Assignment-4: Classical Mechanics

Solutions

1. As there is no force on the $x - y$ plane, the center of mass of the top moves only on a line of the z-axis. Thus, the top has four degrees of freedom, namely the z position of the center of mass and the three Euler angles. We choose the space fixed axes where the $x - y$ plane is the plane of the table and the the body fixed axes as one along the symmetry axis of the top, and the remaining two are arbitrary. Let θ denote the angle between the body fixed z axis and the symmetry axis of the top. Thus the z position of the center of mass is given by $z_{cm} = l\cos\theta$. The Lagrangian is

$$\begin{aligned} L &= T_{rot} + T_{cm} - mgl\cos\theta \\ &= \frac{1}{2}I_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 + \frac{1}{2}ml^2\dot{\theta}^2\sin^2\theta - mgl\cos\theta \end{aligned}$$

Clearly ϕ and ψ are cyclic co-ordinates; we can use the Routhian to solve for the motion. The momentum conjugate to ϕ and ψ (M_z and M_3 respectively) are constants; the Routhian is

$$R = p_\psi\dot{\psi} + p_\phi\dot{\phi} - L$$

Using the relations $p_\psi = \partial L/\partial\dot{\psi}$ and $p_\phi = \partial L/\partial\dot{\phi}$ and making the relevant substitutions, one finds

$$R = E - (I_1 + ml^2\sin^2\theta)\dot{\theta}^2$$

where E is the total energy, given by

$$E = \frac{p_\psi^2}{2I_3} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{1}{2}\left(I_1 + ml^2\sin^2\theta\right)\dot{\theta}^2 + mgl\cos\theta \tag{1.1}$$

Solving for $\dot{\theta}$ from this, one gets the integral

$$t = \int \frac{d\theta}{F(\theta)} \tag{1.2a}$$

where $F(\theta)$ is given by

$$F(\theta) = \left(\frac{2\left(E - \left(\frac{p_\psi^2}{2I_3} + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + mgl\cos\theta\right)\right)}{I_1 + ml^2 \sin^2\theta} \right)^{1/2} \tag{1.2b}$$

The expression (1.1) shows that the θ motion can be regarded as taking place in one dimension in a field where the "effective potential energy" is

$$V_{eff}(\theta) = \frac{p_\psi^2}{2I_3} + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + mgl\cos\theta$$

A discussion of the qualitative features of this effective potential is given in Landau-Lifshitz, problem 1, pp. $112 - 113$.
Also, to obtain the integrals for ϕ and ψ, one can use the Routhian, which in terms of p_ϕ and p_ψ is

$$R = \frac{p_\psi^2}{2I_3} + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} - \frac{1}{2}\left(I_1 + ml^2 \sin^2\theta\right)\dot{\theta}^2 + mgl\cos\theta$$

Using the Routhian equations (for cyclic co-ordinates) $\dot{q} = \partial R/\partial p$, we get

$$\psi = \int \left(\frac{p_\psi}{I_3} - \cos\theta \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta} \right) dt$$

$$\phi = \int \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta} dt$$

The angles ϕ and ψ are expressed in terms of θ, the solution to which is obtained from (1.2a) and (1.2b).

2. For a planar lamina, we have

$$I_3 = 2I_1 \tag{2.1}$$

Also, from Euler's equations one has for the Eulerian angle ψ

$$\dot{\psi} = M\cos\theta \left(\frac{1}{I_3} - \frac{1}{I_1} \right)$$

For small angles θ as assumed in the problem, one gets, using (2.1)

$$-\dot{\psi} = \frac{M}{2I_1} \tag{2.2}$$

Using the formula for the angular velocity of precession $\Omega_{pr} = M/I_1$ and the relation (2.2), we finally obtain

$$\Omega_{pr} = 2\dot{\psi} = \frac{M}{I_1}$$

Thus we conclude that the angular velocity of the spinning plate about it's axis is approximately half that of the angular velocity of it's precession.

