Assignment-2: Classical Mechanics

Solutions

1. We are given that a one dimensional particle oscillates in a symmetric potential, and that the time period of oscillation is given as a function of the particle's energy by

$$T(E_0) = \frac{\pi}{\alpha}\sqrt{\frac{2m}{E_0 + U}} \tag{1.1}$$

As explained in class, the time period as a function of energy completely determines the (symmetric) potential via the formula

$$\tfrac{1}{2\pi\sqrt{2m}} \int_0^a \tfrac{T(E)dE}{\sqrt{V-E}} = x(V)$$

(eq. 12.2, pg. 29 of Landau-Lifshitz.)
Now

$$\begin{aligned}\int_0^V \frac{dE}{\sqrt{(E+U)(V-E)}} &= \int_0^V \frac{dE}{\sqrt{-\left(E + \frac{U-V}{2}\right)^2 + \left(\frac{U+V}{2}\right)^2}} \\ &= \cos^{-1}\left(\frac{U-V}{U+V}\right)\end{aligned}$$

It follows that

$$2\alpha x(V) = \cos^{-1}\left(\tfrac{U-V}{U+V}\right)$$

Inverting this expression to get V as a function of x we find

$$V(x) = U\tan^2 \alpha x$$

It is instructive to check directly that the time period of oscillation in such a potential is given by (1.1). This is done on pg. 27, problem 2 of Landau

Lifshitz.

2. (a) In a p dimensional space, the area of a sphere depends on the radius as r^{p-1}. Thus, from 'Gauss law', we infer that the force must behave like $r^{-(p-1)}$. Clearly, for this to happen, the potential must go as $r^{-(p-2)}$ in p space and 1 time dimensions. (Equivalently note that the Greens function of the Laplacian operator is $\propto \frac{1}{r^{p-2}}$ in p spatial dimensions). Note this formula reduces to the familiar $1/r$ potential in $p = 3$.

(b) (i) In the case $p = 4$, the potential depends on r as $V(r) = -\alpha/r^2$, where α is a positive constant. As in $p = 3$ the motion is confined to a plane (this may be understood from angular momentum conservation, or, more directly, by noting that the force- hence acceleration - lies in the initial plane of motion, so the particle never leaves the original plane). The motion of the particle is governed by the Lagrangian

$$L = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{\alpha}{r^2} \tag{2.1}$$

where ϕ is the angle in the plane of motion. ϕ is a cyclic coordinate so its conjugate momentum $M = \mu r^2\dot{\phi}$ is conserved (this is simply the angular momentum of the particle in the plane of motion). The energy of the system is also conserved. Inserting the conservation of angular momentum into the conservation of energy yields an effective one dimensional problem in the potential

$$V_{eff}(r) = -\frac{\alpha}{r^2} + \frac{M^2}{2\mu r^2} = \frac{\left(\frac{M^2}{2\mu} - \alpha\right)}{r^2}$$

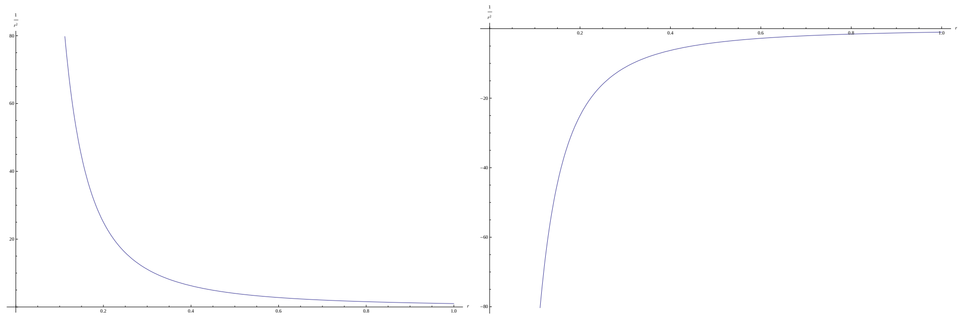
The motion of the radius as a function of time is determined by

$$\frac{\mu}{2}\left(\frac{dr}{dt}\right)^2 + V_{eff}(r) = E$$

The orbit is determined by substituting $dt = (\mu r^2/M)d\phi$ into this relation.

