Lecture 11. 1/10/09.

In this lecture we will consider "vacuum solutions" of the Einstein eqns, namely solutions with $T_{\mu\nu}=0$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \Rightarrow R = 0 \Rightarrow R_{\mu\nu} = 0$$

Our reasons for doing so are:

(i) it's the simplest case (later we will introduce specific $T_{\mu\nu}$'s, Λ, etc)

(ii) it can describe the geometry of spacetime <u>outside</u> any localised matter distribution.

For comparison for a spherically symmetric body ~~By analogy~~, we have electrostatic potentials

$$\varphi_M \sim \frac{e}{r}$$

and gravitational potentials

$$\varphi_G \sim -\frac{GM}{r}$$

which solve $\vec{\nabla}^2 \varphi_M = 0$ or $\vec{\nabla}^2 \varphi_G = 0$ <u>outside</u> the charge or matter distribution.

The second example, and the Newtonian correspondence, actually tells us an approximate soln to the vacuum Einstein eqns. We have that Φ in ~~the~~ Newtonian theory is approximately

$$ds^2 = -(1+2\Phi)\,dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2)$$

Therefore there must be an approximate soln:

$$ds^2 \sim -\left(1-\frac{2GM}{r}\right)dt^2$$
$$+\left(1+\frac{2GM}{r}\right)\left(dr^2+r^2\,d\Omega^2\right)$$

valid at large r. Let us rewrite this approx soln by defining

$$r' = \left(1+\frac{GM}{r}\right)r = r+GM$$

Then
$$r'^2 = (r+GM)^2 \sim r^2+2GMr$$
$$= r^2\left(1+\frac{2GM}{r}\right).$$

and $dr' = dr$,
$$1+\frac{2GM}{r} = 1+\frac{2GM}{r'-GM}$$
$$\simeq 1+\frac{2GM}{r'}$$

So
$$ds^2 \sim -\left(1-\frac{2GM}{r}\right)dt^2+\left(1+\frac{2GM}{r}\right)dr^2$$
$$+\,r^2 d\Omega^2$$
(where we have dropped the prime on r').

How do we extend this to an exact solution of Einstein's eqn valid for finite r, not just $r \gg GM$?

~~Notice that~~ The approximate solution above is spherically symmetric. Let us then assume the most general spherically symmetric metric:

$$ds^2 = -f_1(r,t)\,dt^2 + f_2(r,t)\,dr\,dt$$
$$+ f_3(r,t)\,dr^2 + f_4(r,t)\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

Now, note that we are allowed coordinate transformations

$$r \to r'(r,t)$$
$$t \to t'(r,t).$$

We choose these two functions to set $f_2(r,t) = 0$ and $f_4(r,t) = r^2$. Also we re-name

$$f_1 = e^{2\alpha(r,t)}, \quad f_3(r,t) = e^{2\beta(r,t)}.$$ Thus

$$ds^2 = -e^{2\alpha(r,t)}\,dt^2 + e^{2\beta(r,t)}\,dr^2$$
$$+ r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

Note that $t \to t'(t)$ is still allowed, but is not useful to absorb α since it depends on r as well. (it will be useful soon)

Now $g_{\mu\nu} = \begin{bmatrix} -e^{2\alpha} & & & \\ & e^{2\beta} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{bmatrix}$ so $g^{\mu\nu} = \begin{bmatrix} -e^{-2\alpha} & & & \\ & e^{-2\beta} & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2\sin^2\theta} \end{bmatrix}$

Then we find: ($'$ denotes differentiation in r, and $\dot{}$ " " " t)

$$\Gamma^t_{tr} = \alpha' \qquad \Gamma^t_{tt} = \dot{\alpha} \qquad \Gamma^r_{rt} = \dot{\beta}$$

$$\Gamma^r_{tt} = e^{2(\alpha-\beta)}\alpha' \qquad \Gamma^t_{rr} = e^{2(\beta-\alpha)}\dot{\beta}$$

$$\Gamma^r_{rr} = \beta'$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^r_{\theta\theta} = -re^{-2\beta} \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}$$

$$\Gamma^r_{\varphi\varphi} = -re^{-2\beta}\sin^2\theta \qquad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \qquad \Gamma^\varphi_{\theta\varphi} = \cot\theta$$

