Lecture 5 25/8/09

We have discussed the geodesic equation:

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$

for a particle moving in curved <u>space</u>. Now we would like to discuss its <u>spacetime</u> version. We may guess it to be,

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda = 0$$

Now, however, we need to be more precise about its meaning. Since $x^\mu = (t, x, y, z)$, x^0 is the <u>time</u>. So what does $\dot{x}^\mu$ mean?

In fact $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ where τ is an arbitrary parameter along the world-line. This can make sense only if the equation is invariant under arbitrary nonsingular changes: $\tau \to \tau'(\tau)$, for which $x'^\mu(\tau') = x^\mu(\tau)$.

First we work in Minkowski spacetime: $g_{\mu\nu}(x) = \eta_{\mu\nu}$. To find the action from which ~~the~~ equation $\ddot{x}^\mu = 0$ follows, let us note the following. The naive action

$$S = \int L d\tau \sim K \int d\tau\, \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \qquad (K = \text{const})$$

is incorrect. It is <u>not</u> invariant under $\tau \to \tau'(\tau)$, $x'^\mu(\tau') = x^\mu(\tau)$. Moreover,

$$p_\mu = \frac{dL}{d\dot{x}^\mu} = K \dot{x}_\mu \Rightarrow p^\mu p_\mu = K^2 \dot{x}^\mu \dot{x}_\mu$$

while we expect $p^\mu p_\mu = m^2$ (Fixed!)

Both problems are solved if we take the action

$$S = \int L\, d\tau = K \int d\tau \sqrt{-\eta^{\mu\nu} \dot{x}_\mu \dot{x}_\nu}$$

Under $\tau \to \tau'(\tau)$,

$$d\tau' = \frac{d\tau'}{d\tau} d\tau$$

$$\frac{dx'^\mu}{d\tau'} = \frac{d\tau}{d\tau'} \frac{dx^\mu}{d\tau}$$

so S is indeed invariant. Moreover,

$$p_\mu = \frac{dL}{d\dot{x}^\mu} = \frac{-K \dot{x}_\mu}{\sqrt{-\eta_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}}$$

So $$p^\mu p_\mu = K^2 \frac{\dot{x}^\mu \dot{x}_\mu}{\left(\sqrt{-\dot{x}^\alpha \dot{x}_\alpha}\right)^2} = -K^2$$

We expect $p^\mu p_\mu = -m^2$, so $K = \pm m$. ~~tre~~

We can even fix the sign. Using the reparametrization invariance of τ, we can choose:

$$x^0 = \tau$$

and we have:

$$S = K \int d\tau \sqrt{-\eta^{00} \cancel{*} - \eta^{ij} \dot{x}^i \dot{x}^j}$$

$$= K \int d\tau \sqrt{1 - \dot{x}^i \dot{x}^i} \sim -\frac{1}{2} K \dot{x}^i \dot{x}^i + \text{const.}$$

for low velocities $\dot{x}^i$. Thus $K = -m$.

So now let's look at the action

$$S = \int L\, d\tau = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu}$$

Its equation of motion is

$$\frac{d}{d\tau}\frac{dL}{d\dot{x}^\mu} = 0 \Rightarrow \frac{\ddot{x}^\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}} = 0 \Rightarrow \ddot{x}^\mu = 0$$

$$(\text{if } \dot{x}^\mu \dot{x}_\mu \neq 0).$$

Moreover $\quad p_\mu \equiv \dfrac{dL}{d\dot{x}^\mu} = \dfrac{m\dot{x}^\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}$

In the "physical" gauge $x^0 = \tau$ we have

$$\sqrt{-\dot{x}^\nu \dot{x}_\nu} = \sqrt{1 - \dot{x}^i \dot{x}^i} = \sqrt{1-\vec{v}^{\,2}}$$

So $\quad E = \dfrac{m}{\sqrt{1-\vec{v}^{\,2}}}\ , \quad \vec{p} = \dfrac{m\vec{v}}{\sqrt{1-\vec{v}^{\,2}}}\ $: familiar.

Note also that over a small interval $d\tau$, the length ds of the trajectory is:

$$ds = \sqrt{-\eta^{\mu\nu} dx_\mu dx_\nu}$$

so in fact $\quad S = -m\int d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu}$

$$= -m \int \sqrt{-dx^\mu dx_\mu}$$

and it's proportional to a geometric quantity: the invariant length of the trajectory.