(ii) The above effective potential is qualitatively different depending on whether $M > M_c = \sqrt{2\mu\alpha}$ or $M < M_c$. In the first case the effective potential is everywhere positive, and the particle necessarily has positive energy (the effective potential is sketched in the first of the figures below). In the second case the effective potential is everywhere negative, and the energy of the system could be either positive or negative (the potential is schematically sketched in the second of the figures below).

(iii) In the $M^2 > 2\mu\alpha$ case, the particle starts out from infinity, reaches a



minimum radius r_{min}, and the heads back to ∞. The particle never reaches the center. In the case $M^2 < 2\mu\alpha$, a particle moving inwards *always* 'falls' to the attractive center and hits it. The time taken by a particle of energy E (E can be either positive or negative) to hit the center starting from $r = r_0$ is given by

$$\int_0^T dt = \int_0^{r_0} \sqrt{\frac{\mu}{2}} \frac{dr}{\sqrt{E + \frac{\alpha - M^2/2\mu}{r^2}}}$$

Solving the above integral by elementary techniques gives the time of fall as

$$T_{fall} = \frac{1}{E}\left(\sqrt{\frac{\mu\alpha}{2} - \frac{M^2}{4} + \frac{\mu E r_0^2}{2}} - \sqrt{\frac{\mu\alpha}{2} - \frac{M^2}{4}}\right) \tag{2.2}$$

If the total energy E is negative the maximum possible value of r_0 is given by $r_{max}^2 = \frac{\alpha - \frac{M^2}{2\mu}}{|E|}$ and the maximum time of fall is obtained by plugging $r = r_{max}$ into (2.2) giving $T_{max} = \frac{1}{|E|}\sqrt{\frac{\mu\alpha}{2} - \frac{M^2}{4}}$

(iv) The equation of orbit is given by the integral

$$\phi(r) = \int \frac{M dr/r^2}{\sqrt{2\mu(E + \frac{\alpha}{r^2}) - \frac{M^2}{r^2}}}$$

Here we have the two cases $E > 0$ and $E < 0$. The above integral can be performed easily by substituting $u = 1/r$. In the case that $M > M_c$ (when the energy of motion is always positive) we obtain

$$\frac{1}{r} = \sqrt{\frac{2\mu E}{M^2 - 2\mu\alpha}} \sin\left(\phi\sqrt{\left(1 - \frac{2\mu\alpha}{M^2}\right)}\right)$$

Consider, in this case, a particle that comes in from infinity toward the attractive center and goes back to infinity. The particle reaches its minimum radius when the argument of the sin function above is $\frac{\pi}{2}$ i.e. when

$$\phi = \frac{\pi}{2} \frac{M}{\sqrt{M^2 - 2\mu\alpha}} \tag{2.3}$$

[1] The total angle traversed is thus twice the above value, i.e., from ∞ to r_{min} and from r_{min} back off to ∞. The total angle is thus $\phi_{tot} = \frac{\pi M}{\sqrt{M^2-2\mu\alpha}}$. As $M \to \infty$, the particle whizzes past our deflecting center far away and so almost undeflected, so the total angle traversed is π as one would expect. As M is decreased the particle impact parameter decreases and the deflection angle increases. Note that the deflection angle could be made as large as one likes by tuning the value of M^2 closer and closer to $2\mu\alpha$. Thus the particle can wind around the center any number of times while executing its orbit. The number of times the particle winds around the center is given by

$$n = \left[\frac{M}{2\sqrt{M^2 - 2\mu\alpha}} \right]$$

where $[x]$ denotes the integer part of x. On the other hand when $M < M_c$ the orbit is given by

