Lecture 2 11/8/09

We start by reviewing some facts about Newtonian gravity as well as special relativity, since we are trying to reconcile the two.
Although it is not needed, comparison of Newton's gravity to electrostatics will be very helpful in attaining our goal.

Newtonian gravity is defined by a scalar "gravitational potential" $\varphi_G(x)$. The force due to such a potential is ~~$m\nabla\varphi_G$~~

$$F_i = -m\,\partial_i\varphi_G$$

Therefore the equation $F_i = m a_i$ becomes

$$m\ddot{x}_i = -m\,\partial_i\varphi_G \quad \text{or} \quad \ddot{x}_i = -\partial_i\varphi_G .$$

How is φ_G obtained? It arises from a matter distribution:

$$\partial_i\partial_i\varphi_G = 4\pi G_N \rho_G$$

$6.67 \times 10^{-11}\ m^3/kg\,s^2$
G_N = Newton's constant

where ρ_G is the matter density (mass per unit volume). As an example, suppose

$$\rho_G = M\,\delta^3(\vec{x})$$

describing a point mass M at the origin.

Then $\varphi_G = \cancel{G_N M \frac{x^i}{r^3}} \quad \frac{G_N M}{r}$

where $r = |\vec{x}|$.

Hence ~~E~~ $\ddot{x}^i = G_N M \frac{x^i}{r^3}$ as expected.

Note that $\partial_i \partial_i \varphi_G = 4\pi G_N \rho$ is not relativistically invariant. That is precisely the problem we wish to address.

First, let us compare this with the equations for electrostatics. Here we have a scalar potential φ_E and

$$m \ddot{x}^i = - q \, \partial_i \varphi_E$$

where q is the electric charge of the particle.

Also, φ_E is determined from a charge distribution by:

$$\partial_i \partial_i \varphi_E = 4\pi G_E \rho_E$$

where G_E is some constant and ρ_E is the electric charge density. * For $\rho_E = Q \, \delta^3(\vec{x})$ we get

$$\varphi_E = \frac{G_E Q}{r}$$

from which we can identify G_E with $\frac{1}{4\pi\epsilon_0}$.

For our brief review of special relativity we will only recall the concept of Lorentz invariance. This says that the laws of physics are unchanged by a linear transformation of space and time : $x^\mu\ (t, \vec{x}) \to$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

where $\Lambda^\mu{}_\nu$ is a 4×4 matrix satisfying

$$\Lambda^T \eta \Lambda = \eta$$

with $\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Let us understand this bare-bones statement in more detail. First note that along the trajectory of a light ray we have

$$-c^2 dt^2 + dx^2 + dy^2 + dz^2 = 0$$

Choose units where $c = 1$. This means we measure time in length units.

Now the above equation is the same as

$$dx^\mu \cdot \eta_{\mu\nu} dx^\nu = 0$$

Under $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, we have

$$dx^\mu \eta_{\mu\nu} dx^\nu = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\, dx^\alpha dx^\beta \eta_{\mu\nu} = (\Lambda^T \eta \Lambda)_{\alpha\beta}\, dx^\alpha dx^\beta$$

Then with $\Lambda^T \eta \Lambda = \eta$, this becomes

$$\eta_{\alpha\beta}\, dx^\alpha dx^\beta$$ as expected.

Another simple physical consequence of ~~these~~ the abstract definition is as follows. Suppose Λ is of the form

$$\left(\begin{array}{cc|cc} a & b & \multicolumn{2}{c}{0} \\ c & d & & \\ \hline \multicolumn{2}{c|}{0} & 1 & 0 \\ & & 0 & 1 \end{array}\right)$$

Then $$\Lambda^T \eta \Lambda = \left(\begin{array}{cc|c} a & c & 0 \\ b & d & \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right) \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \left(\begin{array}{cc|c} a & b & 0 \\ c & d & \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} a & c & \\ b & d & \\ \hline & & 1 \end{array}\right) \left(\begin{array}{cc|c} -a & -b & \\ c & d & \\ \hline & & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -a^2+c^2 & -ab+cd & \\ -ab+cd & -b^2+d^2 & \\ \hline & & 1 \end{array}\right)$$

Thus $ab = cd$, $\quad -a^2+c^2 = -1$

$-b^2+d^2 = +1$

So $a = d = \operatorname{Cosh} \eta$, $\quad c = \operatorname{Sinh} \eta = b$

$\cancel{d = \operatorname{Cosh} \chi, \quad b = \operatorname{Sinh} \chi}$

Thus $\Lambda = \left(\begin{array}{cc|c} \cosh\eta & \sinh\eta & \\ \sinh\eta & \cosh\eta & \\ \hline & & 1 \end{array}\right)$

Parametrise $\cosh\eta = \frac{1}{\sqrt{1-v^2}}$

$\sinh\eta = \frac{-v}{\sqrt{1-v^2}}$

then $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ is

$$t \to \frac{t - vx}{\sqrt{1-v^2}}, \quad x \to \frac{x - vt}{\sqrt{1-v^2}}$$ as is well known

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Now we can address the incompatibility of electrostatic Coulomb's law and Newton's law with special relativity. We start with ~~Coulomb's law~~ electrostatics

$$\partial_i \partial_i \varphi_E = 4\pi G_E \rho_E(x)$$

Is this equation relativistically invariant? The operator $\vec{\nabla}^2 = \partial_i \partial_i$ (Laplacian) is not. However the generalisation

$$\Box = \partial_\mu \partial^\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu$$

is invariant.

Next, notice that $\rho_E(x)$ is a charge density, ie charge / unit volume. Now charge is unaffected by Lorentz transformations, but spatial volume does transform.

Indeed ρ_E transforms like the time component of a 4-vector:

$$\rho_E = \frac{q}{\text{vol}} = \frac{q \times \text{time}}{4\text{-vol}}$$

So we must make the replacement

$$\rho_E \longrightarrow J_\mu$$

where $J_\mu = (\rho_E, 0, 0, 0)$ is a special case.

~~Then~~ For the LHS to make sense we correspondingly have

$$\varphi_E \rightarrow A_\mu$$

where $A_\mu = (\varphi_E, 0, 0, 0)$ is a special case.

Thus: $\Box A_\mu = 4\pi G_E J_\mu$ is our equation and it is indeed relativistically covariant.

If J_μ is a conserved current then $\partial^\mu J_\mu = 0$ (eqn of continuity: $\dot{\rho} + \vec{\nabla} \cdot \vec{J} = 0$). Hence

$$\Box(\partial^\mu A_\mu) = 0 \longrightarrow \partial^\mu A_\mu = 0$$

Now we can replace $\Box A_\mu$ by $\partial^\nu(\partial_\nu A_\mu - \partial_\mu A_\nu)$ $= \partial^\nu F_{\nu\mu}$.

Hence finally $\boxed{\partial^\nu F_{\nu\mu} = 4\pi G_E J_\mu}$