Lecture 10. 29/9/09

Conservation laws and Killing vectors.

In pre-relativistic physics it was clear that if a system is time-translation invariant then energy is conserved. But in special relativity, and even more so in general relativity, there can be different definitions of time. So what exactly is conserved?

Let's go back to the mechanics of a particle on a curved space:

$$L = \frac{1}{2} m \int dt \; g_{ij}(x) \, \dot{x}^i \dot{x}^j$$

Suppose for some region the metric $g_{ij}(x,y,z)$ is independent of z. Then the z-momentum is conserved:

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m \, g_{ij} \dot{x}^j$$

and $p_z = p_3$ satisfies $\dfrac{dp_z}{dt} = 0$.

Similar reasoning tells us that for a spacetime metric $g_{\mu\nu}$ that is independent of time ("stationary"),

$$\frac{d}{d\tau}\left(g_{0\nu} \dot{x}^\nu \right) = 0 = \frac{d}{d\tau}\left(g_{0\nu} p^\nu \right)$$

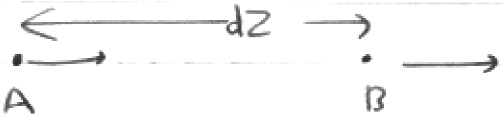
where $p^\nu = m \dfrac{dx^\nu}{d\tau}$.

The fact that it is precisely $g_{0\nu} p^\nu$ that is conserved is one way to understand the famous gravitational redshift. It arises from a conflict between observed energy and conserved energy. Let us first demonstrate this and then return to the question of what happens to conserved quantities when we change our definition of time.

Gravitational redshift

First let us re-visit how this looks in terms of the equivalence principle.

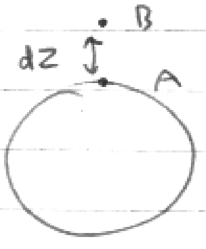
Consider two objects A, B accelerating in the same direction:



At time t_0, A emits a photon of wavelength λ_0. The separation of A and B is a constant z, so the photon reaches B after a time $\Delta t = \frac{dz}{c}$. In this time B has picked up an extra velocity $\Delta v = a \Delta t = \frac{a\,dz}{c}$. Thus the photon is redshifted by

$$\frac{\Delta \omega}{\omega_0} = -\frac{\Delta v}{c} = -\frac{a\,dz}{c^2}.$$

The principle of equivalence says this must also happen if we put A and B at ~~rest~~ rest on a planet.



where the acceleration due to gravity is $\vec{a}$.

Now $\vec{a} = \vec{\nabla}\Phi$ where Φ is the Newtonian potential.

Thus $$\frac{\Delta\omega}{\omega_0} = -\frac{1}{c^2}\, dz\, \partial_z \Phi$$

$$\int \frac{\Delta\omega}{\omega_0} = -\frac{1}{c^2}\int_A^B dz\,\partial_z\Phi = \frac{\Phi_A - \Phi_B}{c^2}$$

Now consider a general ~~stationary~~ static metric:

$$ds^2 = g_{00}\, dt^2 + g_{ij}\, dx^i dx^j$$

and an observer at rest : $u_\mu = (u_0, 0, 0, 0)$

Now $u_\mu u^\mu = -1 \Rightarrow g^{00}(u_0)^2 = g_{00}(u^0)^2 = -1$

$$\Rightarrow \quad u^0 = \frac{1}{\sqrt{-g_{00}}}$$

Consider trying to measure the frequency of a photon in a gravitational field. The photon travels on a null geodesic:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0, \qquad g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0.$$

Now the energy of a photon as measured in the frame of an observer with 4-velocity U^μ is

$$E_{obs} = - g_{\mu\nu} U^\mu p^\nu$$

where p^ν is the photon 4-momentum.

(For massive particles $p^\mu = m \frac{dx^\mu}{d\tau}$ where τ is the proper time. For photons, we can take $p^\mu = \alpha \frac{dx^\mu}{ds}$ where s is an arbitrary parameter).

So $E_{obs} = -\alpha g_{\mu\nu} U^\mu \frac{dx^\nu}{ds}$

$= -\alpha g_{00} U^0 \frac{dx^0}{ds}$. for a static metric.