$$\begin{array}{ll} E > 0 \text{ case:} & \frac{1}{r} = \sqrt{\frac{2\mu E}{2\mu\alpha - M^2}} \sinh\left(\phi\sqrt{\left(\frac{2\mu\alpha}{M^2} - 1\right)}\right) \\ E < 0 \text{ case:} & \frac{1}{r} = \sqrt{\frac{2\mu|E|}{2\mu\alpha - M^2}} \cosh\left(\phi\sqrt{\left(\frac{2\mu\alpha}{M^2} - 1\right)}\right) \end{array}$$

As we have seen above, when $M < M_c$ the particle always falls to the center (in a finite time, starting from any finite position). From the formulas above we see that $\phi \to \infty$ as $r \to 0$, so that the particle winds the origin an infinite number of times before hitting it (see also prob no. 2, pg. 40 of Landau-Lifshitz.)

---

[1] This may be directly checked as follows:

$$\phi(r) = \int_{\infty}^{r_{min}} \frac{M dr/r^2}{\sqrt{2\mu(E + \frac{\alpha}{r^2}) - \frac{M^2}{r^2}}}$$

Making the substitution $u = 1/r$ one obtains (2.3)

3. The given potential is

$$V(x,y) = a(x^4 + y^4)$$

(a) Clearly the motion in x and y are separate and can be solved independently of each other. The motion is periodic in each direction and the energy in each direction is separately conserved.Parameterizing the energy in each direction as $E_x = ax_0^4$ and $E_y = ay_0^4$, we can write for the orbits the equations

$$\begin{aligned} t + c_1 &= \int_0^x \sqrt{\tfrac{m}{2}} \frac{dx'}{\sqrt{a(x_0^4 - x'^4)}} \\ \text{and} \quad t + c_2 &= \int_0^y \sqrt{\tfrac{m}{2}} \frac{dy'}{\sqrt{a(y_0^4 - y'^4)}} \end{aligned}$$

Here, c_1 and c_2 are integration constants.
(b) The nature of the motion is oscillatory in x and y. Also, we can use scaling to understand the dependence of the time period on the energy as follows. If $x \to \lambda x$, from the scaling relations for the times in the problem, we get $t \to \lambda^{-1} t$, as here the exponent in the potential is 4. Thus the time period T scales as x_0^{-1} and similarly for y_0, where x_0 and y_0 are the amplitudes in each direction.
(c) The orbits of motion are clearly always bounded in a rectangular box of side $2x_0$ in the x direction and side $2y_0$ in the y direction. The detailed shapes traversed by the orbits can, however, be very complicated. Recall that the time period of x oscillation is given by $T \sim 1/x_0$, while an analogous formula holds for y oscillations It follows that, in general, if x_0 and y_0 aren't rationally related, the orbits are not closed and will be very complicated.
Of course special values of x_0 and y_0 lead to more regular orbits. In particular, whenever x_0 and y_0 are rationally related the orbits are closed. Let us consider the study the special case $x_0 = y_0$ in more detail. Here we have two specially simple extreme cases. The first $c_1 = c_2$. In this case it is apparent from the orbit relations above the $x(t) = y(t)$ for all times and the particle undergoes equal linear oscillations. (along a straight line).
On the other hand $x_0 = y_0 = \alpha$ but $c_1 = c_2 + \frac{1}{4}T$ (where T is the time period of oscillations), gives an orbit with Z_4 symmetry (the Z_4 is generated by $\frac{\pi}{2}$ rotations in the x, y plane). The orbit passes through the point $(0, \alpha)$. It is instructive to determine the shape of the orbit in the neighbourhood of this point in more detail. This may be achieved by the use of perturbation

theory. We will use perturbation theory. Let us first define

$$F(\beta) = \int_0^\beta \frac{dk}{\sqrt{(\alpha^4 - k^4)}}$$

It follows that

$$\begin{array}{rl} t + c_1 = \int_0^x \sqrt{\frac{m}{2}} \frac{dx'}{\sqrt{a(\alpha^4 - x'^4)}} & = \sqrt{\frac{m}{2a}} F(x) \\ \text{and} \quad t + c_2 = \int_0^y \sqrt{\frac{m}{2}} \frac{dy'}{\sqrt{a(\alpha^4 - y'^4)}} & = \sqrt{\frac{m}{2a}} F(y) \end{array}$$