3. (a) Let the z-axis point vertically upwards and the x-axis point north. The first case is just the case of projectile motion in a gravitational field considered in an inertial frame. Using the above co-ordinate system and simple kinematic relations, one obtains the landing point of the ball as $2v^2 \cos\alpha \sin\alpha / g(\cos\beta, \sin\beta, 0)$.
(b) (i) In this case one can use the equations for motion in a non-inertial frame which take into account the effects of Coriolis force due to the rotation of the frame of reference. Ignoring the effects of the centrifugal force, which come from terms of second order in Ω, the angular velocity of the rotation of the earth, we get the equation of motion (eq. 39.9 of Landau-Lifshitz)

$$\dot{\mathbf{v}} = 2\mathbf{v} \times \mathbf{\Omega} + \mathbf{g} \tag{3.1}$$

As in prob. 1, pg. 129 of Landau-Lifshitz, we solve the above equation by successive approximation. In our co-ordinate system, the components of the vectors are $g_x = g_y = 0$, $g_z = -g$; $\Omega_x = \Omega\cos\theta$, $\Omega_y = 0$, $\Omega_z = \Omega\sin\theta$. The solution to (3.1) using successive approximation is (see eq. (2), Landau-Lifshitz)

$$\mathbf{r} = \mathbf{h} + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + \frac{1}{3}t^3\mathbf{g} \times \mathbf{\Omega} + t^2\mathbf{v}_0 \times \mathbf{\Omega} + \mathcal{O}(\mathbf{\Omega}^2)$$

where $\mathbf{h}$ is the initial radius vector of the particle and $\mathbf{v}_0$ is the initial velocity vector. In our case, $\mathbf{h} = \mathbf{0}$ and $\mathbf{v}_0 = (v\sin\alpha\cos\beta, v\sin\alpha\sin\beta, v\cos\alpha)$. Solving for the time of flight (by setting the z co-ordinate to zero), one obtains

$$t_f = \frac{v\cos\alpha}{v\Omega\sin\alpha\sin\beta\cos\theta + \frac{g}{2}}$$

Expanding the above solution to $\mathcal{O}(\Omega)$ (this is the consistent order at which we are working), one finds the time of flight as

$$t_f = \frac{2v\cos\alpha}{g}\left(1 - \frac{2v\Omega}{g}\sin\alpha\sin\beta\cos\theta + \mathcal{O}(\Omega^2)\right)$$

Plugging in the above time of flight, one easily finds the following $\mathcal{O}(\Omega)$ corrections to the co-ordinates of the projectile as

$$\begin{aligned}
\delta x &= \frac{4v^3\Omega}{g^2}\left(\cos\alpha\sin\theta - \sin\alpha\cos\beta\cos\theta\right)\cos\alpha\sin\alpha\sin\beta \\
\delta y &= -\frac{4v^3\Omega}{g^2}\left(\sin^2\alpha\sin^2\beta\cos\alpha\cos\theta + \cos^2\alpha(\sin\alpha\cos\beta\sin\theta - \frac{1}{3}\cos\alpha\cos\theta)\right)
\end{aligned}$$

(ii) To obtain the solution in this case, let us choose a coordinate system which has its origin as the center of the earth and work in spherical polar co-ordinates. This system remains fixed (with respect to the “fixed stars“ as the earth rotates). In this system, we have to compute the initial velocity of the projectile through the velocity addition formula. Thus the $(\hat{r}, \hat{\theta}, \hat{\phi})$ components of the initial velocity vector are

$$\begin{aligned}
v_r &= v\cos\alpha \\
v_\theta &= -v\sin\alpha\cos\beta \\
v_\phi &= -v\sin\alpha\sin\beta + R\Omega\cos\theta
\end{aligned}$$

where R is the radius of the earth. We use the equations of motion in spherical polar co-ordinates to determine the time of flight and compute the changes to the co-ordinates of the projectile to first order in Ω. This can be done as follows. The components of the angular velocity of the projectile are clearly

$$\begin{aligned}
\dot{r} &= v\cos\alpha \\
\dot{\theta} &= -\frac{v\sin\alpha\cos\beta}{R} \\
\dot{\phi} &= -\frac{v\sin\alpha\sin\beta}{R\cos\theta} + \Omega
\end{aligned}$$

The action of the projectile in these co-ordinates is

$$S = \int\left(\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\dot{\phi}^2\right) - mgr\right)dt$$