Then

$$R^t{}_{rtr} = e^{2(\beta-\alpha)}\ddot{\beta} + 2(\dot{\beta}-\dot{\alpha})\dot{\beta}e^{2(\beta-\alpha)} - \alpha''$$

$$+ e^{2(\beta-\alpha)}\dot{\alpha}\dot{\beta} + \alpha'\beta' - \alpha'^2 - e^{2(\beta-\alpha)}\dot{\beta}^2$$

$$= \cancel{(\dot{\beta}-\dot{\alpha})\dot{\beta}e^{2(\beta-\alpha)}}$$

$$= e^{2(\beta-\alpha)}\left(\dot{\beta}(\dot{\beta}-\dot{\alpha}) + \ddot{\beta}\right) + \alpha'\beta' - \alpha'^2 - \alpha''$$

$$R^t{}_{\theta t\theta} = -r\alpha' e^{-2\beta} \qquad R^t{}_{\theta r\theta} = -r\dot{\beta}e^{-2\alpha}$$

$$R^t{}_{\varphi t\varphi} = -r\alpha'\sin^2\theta\, e^{-2\beta} \qquad R^t{}_{\varphi r\varphi} = -r\dot{\beta}e^{-2\alpha}\sin^2\theta$$

$$R^r{}_{\theta r\theta} = r\beta' e^{-2\beta}$$

$$R^r{}_{\varphi r\varphi} = r\beta' e^{-2\beta}\sin^2\theta$$

$$R^\theta{}_{\varphi\theta\varphi} = (1-e^{-2\beta})\sin^2\theta$$

(all others zero)

Thus the Ricci tensor is:

$$R^t{}_t = g^{\theta\theta} R^t{}_{\theta t \theta} + g^{\varphi\varphi} R^t{}_{\varphi t \varphi} + g^{rr} R^t{}_{r t r}$$

$$= -\frac{2}{r}\alpha' e^{-2\beta} + e^{-2\beta}\Big(e^{2(\beta-\alpha)}\big(\dot\beta(\dot\beta-\dot\alpha)+\ddot\beta\big) + \alpha'\beta' - \alpha'^2 - \alpha''\Big)$$

$$= e^{-2\beta}\left(\alpha'\beta' - \alpha'^2 - \alpha'' - \frac{2}{r}\alpha'\right) + e^{-2\alpha}\big(\dot\beta(\dot\beta-\dot\alpha)+\ddot\beta\big)$$

$$R^r{}_r = g^{tt} R^r{}_{trt} + g^{\theta\theta} R^r{}_{\theta r \theta} + g^{\varphi\varphi} R^r{}_{\varphi r \varphi}$$

$$= g^{rr} R^t{}_{rtr} + \text{"} + \text{"}$$

$$\cancel{= e^{-2\beta}\Big(e^{2(\beta-\alpha)}\big(\dot\beta(\dot\beta-\dot\alpha)+\ddot\beta\big)}$$

$$= e^{-2\alpha}\big(\dot\beta(\dot\beta-\dot\alpha)+\ddot\beta\big) + e^{-2\beta}\left(\alpha'\beta' - \alpha'^2 - \alpha''\right) + \frac{2\beta'}{r} e^{-2\beta}$$

$$= e^{-2\beta}\left(\alpha'\beta' - \alpha'^2 - \alpha'' + \frac{2\beta'}{r}\right) + e^{-2\alpha}\big(\dot\beta(\dot\beta-\dot\alpha)+\ddot\beta\big)$$

$$R^\theta{}_\theta = g^{rr} R^\theta{}_{r\theta r} + g^{\varphi\varphi} R^\theta{}_{\varphi\theta\varphi} + g^{tt} R^\theta{}_{t\theta t}$$

$$= g^{\theta\theta} R^r{}_{\theta r\theta} + g^{\varphi\varphi} R^\theta{}_{\varphi\theta\varphi} + g^{\theta\theta} R^t{}_{\theta t\theta}$$

$$= \frac{1}{r^2}\left(r\beta' e^{-2\beta} - r\alpha' e^{-2\beta}\right) + \frac{1}{r^2 \sin^2\theta}\left(1 - e^{-2\beta}\right)\sin^2\theta$$