In order to determine the trajectory in the neighbourhood of $(0, \alpha)$ we need to determine the first terms in the expansion of $F(x)$ about 0 and α. It is not difficult to show that

$$F(x) = \frac{x}{\alpha^2} + \mathcal{O}(x^2)$$

and

$$F(a - \epsilon) = \frac{\sqrt{\epsilon}}{\alpha^{\frac{3}{2}}} + \mathcal{O}(\epsilon^{\frac{3}{2}})$$

[2] Equating the leading terms on both sides (the time for both are the same) to find the relation between x and y in the trajectory gives

$$\frac{x}{\alpha^2} = \frac{\sqrt{\alpha - y}}{\alpha^{3/2}}$$

---

[2]The first of these equations follows immediately by differentiating the expression for F. The second expression is more subtle to derive, as it is not analytic in ϵ and so cannot be obtained by expanding the answer in a power series in ϵ. This result may instead be obtained as follows. We wish to evaluate

$$\int_{\alpha-\epsilon}^{\alpha} \frac{dy}{\sqrt{\alpha^4 - y^4}}$$

for small ϵ. Setting $y = \alpha - \zeta$ where ζ is a new integration variable, we can write the above integral as

$$\int_0^\epsilon \frac{d\zeta}{\sqrt{\alpha^4 - (\alpha - \zeta)^4}} = \int_0^\epsilon \frac{d\zeta}{\sqrt{4\alpha^3\zeta - 6\alpha^2\zeta^2 + 4\alpha\zeta^3 - \zeta^4}}$$

here $\zeta < \epsilon$ is small (as ϵ is small). We can expand the denominator of the integrand to leading order in the variable ζ; this gives

$$\int_0^\epsilon \frac{d\zeta}{\sqrt{4\alpha^3\zeta - 6\alpha^2\zeta^2 + 4\alpha\zeta^3 - \zeta^4}} = \frac{\sqrt{\epsilon}}{\alpha^{3/2}} + \mathcal{O}(\epsilon^2)$$

We thus see that to leading order, the integral is proportional to $\sqrt{\epsilon}$, which is why a naive therefore,

$$x^2 = \alpha(\alpha - y) \tag{3.1}$$

It is instructive to compare this trajectory to a circle of radius α. In the neighbourhood of $(0, \alpha)$ the circle $x^2 + y^2 = \alpha^2$ reduces to

$$x^2 = 2\alpha(\alpha - y)$$

It follows that the orbit we have constructed is tangent to this circle at $(0, \alpha)$, but lies entirely within the circle. By symmetry, this behaviour is the same for all the other three extreme points $(-\alpha, 0)$, $(0, -\alpha)$ and $(\alpha, 0)$. [3]

4. (a) We have from the virial theorem

$$2\overline{T} = \alpha\overline{V}$$

where $\alpha = -1$ in this case. Therefore, we get

$$2\overline{T} = |\overline{V}|$$

This gives $\overline{T} = |\overline{E}|$. Consequently, $|\overline{V}| = 2|E|$. The abrupt change in the mass of the star doesn't cause any change in the kinetic energy of the planet. Thus, if $M \to \lambda M$, we must have $T' = T$, but $\overline{V'} = -2\lambda\overline{T}$, because of the virial theorem. Thus, the total energy after the change is $\overline{E'} = (1 - 2\lambda)\overline{T}$. For the cases listed in the problem, we have thus

$$\begin{array}{ll} \lambda = 2 & E' < 0 \text{ elliptic,} \\ \lambda = \frac{1}{2} & E' = 0 \text{ parabolic,} \\ \lambda = \frac{1}{4} & E' > 0 \text{ hyperbolic} \end{array}$$