Note here that θ is the latitude and is related to the conventional polar angle δ by $\theta = \frac{\pi}{2} - \delta$. The equations of motion are

$$\begin{aligned}
&\ddot{r} - r\left(\dot{\theta}^2 + \cos^2\theta\dot{\phi}^2\right) = -g \\
&r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\cos\theta\sin\theta = 0 \\
&r\ddot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\cos\theta + 2r\dot{\theta}\dot{\phi}\sin\theta = 0
\end{aligned}$$

One has to solve the equations of motion for all the three co-ordinates simultaneously. We will be working, however, in the approximation where we take the radius of the earth to be infinite; we consider the earth to be essentially flat. More precisely, we work to leading order in $1/R$ and drop all terms that are $\mathcal{O}(1/R)$ (or higher). With this understanding, we can approximate the velocities $\dot{\theta}$ and $\dot{\phi}$ to be almost constant over the time of flight. This is because the change in $\dot{\theta}$ during the time of flight, which is $\ddot{\theta}(2v\cos\alpha)/g$ is $\mathcal{O}(1/R)$ as can be seen from the above equations and hence can be neglected. This gives

$$\ddot{r} - \frac{v^2\sin^2\alpha}{R} + 2v\Omega\sin\alpha\sin\beta\cos\theta = -g$$

where we have neglected terms of $\mathcal{O}(\Omega^2)$ coming from expanding $\dot{\phi}^2$. Solving this and imposing the initial conditions appropriately, and dropping the $1/R$ term in accordance with the approximation, one finds that

$$r(t) = -v\Omega\sin\alpha\sin\beta\cos\theta t^2 - \frac{g}{2}t^2 + v\cos\alpha t + R$$

Setting $r(t_f) = R$ for computing the time of flight t_f, one finds that the time of flight is

$$t_f = \frac{v\cos\alpha}{v\Omega\sin\alpha\sin\beta\cos\theta + \frac{g}{2}}$$

Expansion to $\mathcal{O}(\Omega)$ gives

$$t_f = \frac{2v\cos\alpha}{g}\left(1 - \frac{2v\Omega}{g}\sin\alpha\sin\beta\cos\theta + \mathcal{O}(\Omega^2)\right)$$

which is in agreement with the t_f computed in part (i) above.
Next, we solve the equation of motion for the θ co-ordinate, which is

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\cos\theta\sin\theta = 0$$

solving the above equation and making the same approximation as above and remembering that $\delta x = -R\delta\theta$, one obtains for the change in the x co-ordinate

$$\delta x = \frac{4v^3\Omega}{g^2}\left(\cos\alpha\sin\theta - \sin\alpha\cos\beta\cos\theta\right)\cos\alpha\sin\alpha\sin\beta$$

which is in agreement with our previous result.
It now remains to solve the equation of motion for the ϕ co-ordinate, which reads

$$r\ddot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\cos\theta + 2r\dot{\theta}\dot{\phi}\sin\theta = 0$$

Solving this equation and making the approximations mentioned and remembering that $\delta y = -R\cos\theta\delta\phi$, one finally obtains

$$\begin{aligned}\delta y = &-\frac{4v^3\Omega}{g^2}\left(\sin^2\alpha\sin^2\beta\cos\alpha\cos\theta + \cos^2\alpha(\sin\alpha\cos\beta\sin\theta - \frac{1}{3}\cos\alpha\cos\theta)\right)\\ &-\frac{2R\Omega v}{g}\cos\theta\cos\alpha\end{aligned}$$

the last term in the above expression gives the amount by which the earth has rotated during the time the projectile was in flight; to get the change in the co-ordinate with respect to the earth, we must subtract out this contribution. Thus one finally obtains

$$\delta y = -\frac{4v^3\Omega}{g^2}\left(\sin^2\alpha\sin^2\beta\cos\alpha\cos\theta + \cos^2\alpha(\sin\alpha\cos\beta\sin\theta - \frac{1}{3}\cos\alpha\cos\theta)\right)$$

This is clearly in agreement with what was found in (i) above.

