Lecture 9 17/9/09

Freely falling frames

In order to do physics in pseudo-Riemannian spacetime, it is useful to consider coordinates that are specially adapted. A simple example is the freely falling frame.

We saw that $\Gamma^{\lambda}_{\mu\nu}$ can be made to vanish at a point in spacetime. But in fact, it can be made to vanish all along a given geodesic. However Γ will not vanish for any infinitesimal variation away from the geodesic. Indeed the existence of such a frame is the key to the principle of equivalence: a freely falling body experiences no gravity!

In a freely falling frame, the time coordinate gets identified with the proper time along the geodesic. Of course at each time instant, the spatial coordinates also have to be chosen suitably to make $g_{\mu\nu,\lambda} = 0$ along the geodesic.

This is done as follows. We want to define a set of orthonormal vectors $e^{\mu}_{\hat{\alpha}}$ to define the coordinate axes of the freely falling frame:

$e^{\mu}_{\hat{0}}$: time vector, $e^{\mu}_{\hat{\lambda}}$: 3 space vectors.

So define $e^{\mu}_{\hat{0}} = \frac{dx^{\mu}}{d\tau}$

If τ is the proper time then $d\tau^2 = -dx^{\mu}dx^{\nu}g_{\mu\nu}$

so $g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = -1 \Rightarrow g_{\mu\nu}e^{\mu}_{\hat{0}}e^{\nu}_{\hat{0}} = -1$

so $e^{\mu}_{\hat{0}}$ is correctly normalised.

Next to define $e^{\mu}_{\hat{\lambda}}$, require $\frac{dx^{\nu}}{d\tau}D_{\nu}e^{\mu}_{\hat{\lambda}} = 0$

ie $\frac{de^{\mu}_{\hat{\lambda}}}{d\tau} + \Gamma^{\mu}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}e^{\lambda}_{\hat{\lambda}} = 0$

Since $u^{\mu} = \frac{dx^{\mu}}{d\tau}$ also satisfies $u^{\nu}D_{\nu}u^{\mu} = 0$,

all in all we have $u^{\nu}D_{\nu}e^{\mu}_{\hat{\alpha}} = 0$ as the definition of freely falling coordinates.

Physical observations in a metric

The metric affects how we perceive time and space intervals. Consider first, two events occurring at the same point in space, ~~(...)~~ but at different times. Then

$$d\tau^2 = -g_{00}\,dt^2$$

So $d\tau = \sqrt{-g_{00}}\,dt$

$\Rightarrow \tau = \int\sqrt{-g_{00}}\,dt$. (where $t = x^0$ and we have put $c=1$).

Next let's understand how the metric affects spatial distances. For this, consider two spatial points A at x^i and B at $x^i + dx^i$.
A light signal ~~from~~ leaves B at time $x^0 + dx^{0(1)}$, reaches A at x^0, and is reflected back to B at $x^0 + dx^{0(2)}$. We want to find the time separation $dx^{0(2)} - dx^{0(1)}$ as a function of the spatial separation dx^i.

Since $ds^2 = 0$ for light,

$$g_{00} (dx^0)^2 + 2 g_{0i} \, dx^0 dx^i + g_{ij} \, dx^i dx^j = 0$$

$$\Rightarrow \quad dx^0 = \frac{1}{g_{00}} \left(-g_{0i} \, dx^i \pm \sqrt{(g_{0i} g_{0j} - g_{ij} g_{00}) \, dx^i dx^j} \right)$$

The two roots are $dx^{0(1)}$, $dx^{0(2)}$ so

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}} \sqrt{(g_{0i} g_{0j} - g_{ij} g_{00}) \, dx^i dx^j}$$

From this we get a measure of the proper time which in turn is an invariant measure of the distance between A and B:

$$d\tau = \frac{1}{2} \sqrt{g_{00}} \left(dx^{0(2)} - dx^{0(1)} \right)$$

$$= \sqrt{\left(-\frac{g_{0i} g_{0j}}{g_{00}} + g_{ij} \right) dx^i dx^j}$$

$$= \sqrt{\gamma_{ij} \, dx^i dx^j}$$

where $\gamma_{ij} = g_{ij} - \dfrac{g_{0i} g_{0j}}{g_{00}}$

Notice that if γ^{ij} is defined as the inverse of γ_{ij}, then

$$\gamma^{ij} = g^{ij}$$

(where g^{ij} are the space components of $g^{\mu\nu}$ which is the inverse of $g_{\mu\nu}$).

Only if γ_{ij} is independent of time, this infinitesimal relation can be integrated to give a finite physical distance.

We can now consider the synchronisation of two clocks located at different points. The basic problem is that at the two points ~~the~~ (A and B as before), x^0 is not defined in the same way. The thought-expt just performed above, however, provides a way. We say that the instant x^0 at A is synchronous with $x^0 + \underbrace{\frac{1}{2}(dx^{0(1)} + dx^{0(2)})}_{\Delta x^0}$ at B.

Then we find $\Delta x^0 = -\dfrac{g_{0i}\, dx^i}{g_{00}}$.

In this way we can synchronise clocks along any open contour. But what if the contour returns to its starting point? If we find $\Delta x^0 \neq 0$ then we are in trouble! So synchronisation of all clocks in a finite region is possible only if $g_{0i} = 0$.

This is not a physical requirement, rather it is possible to achieve by a suitable choice of coordinate system: synchronous coordinates. $\left(\begin{matrix} g_{00} = 1 \\ g_{0i} = 0 \end{matrix}\right)$

Notice that if we have two synchronized events at points A, B at times x^0_A and x^0_B, and if after some time there are two more events at $x^0_A + dx^0_A$ and $x^0_B + dx^0_B$ (also synchronised) then ~~[illegible]~~ in general

$$d\tau_A = \sqrt{-g_{00}}\, dx^0_A \neq d\tau_B = \sqrt{-g_{00}}\, dx^0_B$$

(time lapse between pairs of synchronised events is not equal).

Time independence

In physics we distinguish between time-independent and time-dependent processes/quantities. But in GR there seems to be a problem. Any time independent function, or tensor, can be made time-dependent by a (time-dependent) coordinate transformation !. Accordingly we must be careful and precise with our definitions.

(i) A metric $g_{\mu\nu}(x)$ is called <u>stationary</u> if in <u>some</u> coordinate system, all components $g_{\mu\nu}$ are independent of t.

(ii) A metric $g_{\mu\nu}(x)$ is called <u>static</u> if ~~it~~ in some coordinate system, $g_{\mu\nu}$ is indep of ~~t~~ <u>and</u> $g_{0i} = 0$.