(b) We suppose that the central potential, before the sun changes mass, was given by $-\frac{\alpha}{r}$. Equating the centripetal and gravitational forces for the

Taylor expansion couldn't work. Finally, we have thus to leading order in x and $\epsilon = \alpha - y$

$$t = \sqrt{\frac{m}{2a}}F(x) = \sqrt{\frac{m}{2a}}\frac{x}{\alpha^2} + \mathcal{O}(x^2)$$

$$\text{and } t + \sqrt{\frac{m}{2a}}F(\alpha) = \sqrt{\frac{m}{2a}}F(y) = \sqrt{\frac{m}{2a}}F(\alpha) + \sqrt{\frac{m}{2a}}\sqrt{\frac{\alpha - y}{\alpha^3}} + \mathcal{O}(\epsilon^2)$$

[3] It is interesting to note and easy to check that by repeating the same procedure for a potential $a(x^2 + y^2)$, and using the same conditions as above, one indeed gets the equation for the trajectory of a circle which was found above.

original system, we get $\frac{mv^2}{R} = \frac{\alpha}{R^2}$, which gives $mv^2 = \frac{\alpha}{R}$; this can also be seen clearly from the virial theorem. It follows that the original total energy is $E = -\frac{\alpha}{2R}$ and the original angular momentum of the system is $M = mvR = \sqrt{\alpha m R}$. After the abrupt change of mass we have

$$\begin{array}{ll} V(r) & = -\frac{\alpha'}{r} = \frac{\alpha}{2r} \\ E' & = (2\lambda - 1)E = (1-2\lambda)\frac{\alpha}{2R} = \frac{1-2\lambda}{\lambda}\frac{\alpha'}{2R} \\ M' & = M = \sqrt{\alpha m R} = \frac{\alpha' m R}{\lambda} \end{array}$$

The equation of the trajectory is

$$p/r = 1 + e\cos(\phi + c)$$

where

$$p = M^2/m\alpha',\ e = \sqrt{1 + (2E'M^2/m\alpha'^2)}$$

and c is a constant set by initial conditions (eqs. 15.4 and 15.5, pg. 36 of Landau-Lifshitz.)
In using these formulae one must be careful to use the new constant $\alpha' = \lambda\alpha$. We find $e = 1/2$, $e = 1$ and $e = 3$ and $p = R/2$, $p = 2R$ and $p = 4R$ for the three cases (i), (ii) and (iii) respectively.
Let us assume that our planet was at $\phi = 0$ when the sun changed mass. It follows that the subsequent orbit of the planet is given by

$$\lambda = 2 \quad \frac{1}{r} = \frac{2}{R}(1 + \frac{1}{2}\cos(\phi + \pi)) = \frac{2}{R}(1 - \frac{1}{2}\cos\phi)$$

$$\lambda = \frac{1}{2} \quad \frac{1}{r} = \frac{1}{2R}(1 + \cos\phi) \text{ and}$$

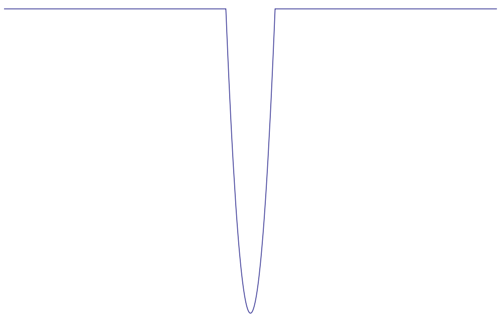
$$\lambda = \frac{1}{4} \quad \frac{1}{r} = \frac{1}{4R}(1 + 3\cos\phi)$$

where we have determined the constant c, in every case, by the requirement that the radius is r when $\phi = 0$ (our initial condition). Note that the initial position of the planet ($\phi = 0$) is the point of furthest approach to the sun in the first case, while is is the point of nearest reach in the other two cases, in accordance with the physical intuition (the planet should fall in if the mass of the sun is increased, but fall out if it is decreased).