4. This is a similar problem to the one in Landau-Lifshitz, pg. 121, problem 2. The Euler equations are

$$d\Omega_1/dt + (I_2 - I_3)\Omega_3\Omega_2/I_1 = 0 \tag{4.1a}$$
$$d\Omega_2/dt + (I_3 - I_1)\Omega_1\Omega_3/I_2 = 0 \tag{4.1b}$$
$$d\Omega_3/dt + (I_1 - I_2)\Omega_2\Omega_1/I_3 = 0 \tag{4.1c}$$

As explained in the problem in the text, this case corresponds to the movement of the tip of $\mathbf{M}$ along a curve through the x_2-axis. In this case, we get the following expressions for Ω_1 and Ω_3 in terms of Ω_2

$$\Omega_1^2 = [(M^2 - 2EI_3) - I_2(I_2 - I_3)\Omega_2^2]/I_1(I_1 - I_3)$$
$$\Omega_3^2 = [(2EI_1 - M^2) - I_2(I_1 - I_2)\Omega_2^2]/I_3(I_1 - I_3)$$

Substituting these relations into the Euler equation for Ω_2 in (4.1b), one obtains

$$\begin{aligned} d\Omega_2/dt &= (I_3 - I_1)\Omega_3\Omega_1/I_2 \\ &= (2E - I_2\Omega_2^2)\sqrt{(I_1 - I_2)(I_2 - I_3)}/I_2\sqrt{I_1 I_3} \end{aligned} \tag{4.2}$$

Defining $\tau = t\sqrt{(I_1 - I_2)(I_2 - I_3)/I_1 I_3}\Omega_0$ and $s = \Omega_2\sqrt{I_2/2E} = \Omega_2/\Omega_0$, where $\Omega_0 = M/I_2 = 2E/M$, one obtains for (4.2) in terms of τ and s

$$\frac{ds}{d\tau} = 1 - s^2$$

solving the above equation, one obtains

$$\Omega_1 = \Omega_0\sqrt{I_2(I_2 - I_3)/I_1(I_1 - I_3)}\mathrm{sech}\tau \tag{4.3a}$$
$$\Omega_2 = \Omega_0\mathrm{tanh}\tau \tag{4.3b}$$
$$\Omega_3 = \Omega_0\sqrt{I_2(I_1 - I_2)/I_3(I_1 - I_3)}\mathrm{sech}\tau \tag{4.3c}$$

To obtain the time dependence of the Eulerian angles, we use the relations between the components of the vector $\mathbf{\Omega}$ and the Eulerian angles. Proceeding as given in Landau-Lifshitz and making the appropriate changes in the suffixes of the relations, one obtains $\cos\theta = \mathrm{tanh}\tau$, $\phi = \Omega_0 t + \mathrm{const}$, $\tan\psi = \sqrt{I_1(I_2 - I_3)/I_3(I_1 - I_2)}$.

5. The effective potential, from eq.(6) on page 112 of Landau-Lifshitz for the case $M_3 = M_z (= M)$ is

$$U_{eff} = \frac{M^2(1 - \cos\theta)^2}{2I_1' \sin^2\theta} - \mu g l(1 - \cos\theta)$$

On the analysis of the minima of this effective potential, one finds the following expression for the first and second derivatives respectively

$$U'_{eff}(\theta) = \frac{2\alpha(1 - \cos\theta)^2}{\sin^3\theta} - \beta\sin\theta$$
$$U''_{eff}(\theta) = \frac{2\alpha(2 - \cos\theta)(1 - \cos\theta)^2}{\sin^4\theta} - \beta\cos\theta$$

where $\alpha = M^2/2I'_1$ and $\beta = \mu g l$. Setting the first derivative to zero gives the following condition

$$2\alpha(1-\cos\theta)^2 = \beta\sin^4\theta \tag{5.1}$$

It is clear that $\theta = 0$ solves the above condition. But for $\theta = 0$ to be a true minima of the potential, we require the second derivative to be positive around that point. For this, we expand $U''(\theta)$ about $\theta = 0$ and require positivity, which can easily be seen to lead to the following condition

$$\frac{M^2}{I'_1} > 4\mu g l \tag{A}$$

It can be seen by plotting the curve or by other means that when this condition is met, $\theta = 0$ is the only stable minimum. However, there could be other non-zero solutions to the minimization condition (5.1). If we want to look for a solution with a non-zero solution, one can easily simplify the condition to get the following