$$= \frac{1}{r^2}\left\{ e^{-2\beta}\left(r(\beta'-\alpha') - 1\right) + 1 \right\}$$

$$R^{\varphi}{}_{\varphi} = g^{tt} R^{\varphi}{}_{t\varphi t} + g^{\theta\theta} R^{\varphi}{}_{\theta\varphi\theta} + g^{rr} R^{\varphi}{}_{r\varphi r}$$

$$= g^{\varphi\varphi} \left(R^{t}{}_{\varphi t\varphi} + R^{\theta}{}_{\varphi\theta\varphi} + R^{r}{}_{\varphi r\varphi} \right)$$

$$= R^{\theta}{}_{\theta} \quad \text{as one can easily show.}$$

Finally $R^{t}{}_{r} = g^{\theta\theta} R^{t}{}_{\theta r\theta} + g^{\varphi\varphi} R^{t}{}_{\varphi r\varphi}$

$$= -\frac{2}{r} e^{-2\alpha} \dot{\beta}$$

We want to set $R^{\mu}{}_{\nu} = 0$ for all μ, ν. In particular,

$$R^{t}{}_{r} = 0 \implies \dot{\beta} = 0 \implies \beta = \beta(r).$$

Next, the combination $R^{t}{}_{t} - R^{r}{}_{r} = 0$

$$\implies \alpha' + \beta' = 0$$

$$\implies \alpha + \beta = f(t)$$

Now under $t \to t'(t)$, which is still allowed, we have

$$-e^{2\alpha} dt^2 \to -e^{2\alpha} \left(\frac{dt'}{dt}\right)^2 dt^2$$

which shifts α by an arbitrary fn of t. In this way we can make $f(t) = 0$.

Thus $\alpha + \beta = 0$, but since $\dot{\beta} = 0$ it follows that $\dot{\alpha} = 0$.

Hence we have proved that the metric is static!

$$ds^2 = -e^{2\alpha(r)}\,dt^2 + e^{2\beta(r)}\,dr^2 + r^2 \underbrace{d\Omega_2^2}_{d\theta^2 + \sin^2\theta\, d\varphi^2}$$

Next using $R^\theta{}_\theta = 0$ we have:

$$2r e^{2\alpha}\alpha' + e^{2\alpha} = 1$$

i.e. $$(r e^{2\alpha})' = 1$$

$$\Rightarrow \quad r e^{2\alpha} = r - \underbrace{r_0}_{\text{const.}}$$

$$\therefore \quad e^{2\alpha} = 1 - \frac{r_0}{r}$$

But we have already seen that for large r,

$$e^{2\alpha} \simeq 1 - \frac{2GM}{r}$$

$$\Rightarrow \boxed{r_0 = 2GM}$$

where M is the mass of the matter distribution outside which we have determined the metric.

Finally $\beta = -\alpha \Rightarrow e^{2\beta} = \frac{1}{e^{2\alpha}} = \frac{1}{1 - \frac{2GM}{r}}$.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)}dr^2 + r^2\,d\Omega^2$$

$\rightarrow$ Schwarzschild metric.

How do we interpret the Schwarzschild metric? It is the metric <u>outside</u> any spherically symmetric body. So it is <u>invalid</u> if $r < \tilde{r}$ where $\tilde{r}$ is the physical size of the body.

For example the gravitational field of the earth is correctly described by it for $r > 6378$ km (or a bit more if you want to exclude the atmosphere). (here we need to ignore the earth's rotation)

The Schwarzschild metric appears to have some strange properties for $r = 2GM$. We see that at this point $g_{00} = 0$: "infinite redshift". Also $g_{rr} \to \infty$.

However this is not a problem as long as the physical radius of the matter distribution, $\tilde{r}$, is greater than $2GM$. For example the earth has

$$2GM \simeq 9 \text{ mm}$$
$$\tilde{r} \simeq 6378 \text{ km}$$

Since $\tilde{r} \gg 2GM$, the question of continuing the Schwarzschild metric to $r = 2GM$ does not arise. It ceases to be valid long before that!

However if some object has a mass and size such that $\tilde{r} \lesssim 2GM$ then there is an issue. In this case we are allowed to take r close to $2GM$ and the resulting analysis is extremely subtle & interesting. Such an object is called a black hole.