5. The qualitative motion of a particle in the potential described in this problem set is clear. In every case the potential vanishes for $x > |a|$. If

$k > 0$, the potential for $|x| < |a|$ is negative (and a piece of a harmonic oscillator potential). A particle incident on this potential will go straight through. As the particle moves faster in the potential than it would have in the absence of a potential, at late times its position as a function of time will be given by $x = vt + \alpha$ for some positive α. On the other hand, if $k < 0$ the potential is positive for $|x| < a$. In this case we have two qualitatively different motions. If $\frac{mv^2}{2} > \frac{|k|a^2}{2}$, the particle sails over the potential barrier. Its late time motion is given by $x = vt - \alpha$ for some positive α. On the other hand, if $\frac{mv^2}{2} < \frac{|k|a^2}{2}$ the particle is reflected by the potential, and its late time motion is given by $x = -vt + \alpha$. We will see below that α turns out to be negative. We now turn to the detailed solution to the problem.

(a) (i) The potential for this case is shown above. We can solve this prob-



lem by using the fact that the motion of the particle, for $|x| < a$, is harmonic. Let us use a convention in which harmonic motion is described by $x(t) = A \sin \phi(t)$ where $\phi(t) = \sqrt{\frac{k}{m}} t + \alpha$ is the phase of the harmonic motion. Let $\sqrt{k}m = \omega$. The phase at $x = -a$ is given by $\tan \phi = \frac{\omega x}{v} = \frac{-\omega a}{v}$. The phase at $x = a$ is given by $\tan \phi = \frac{\omega x}{v} = \frac{\omega a}{v}$. The time the particle takes to traverse the interval $(-a, a)$ is consequently given by

$$t = \frac{2}{\omega} \tan^{-1} \left(\frac{\omega a}{v} \right)$$

Note that this time period reduces to $\frac{2a}{v}$ in the limit $\omega \to 0$, but, in general, is smaller than this quantity (we use since $x < \tan x$ for any x). It follows that the late time motion of the particle is given by

$$x(t) = v \left(t + 2\sqrt{\frac{m}{k}} \tan^{-1} \left(\frac{a}{v} \sqrt{\frac{k}{m}} \right) - \frac{2a}{v} \right)$$

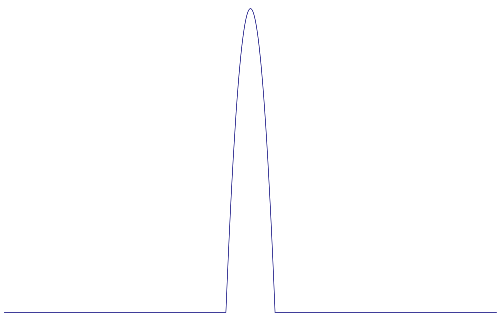
The above result can also be arrived at by using the formula for time traversal through a potential

$$T = \sqrt{\frac{m}{2}} \int_{-a}^{a} \frac{dx}{\sqrt{\left(\frac{mv^2+ka^2}{2} - \frac{kx^2}{2}\right)}}$$

which gives

$$T = 2\sqrt{\frac{m}{k}} \sin^{-1}\left(a\sqrt{\frac{k}{mv^2+ka^2}}\right) = 2\sqrt{\frac{m}{k}} \tan^{-1}\left(\frac{a}{v}\sqrt{\frac{k}{m}}\right)$$

Which is exactly the same as the result obtained from the previous method. (ii) As we have explained above, in this case there are two scenarios: $\frac{mv^2}{2} < \frac{|k|a^2}{2}$ and $\frac{mv^2}{2} > \frac{|k|a^2}{2}$. In the former case the particle hits the potential 'hump' and gets reflected back, and in the latter case the particle just flies past the obstacle. The potential can be schematically drawn as shown above.