Now, notice that $E_{cons} = -\alpha g_{00} \frac{dx^0}{ds} = - g_{\mu\nu} p^\nu \Big|_{\mu=0}$

is the <u>conserved</u> energy. Hence

$$E_{obs} = \hbar \omega_{obs} = U^0 E_{cons}$$

$$= \frac{1}{\sqrt{-g_{00}}} E_{cons}.$$

Therefore $\frac{\omega_B}{\omega_A} = \sqrt{\frac{g_{00}(A)}{g_{00}(B)}}$

In Newtonian appr,
Since $g_{00} \sim 1 + 2\Phi$, $\sqrt{\frac{g_{00}(A)}{g_{00}(B)}} \sim \sqrt{\frac{1+2\Phi_A}{1+2\Phi_B}}$

$\simeq 1 + \Phi_A - \Phi_B$

So $\frac{\omega_B}{\omega_A} \sim \frac{\omega_A + \Delta\omega}{\omega_A} = 1 + \frac{\Delta\omega}{\omega_A} = 1 + \Phi_A - \Phi_B$ as seen earlier.

~~Killing vectors~~

Killing vectors

In the previous derivation, we used that

$$E_{cons} = -g_{0\nu} p^\nu = -g_{00} p^0 \quad \text{(static case)}$$

This statement requires us to be in the coordinate system where the stationary metric is explicitly time-independent and $g_{0i} = 0$.

However, corresponding statements can be made in general coordinates. Consider the 4-momentum of a massive particle:

$$p^\mu = m \frac{dx^\mu}{d\tau} = m u^\mu$$

Clearly p^μ satisfies the geodesic eqn

$$\frac{dp^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} u^\nu p^\lambda = 0$$

Therefore:

$$\frac{dp_\mu}{d\tau} = \frac{d}{d\tau}\left(g_{\mu\nu} p^\nu \right) = g_{\mu\nu} \frac{dp^\nu}{d\tau} + g_{\mu\nu,\lambda} u^\lambda p^\nu$$

$$= -g_{\mu\nu} \Gamma^\nu_{\lambda\rho} u^\lambda p^\rho + g_{\mu\rho,\lambda} u^\lambda p^\rho$$

$$= u^\lambda p^\rho \left(-\tfrac{1}{2} g_{\mu\lambda,\rho} - \tfrac{1}{2} g_{\mu\rho,\lambda} + \tfrac{1}{2} g_{\lambda\rho,\mu} + g_{\mu\rho,\lambda} \right)$$

$$= \tfrac{1}{2} u^\lambda p^\rho g_{\lambda\rho,\mu} \quad (\text{since } u^\lambda p^\rho = u^\rho p^\lambda)$$

Thus $\quad \frac{dp_\mu}{d\tau} = \frac{1}{2} U^\lambda p^\rho g_{\lambda\rho,\mu}$

For each μ, this tells us that if $g_{\lambda\rho}$ is indep of ~~xxxx~~ x^μ ~~then~~ then $\frac{dp_\mu}{d\tau}$ is conserved along the geodesic.

More generally let us assume there exists a vector K^μ such that $K^\mu p_\mu$ is conserved:

$$\frac{d}{d\tau}(K^\mu p_\mu) = 0$$

What does this imply?

$$\frac{d}{d\tau}(K^\mu p_\mu) = \frac{dK^\mu}{d\tau} p_\mu + K^\mu \frac{dp_\mu}{d\tau}$$

$$= \partial_\nu K^\mu U^\nu p_\mu + K^\mu \left(\frac{1}{2} g_{\lambda\rho,\mu} U^\lambda p^\rho \right)$$

$$= U^\lambda p^\rho \left(g_{\mu\rho} \partial_\lambda K^\mu + \frac{1}{2} g_{\lambda\rho,\mu} K^\mu \right)$$

~~Now~~ $$= U^\lambda p^\rho \left(\partial_\lambda K_\rho - g_{\mu\rho,\lambda} K^\mu + \frac{1}{2} g_{\lambda\rho,\mu} K^\mu \right)$$

$$= U^\lambda p^\rho \left(\partial_\lambda K_\rho - \Gamma^\mu_{\lambda\rho} K_\mu \right) = U^\lambda p^\rho D_\lambda K_\rho$$

$$= \frac{1}{2} U^\lambda p^\rho \left(D_\lambda K_\rho + D_\rho K_\lambda \right)$$

Hence $\quad D_\lambda K_\rho + D_\rho K_\lambda = 0 \quad$ is the condition for the existence of a conserved quantity! This is called <u>Killing's equation</u> and the vector K^μ is called a Killing vector.

Note that if we make an infinitesimal coordinate transf.

$$x^\mu \to x^\mu + k^\mu$$

then $\delta g_{\mu\nu} = D_\mu k_\nu + D_\nu k_\mu$ (we have already derived this)

Hence a Killing vector can be thought of as generating a coordinate transformation that preserves the form of the metric. Such a transformation is called an isometry.