We can imagine $ds^2 > 0$ (spacelike)

$ds^2 = 0$ (lightlike or null)

$ds^2 < 0$ (timelike)

Since $ds^2 = -dt^2 + d\vec{x}^2$

$$= dt^2\left(-1 + \left(\frac{d\vec{x}}{dt}\right)^2\right) = dt^2(-1+\vec{v}^2)$$

we have (in $c=1$ units)

spacelike: $|\vec{v}| > 1$ (faster than light)

lightlike: $|\vec{v}| = 1$

timelike: $|\vec{v}| < 1$

Physical, massive particles always follow a timelike trajectory.

With all these results, it is easy to guess the generalisation to a curved spacetime with metric $g_{\mu\nu}(x)$.

We have: $ds^2 = -g_{\mu\nu}\,dx^\mu dx^\nu$

so $S = \int d\tau\, L = \int ds$

$$= \int d\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

It is instructive to consider the invariances of this action:

(i) $\tau \to \tau'(\tau)$, $x'^{\mu}(\tau') = x^{\mu}(\tau)$
$g'_{\mu\nu}(x'(\tau)) = g_{\mu\nu}(x(\tau))$ } Re-parametrisation of τ

(ii) $x^{\mu} \to x'^{\mu}(x)$
$g_{\mu\nu} \to g'_{\mu\nu}(x') = \dfrac{\partial x^{\alpha}}{\partial x'^{\mu}} \dfrac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x)$ } spacetime reparametrisation (coordinate transformation)

What is the equation of motion for the action:

$$S = \int d\tau\, L = -m \int d\tau \sqrt{-g_{\mu\nu}(x)\,\dot{x}^{\mu}\dot{x}^{\nu}} \qquad ?$$

We have:

$$\frac{\partial L}{\partial \dot{x}^{\mu}} = m \frac{g_{\mu\nu}\dot{x}^{\nu}}{\sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}}$$

So
$$\frac{d}{d\tau}\left(\frac{g_{\mu\nu}\dot{x}^{\nu}}{\sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}}\right) = -\frac{\partial}{\partial x^{\mu}}\sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}$$
$$= \frac{g_{\alpha\beta,\mu}\dot{x}^{\alpha}\dot{x}^{\beta}}{2\sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}}$$

$\downarrow$

$$\frac{g_{\mu\nu,\lambda}\dot{x}^{\nu}\dot{x}^{\lambda} + g_{\mu\nu}\ddot{x}^{\nu}}{\sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}} \quad - \frac{\frac{1}{2}g_{\nu\lambda,\mu}\dot{x}^{\nu}\dot{x}^{\lambda}}{\sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}}$$
$$= -g_{\mu\nu}\dot{x}^{\nu}\frac{d}{d\tau}\left(\frac{1}{\sqrt{-g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}}\right)$$

Thus $\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = \dot{x}^\mu \frac{d}{d\tau} \ln \sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}$

This is not quite what we expected! However it is of the form

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = f(x)\,\dot{x}^\mu$$

and we can always re-parametrize a curve to get rid of the RHS.

(Concretely, choose τ' such that

$$\frac{d\tau'}{d\tau} = \sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} \quad)$$

Then we finally get $\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$

Now we are only allowed reparametrizations for which

$$\frac{d^2\tau'}{d\tau^2} = 0$$

(the class of τ's related like this are called "affine parameters").

Examples of geodesics:

i) Space. Consider S^2 with coords θ, φ.

The metric is

$$ds^2 = R^2(d\theta^2 + \cos^2\theta \, d\varphi^2) \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq 2\pi$$

Calculate Γ's:

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta}(g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta})$$

$$= 0.$$

$$\Gamma^\theta_{\theta\varphi} = \frac{1}{2} g^{\theta\theta}(g_{\theta\theta,\varphi} + g_{\theta\varphi,\theta} - g_{\theta\varphi,\theta})$$

$$= 0.$$

$$\Gamma^\theta_{\varphi\varphi} = \frac{1}{2} g^{\theta\theta}(g_{\theta\varphi,\varphi} + g_{\theta\varphi,\varphi} - g_{\varphi\varphi,\theta})$$