For the case $\frac{mv^2}{2} > \frac{|k|a^2}{2}$, the solution for the shift in the time constant can be obtained in a similar manner. Thus we solve the equation

$$T = \sqrt{\frac{m}{2}} \int_{-a}^{a} \frac{dx}{\sqrt{\left(\frac{mv^2-|k|a^2}{2} + \frac{|k|x^2}{2}\right)}}$$

The result is

$$T = 2\sqrt{\frac{m}{|k|}} \sinh^{-1}\left(a\sqrt{\frac{|k|}{mv^2-|k|a^2}}\right) = 2\sqrt{\frac{m}{|k|}} \tanh^{-1}\left(\frac{a}{v}\sqrt{\frac{|k|}{m}}\right)$$

We can clearly see that all the qualitative features of the motion for late times remains the same for this case, except that the functions in this case are hyperbolic. We recover the expected behaviour in the case $|k| \to 0$. The late time trajectory in this case ($k < 0$ and $mv^2 > |k|a^2$) is

$$x(t) = v\left(t + 2\sqrt{\frac{m}{|k|}}\tanh^{-1}\left(\frac{a}{v}\sqrt{\frac{|k|}{m}}\right) - \frac{2a}{v}\right)$$

We now proceed to the remaining case $k < 0$ and $mv^2 < |k|a^2$. In this case, the particle hits the barrier at $x = -a$, climbs a certain distance $x = x_{max}$ and then rolls down. We wish to determine the time taken for this. This is clearly just the time taken for the particle to climb up to the point x_{max} multiplied by two. The equation of motion is

$$T = 2\sqrt{\frac{m}{2}}\int_{-a}^{x_{max}} \frac{dx}{\sqrt{\frac{mv^2}{2} + \frac{|k|}{2}(x^2 - a^2)}}$$

Performing the integral gives immediately

$$T = 2\sqrt{\frac{m}{|k|}}\cosh^{-1}\left(a\sqrt{\frac{|k|}{|k|a^2 - mv^2}}\right) = 2\sqrt{\frac{m}{|k|}}\coth^{-1}\left(\frac{a}{v}\sqrt{\frac{|k|}{m}}\right)$$

We can see that as $|k| \to 0$, we also require $v \to 0$ and we find that v scales as $1/\sqrt{|k|}$. This behaviour is explicit in the above result as one can easily check. The late time trajectory is thus $x(t) = -vt + \alpha$ where we can determine α by requiring that at $t = -\frac{a}{v} + 2\sqrt{\frac{m}{|k|}}\coth^{-1}\left(\frac{a}{v}\sqrt{\frac{|k|}{m}}\right)$ we must have $x = -a$. Thus we find

$$x(t) = -v\left(t + \frac{a}{v} - 2\sqrt{\frac{m}{|k|}}\coth^{-1}\left(\frac{a}{v}\sqrt{\frac{|k|}{m}}\right)\right) - a$$

(b) The critical velocity v_c is clearly $v_c = a\sqrt{|k|/m}$. And the qualitative difference is explained in solution to case (ii) of part (a).
(c) This can be clearly be read off from the solution obtained in case (i) of part (a). The time at which the particle arrives at $x = L$ is clearly $T = (L + 2a)/v - 2\sqrt{\frac{m}{k}}\tan^{-1}\left(\frac{a}{v}\sqrt{\frac{k}{m}}\right)$. The time delay in arriving at $x = L$ is just $\delta t = 2\sqrt{\frac{m}{k}}\tan^{-1}\left(\frac{a}{v}\sqrt{\frac{k}{m}}\right) - \frac{2a}{v}$.