_n^2 - U(x_n)\right) dt$$

A (constant) infinitesimal boost causes the following change in the coordinates and velocities of the particles:

$$\delta x_n^i = V^i t \epsilon$$
$$\delta \dot{x}_n^i = V^i \epsilon$$

where ϵ is a small parameter and V is the velocity of the boost. Here n labels the particle number and i is the component of the vector quantity. The general expression for the conserved charge is:

$$Q = \mathbf{p} \cdot \delta \mathbf{x} - H$$

where p is the generalized momentum and H is the change in the action. The change in the action can be calculated as follows. The potential function U is taken to be translationally invariant, i.e., it is a function of the separations of the particles only. Since a constant Galilean boost is just a time dependent translation, the potential remains unchanged. Thus, the only change is in the kinetic energy term. Thus, the change in the Lagrangian (to $\mathcal{O}(\epsilon)$) is:

$$\delta L = \epsilon \left(\sum_n m_n \dot{\mathbf{x}}_n\right) \cdot \mathbf{V} = \tfrac{d}{dt}\left(\epsilon \left(\sum_n m_n \mathbf{x}_n\right) \cdot \mathbf{V}\right)$$

Thus, $H = \epsilon \left(\sum_n m_n \mathbf{x}_n\right) \cdot \mathbf{V}$. At the same time, we have, for the first term in the expression for the conserved charge $\mathbf{p} \cdot \delta \mathbf{x} = \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \delta \mathbf{x}$

$$\tfrac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \delta \mathbf{x} = \epsilon \left(\sum_n m_n \dot{\mathbf{x}}_n\right) \cdot \mathbf{V} t$$

substituting for the above quantities in the general expression for Q, we get:

$$Q = \epsilon \left(\sum_n m_n(\dot{\mathbf{x}}_n t - \mathbf{x}_n)\right) \cdot \mathbf{V}$$

Thus, the conserved charge is:

$$\sum_n m_n(\dot{\mathbf{x}}_n t - \mathbf{x}_n)$$

As a consequence,

$$\frac{d}{dt}\left(\sum_n m_n(\dot{x}^i_n t - x^i_n)\right) = t\frac{d}{dt}\left(\sum_n m_n \dot{x}^i_n\right) = 0$$

as the total momentum (the quantity in the brackets) is conserved. Thus, there is no new conserved quantity one can derive from Galilean invariance.

2. (a) The equation of motion is easily found to be:

$$m\ddot{x} = -10kx^9$$

(b) Substituting the given scaling in the equation of motion, we get on the RHS λ^8 and on the LHS we get $\lambda^{2\beta}$. Thus, $\beta = 4$ for $\lambda x(\lambda^\beta t)$ to be a solution.
(c)We have $\lambda x(\lambda^4 t)$ as a solution, and from the scaling arguments for the time period given in class, we easily find that the time period scales as $k = -4$.
(d) Applying the virial theorem as discussed in the class, one finds:

$$2\bar{T} = \alpha\bar{U}$$

Where $\bar{A}$ indicates the time average of the quantity A. T and U are the kinetic and potential energies respectively and α is the exponent in the scaling of the potential. We easily find thus $\bar{T}/\bar{U} = 5$.

3. (a) We fix a system of coordinates in the body fixed frame of the earth. Note that this is not an inertial frame of reference. Thus, by denoting ϕ as the longitude, θ the latitude and h the height above the surface of the earth, we have the Lagrangian for the particle:

$$L = T - V = \frac{m}{2}(R^2(\sin^2\theta(\dot{\phi} + \omega)^2 + \dot{\theta}^2) + \dot{h}^2) - mgh$$

We note that by solving the ϕ equation of motion, one can derive the familiar Coriolis force on objects due to the rotation of the earth.
(b) (i) First, we must specify the number of degrees of freedom of the suspended ball. For this purpose, let us first work in a system of coordinates fixed to the rotating turntable; this is not an inertial system. We choose the z axis along the axis of the turntable, and the xy plane so that the plane

contains the point from which the rod is suspended from the pole; i.e., at $z = 0$. We can observe that there are clearly two 'directions' in which the ball can freely move: one can be defined by the angle that the rod suspending the ball subtends with respect to the pole; another resulting from the fact that the ball can spin around the pole. Let us denote these angles (our generalized coordinates) by θ and ϕ respectively. Thus, there are two generalized coordinates; hence the number of degrees of freedom (defined as the number of variables to be given at $t = 0$ to completely determine the evolution) are four.