$$= -\frac{1}{2} g^{\theta\theta} g_{\varphi\varphi,\theta}$$

$$= -\frac{1}{2} \frac{1}{R^2} \cdot R^2 \cdot 2\sin\theta\cos\theta$$

$$= -\frac{1}{2}\sin 2\theta$$

$$\Gamma^\varphi_{\theta\theta} = \frac{1}{2} g^{\varphi\varphi}(g_{\varphi\theta,\theta} + g_{\varphi\theta,\theta} - g_{\theta\theta,\varphi})$$

$$= 0$$

$$\Gamma^\varphi_{\theta\varphi} = \frac{1}{2} g^{\varphi\varphi}(g_{\varphi\theta,\varphi} + g_{\varphi\varphi,\theta} - g_{\theta\varphi,\varphi})$$

$$= \frac{1}{2} \frac{1}{R^2\cos^2\theta} R^2 . 2\sin\theta\cos\theta = \tan\theta$$

$$\Gamma^{\varphi}_{\varphi\varphi} = \frac{1}{2} g^{\varphi\varphi}\left(g_{\varphi\varphi,\varphi} + g_{\varphi\varphi,\varphi} - g_{\varphi\varphi,\varphi} \right)$$

$$= \frac{1}{2} g^{\varphi\varphi} g_{\varphi\varphi,\varphi} = 0.$$

So:
$$\ddot{\theta} + \Gamma^{\theta}_{\varphi\varphi} \dot{\varphi} \dot{\varphi} = 0$$

$$\ddot{\varphi} + 2\Gamma^{\varphi}_{\theta\varphi} \dot{\theta} \dot{\varphi} = 0$$

$$\Rightarrow \quad \ddot{\theta} - \frac{1}{2} \sin 2\theta \, (\dot{\varphi})^2 = 0$$

$$\ddot{\varphi} + 2 \tan\theta \, \dot{\theta} \dot{\varphi} = 0$$

Look for simple solns:

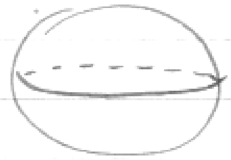
(i) $\theta = \text{const}$. Then $\dot{\theta} = 0$ and:

$$\sin 2\theta \, \dot{\varphi}^2 = 0 = \ddot{\varphi}$$

$$\therefore \quad \varphi = at + b \quad \text{and} \quad \theta = 0$$

(recall $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$)

This is the equator!



(ii) $\varphi = \text{const}$. Now,

$$\ddot{\theta} = 0 \Rightarrow \theta = at + b \quad \text{at any } \varphi!$$

These are great circles of longitude.

(ii) spacetime. Consider

$$ds^2 = R^2\left(-\cosh^2\rho \, dt^2 + d\rho^2\right)$$

This space is called AdS_2. It is obtained from S^2 by ~~the~~ $\theta = i\rho$, $R \to iR$.
$\varphi = t$ $\qquad 0 \leq \rho \leq \infty$

It can be obtained by taking 2+1 d Minkowski spacetime:

$$ds^2 = \cancel{dt^2 - dx^2 + dy^2} - (dx^0)^2 - (dx^1)^2 + (dx^2)^2$$

and imposing:

$$(x^0)^2 + (x^1)^2 - (x^2)^2 \cancel{t^2 + x^2 - y^2} = R^2$$

The eqn is solved by:

$$x^0 = R\cosh\rho\cos t$$
$$x^1 = R\cosh\rho\sin t$$
$$x^2 = R\sinh\rho$$

and we get the above metric.

[We can also consider dS_2, given by starting with

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2$$

and imposing ~~$t^2 + x^2 - y^2 =$~~ ~~$-t^2 + x^2 + y^2 = R^2$~~
$$-(x^0)^2 + (x^1)^2 + (x^2)^2 = R^2$$

we end up with:

$$ds^2 = R^2\left(-dt^2 + \cosh^2 t \, d\theta^2\right)$$]

Now $\Gamma^{\rho}_{tt} = \frac{1}{2} g^{\rho\rho}(-g_{tt,\rho})$

$$= \frac{1}{2}\frac{1}{R^2} \cdot R^2 \cdot +(\cosh^2\rho)'$$

$$= \cosh\rho \sinh\rho$$

$$= \frac{1}{2}\sinh 2\rho$$

$$\Gamma^{t}_{\rho t} = \frac{1}{2} g^{tt}(g_{tt,\rho})$$

$$= \frac{1}{2}\frac{1}{-R^2\cosh^2\rho} \cdot R^2 \cdot -2\cosh\rho\sinh\rho$$

$$= \tanh\rho.$$

Thus: $\ddot{\rho} + \frac{1}{2}\sinh 2\rho\,(\dot{t})^2 = 0$

$$\ddot{t} + 2\tanh\rho\,\dot{t}\dot{\rho} = 0$$

Thus we have the analogous geodesics:

(i) $\rho = $ const. Then $\ddot{t} = 0$ and $\sinh\rho = 0$

So $\rho = 0$ and $t = a\tau + b$

(ii) $t = $ const. Then $\ddot{\rho} = 0 \Rightarrow \rho = a\tau + b$

The first geodesic has

$$\left(\frac{ds}{d\tau}\right)^2 = R^2(-\cosh^2\rho\,\dot{t}^2 + \dot{\rho}^2)$$

$$= -a^2R^2\cosh^2\rho < 0 \Rightarrow \text{timelike}$$

The second has $\left(\frac{ds}{d\tau}\right)^2 = R^2a^2 > 0 \Rightarrow$ spacelike.

(iii) Rindler space in 2d.

$$ds^2 = -X^2 dT^2 + dX^2$$

Here $\Gamma^T_{TX} = \frac{1}{X}$, $\Gamma^X_{TT} = X$, all others $= 0$.

The geodesics are:
$$\ddot{X} + X\dot{T}^2 = 0$$
$$\ddot{T} + \frac{2}{X}\dot{T}\dot{X} = 0$$

With $X = 0$ (risky!) we find the timelike geodesics $T = A\tau + B$. Similarly with $T = $ const we get spacelike geodesics $X = A'\tau + B'$.

It looks like $X = 0$ is a "bad point" (the time intervals shrink to zero size there!).

However, the transformation

$$T = \tanh^{-1}\frac{t}{x}, \quad X = \sqrt{x^2 - t^2}$$
$$t = X\sinh T \qquad x = X\cosh T$$

gives:

$$dT = \frac{x\,dt - t\,dx}{x^2 - t^2}$$

$$dX = \frac{x\,dx - t\,dt}{\sqrt{x^2 - t^2}}$$

So $\quad -X^2 dT^2 + dX^2$

$$= -(x^2 - t^2)\frac{(x\,dt - t\,dx)^2}{(x^2 - t^2)^2} + \frac{(x\,dx - t\,dt)^2}{x^2 - t^2}$$

$$= -\frac{1}{x^2-t^2}\Big(x^2dt^2 + t^2dx^2 - 2xt\,dx\,dt$$

$$- x^2dx^2 - t^2dt^2 + 2xt\,dx\,dt \Big)$$

$$= -\frac{1}{x^2-t^2}\Big((x^2-t^2)\,dt^2 - (x^2-t^2)\,dx^2 \Big)$$

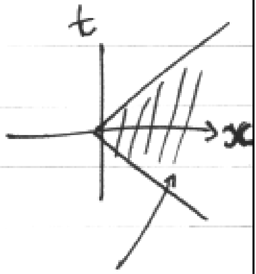
$$= -dt^2 + dx^2 \quad !$$

Thus (at least locally) Rindler space is equivalent to Minkowski space. Then geodesics are trivial:

$$\ddot{x} = 0, \quad \ddot{t} = 0$$

This shows the power of coordinate transformations in finding geodesics more easily.

Note also that in Rindler space, $0 < X < \infty$ and $-\infty < T < \infty$ so:



$$\frac{t}{x} = \tanh T \quad \Rightarrow \quad -1 \le \frac{t}{x} \le 1$$

Also $x^2 - t^2 = X^2 \quad \Rightarrow \quad -x \le t \le x$

which is the same region: "Rindler wedge".

Also $x \geq 0$ since $x = X\cosh T$

We could have alternatively solved the geodesic eqns in Rindler coords by:

$$\ddot{T} + 2\dot{T}\frac{\dot{X}}{X} = 0$$

$$\Rightarrow \quad \frac{d}{d\tau}\ln\dot{T} = -\frac{d}{d\tau}\ln X^2$$

$$\Rightarrow \quad \dot{T} = \frac{A}{X^2}$$

Next, $\ddot{X} + X\dot{T}^2 = 0 \quad \Rightarrow \quad \ddot{X} + \frac{XA^2}{X^4} = 0$

$$\Rightarrow \quad \dot{X}\ddot{X} + \frac{\dot{X}A^2}{X^3} = 0$$

$$\Rightarrow \quad \dot{X}^2 = \frac{A^2}{X^2} + \text{const}.$$