$$\cos\theta = -1 + \sqrt{\frac{2\alpha}{\beta}} \tag{5.2}$$

Plugging in this condition into the second derivative and requiring that it be positive gives the following condition

$$\frac{M^2}{I'_1} < 4\mu g l \tag{B}$$

One can easily check that the above condition also ensures that the condition (5.2) is valid, i.e., $\cos\theta$ has a solution. Thus, we can see that if one requires any other minimum other than zero, then the condition for stability reverses, hence leaving only one true minimum to exist in any given situation. More precisely, if there is a non-zero root of the minimization condition, then $\theta = 0$ is no longer a minimum and vice versa. We can now proceed to analyze the oscillations about the minima in the two (mutually exclusive) scenarios.
Regarding the motion in θ as one dimensional motion in an effective potential, one obtains for small oscillations about $\theta = 0$ the frequency

$$\omega = \sqrt{\frac{M^2}{4I'^2_1} - \frac{\mu g l}{I'_1}}$$

If the stable minima is not at zero, then denoting the stable minima point by $\theta_0 = \cos^{-1}\left(-1 + \sqrt{2\alpha/\beta}\right)$, we can expand the potential about this new point and one then obtains for the frequency

$$\omega = \sqrt{\frac{4\mu g l}{I_1'}\left(1 - \sqrt{\frac{M^2}{4I_1'\mu g l}}\right)}$$

We present two plots that qualitatively illustrate the difference between the two cases (A) and (B) for sample values of the ratio α/β.

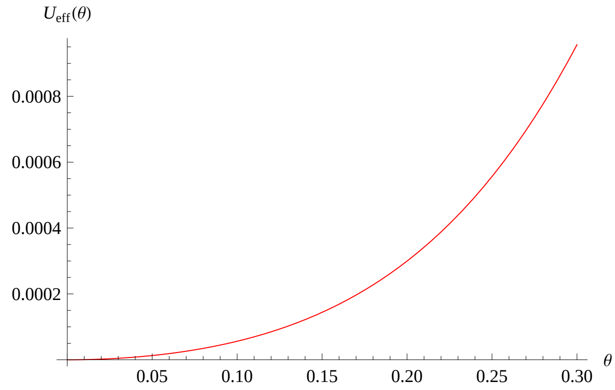

Figure 1: Case (A): $\theta = 0$ is the only stable minimum when $\alpha/\beta > 2$ (plotted here for $\alpha : \beta :: 1 : 0.49$)

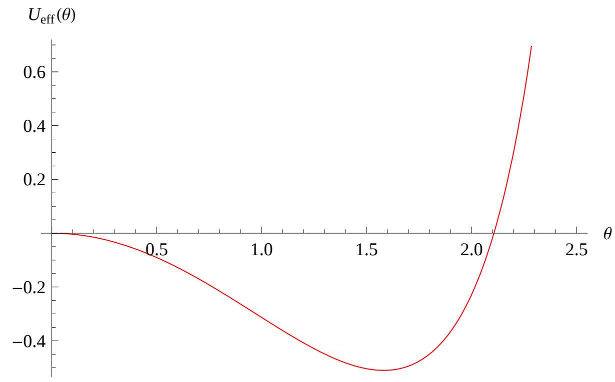

Figure 2: Case (B): $U_{eff}(\theta)$ shows a stable minimum away from $\theta = 0$ when $\alpha/\beta < 2$ (plotted here for $\alpha : \beta :: 0.49 : 1$)

To determine the angular motion of the top, we require to solve the equation for θ which is

$$t = \int \frac{d\theta}{\sqrt{2(E' - U_{eff}(\theta))/I_1'}}$$

for small θ. Expanding $U_{eff}(\theta)$ about $\theta = 0$ (up to $\mathcal{O}(\theta^2)$) and substituting in the above integral, one obtains the following

$$t = \sqrt{\frac{I_1'}{2E'}} \int \frac{d\theta}{\sqrt{1 - \frac{(\alpha-2\beta)\theta^2}{8E'}}}$$

Performing the elementary integral, one obtains as solution for $\theta(t)$

$$\theta(t) = \sqrt{\frac{8E'I_1'}{M^2 - 4\mu g l I_1'}} \sin\left(\sqrt{\frac{M^2}{4I_1'^2} - \frac{\mu g l}{I_1'}}\right) t$$