(ii) To set up the Lagrangian, we need to determine the transformation that relates the body fixed (rotating) frame to a one that is inertial. We choose the inertial system in the same manner as above; the origins of the two systems coincide. Let us call the coordinates of the inertial system as x_{out}, y_{out} and z_{out} and the rotating reference frame x, y and z. Clearly, the inertial system is related to the rotating system by a rotation in the xy plane; thus the relation is:

$$\begin{pmatrix} x_{\text{out}} \\ y_{\text{out}} \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Clearly,

$$z_{\text{out}} = z$$

Note that the rotation angle (the argument of the rotation matrix) could in principle be any function of t; in our specific case it is ωt. It now remains to determine x, y and z in terms of θ and ϕ defined above, which would enable us to write down the Lagrangian in the inertial frame.

We can easily see that with respect to the body fixed frame, the position of the ball is:

$$\begin{aligned} x &= R + (l\sin\theta)\cos\phi \\ y &= (l\sin\theta)\sin\phi \\ z &= -l\cos\theta \end{aligned}$$

Substituting the above relations, after some straightforward algebra, one finds the expression for the Lagrangian:

$$\begin{aligned} L = \frac{m}{2}&(R^2\omega^2 + l^2(\dot\theta^2 + \sin^2\theta(\dot\phi - \omega)^2) - 2R\omega l((\dot\phi - \omega)\sin\theta\cos\phi \\ &+ \dot\theta\cos\theta\sin\phi)) + mgl\cos\theta \end{aligned}$$

4 (a) The translational invariance is clear from the fact that the velocity dependent terms do not contribute to any change in the Lagrangian, whereas the potential depends only on the separations of the particles, hence any constant translation cancels in the term $|\mathbf{r}_1 - \mathbf{r}_2|^2$. Similarly, we can argue that the Lagrangian is rotationally invariant as it is function of v^2 and $|\mathbf{r}_1 - \mathbf{r}_2|^2$, which are unchanged under rotations. Similarly, time translational invariance is guaranteed as the Lagrangian is not explicitly time dependent.
But the v^4 term in the Lagrangian clearly violates Galilean invariance as the velocities change under a Galilean boost and the v^4 term cannot be transformed in any way to give rise to a total derivative as was shown in class, and can be easily checked. The same holds true for the equations of motion following from the given Lagrangian, as it contains terms which are third degree in the velocity.
(b) We can use the general expression given in problem 1 to compute the conserved charges following from the mentioned invariances of the problem. The conserved charge for the translational invariance is the generalized momentum; this is just $\frac{\partial L}{\partial \mathbf{v}_i}$ for each particle. Thus, we find for spatial translational invariance the following conserved charge:

$$\mathbf{p}_i = 2m_i v_i^2 \mathbf{v}_i$$

The conserved charge corresponding to time translational invariance is the energy; the expression for energy is $E = \sum_i \frac{\partial L}{\partial \mathbf{v}_i} \cdot \dot{\mathbf{r}}_i - L$. Thus, we find:

$$E = \sum_i 2m_i v_i^2 \mathbf{v}_i \cdot \dot{\mathbf{r}}_i - \left(\tfrac{1}{2}m_1 v_1^4 + \tfrac{1}{2}m_2 v_2^4 - |\mathbf{r}_1 - \mathbf{r}_2|^2\right) = \tfrac{3}{2}m_1 v_1^4 + \tfrac{3}{2}m_2 v_2^4 + |\mathbf{r}_1 - \mathbf{r}_2|^2$$

Please note that the names 'momentum' and 'energy' are misnomers; in this case they are just conserved charges corresponding to the symmetries of the given problem; the names are purely conventional. Finally, the charge related to rotational invariance can be obtained in a similar way. In this case, $\delta x^a = \epsilon^{abc}\delta\theta^b x^c$, where $\delta\theta^a$ is the infinitesimal angle of rotation. Thus, this gives the conserved charge to be:

$$Q = 2m_i v_i^2 v_i^a \epsilon^{abc} \delta\theta^b x_i^c$$

Thus, the conserved quantity is the 'angular momentum', $\sum_i 2m_i v_i^2 (\mathbf{x}_i \times \mathbf{v}_i)$. Here, i labels the particle number $i = 1, 2$ and a, b, c denote the vector components.