To obtain the angular motion, one uses the equations for $\dot{\phi}$ and $\dot{\psi}$,

$$\dot{\phi} = M(1-\cos\theta)/I_1' \sin^2\theta \tag{5.3a}$$

$$\dot{\psi} = \frac{M}{I_3} - \cos\theta \frac{M(1-\cos\theta)}{I_1' \sin^2\theta} \tag{5.3b}$$

For small θ (we must expand the above equations near $\theta = 0$), we get

$$\dot{\phi} = M/2I_1'$$
$$\dot{\psi} = M\left(\frac{1}{I_3} - \frac{1}{2I_1'}\right)$$

Thus, to lowest order we find that the angular motion is just $\phi = (M/2I_1')t +$ const and $\psi = M\left(\frac{1}{I_3} - \frac{1}{2I_1'}\right)t +$const. The motion in this case is reproduced by Fig. 49a on page 113 of Landau Lifshitz. This is because in this case $\dot{\phi}$ never changes sign.

Next, we determine the oscillations about the minimum θ_0 defined above, in which case this is the only stable equilibrium. Expanding $U_{eff}(\theta)$ about this new point θ_0 and redefining the zero of the potential, i.e., taking $U_{eff}(\theta_0) = 0$, and defining

$$\begin{aligned} E'' &= E' - U_{eff}(\theta_0) \\ &= E' - \left(-\frac{M^2}{2I_1'} - 2\mu g l + \sqrt{\frac{4M^2\mu g l}{I_1'}}\right) \end{aligned}$$

one obtains for the oscillations

$$\theta(t) = \theta_0 + \sqrt{\frac{E''}{2\mu gl\left(1 - \sqrt{\frac{M^2}{4I_1'\mu gl}}\right)}} \sin\left(\sqrt{\frac{4\mu gl}{I_1'}\left(1 - \sqrt{\frac{M^2}{4I_1'\mu gl}}\right)}\right) t$$

It now remains to obtain the angular motion of the top from equations (5.3a) and (5.3b). Expanding in a Taylor series about the point θ_0 to $\mathcal{O}(\theta - \theta_0)$, one finds the following expressions for $\dot{\phi}$ and $\dot{\psi}$

$$\dot{\phi} = \frac{M(1-\cos\theta_0)}{I_1'\sin^2\theta_0} + \frac{M(1-\cos\theta_0)^2}{I_1'\sin^3\theta_0}(\theta - \theta_0) + \mathcal{O}((\theta-\theta_0)^2)$$

$$\dot{\psi} = \frac{M}{I_3} - \frac{M\cos\theta_0(1-\cos\theta_0)}{I_1'\sin^2\theta_0} + \frac{M(1-\cos\theta_0)^2}{I_1'\sin^3\theta_0}(\theta - \theta_0) + \mathcal{O}((\theta-\theta_0)^2)$$

Plugging in the value of $\cos\theta_0 = -1 + \sqrt{2\alpha/\beta}$ and simplifying, one finds the following solution for ϕ

$$\phi(t) = \sqrt{\frac{\mu gl}{I_1'}}t - \left(\frac{E'' I_1'/2M^2}{\sqrt{\frac{4I_1'\mu gl}{M^2}} - 1}\right)^{1/2} \cos\left(\sqrt{\frac{4\mu gl}{I_1'}\left(1 - \sqrt{\frac{M^2}{4I_1'\mu gl}}\right)}\right) t$$

On similar lines, one can easily obtain the solution for $\psi(t)$. The above solution for $\phi(t)$ shows that the motion is a constant precession superposed on oscillation. As long as the amplitude of oscillations are small compared to the constant term, $\dot{\phi}$ doesn't change sign and the motion is still described by Fig. 49a of Landau-Lifshitz. One can obtain an estimate of when the amplitude of oscillation can exceed the constant precession term, hence leading to behaviour as in Fig. 49b of Landau-Lifshitz. Requiring that the (maximum) amplitude of oscillation exceed the constant precession (so that $\dot{\phi}$ can turn negative) gives us $E'' > \sqrt{M^2\mu gl/I_1'}$. This estimate cannot be trusted completely and more analysis may be required as in this regime the small oscillation approximation under which the above results were derived break down.