Lecture 12. 6/10/09

Particle motion in a Schwarzschild metric.

To study this, let us write down the christoffel symbols in a Schwarzschild metric:

$$\Gamma^t_{tr} = \alpha' = \frac{GM}{r(r-2GM)} \qquad \Gamma^r_{tt} = e^{4\alpha}\alpha' = \frac{GM}{r^3}(r-2GM)$$

$$\Gamma^r_{rr} = -\alpha' = \frac{-GM}{r(r-2GM)}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r}, \qquad \Gamma^r_{\theta\theta} = -re^{-2\beta} = -(r-2GM)$$

$$\Gamma^\varphi_{r\varphi} = \frac{1}{r} \qquad \Gamma^r_{\varphi\varphi} = -(r-2GM)\sin^2\theta$$

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \qquad \Gamma^\varphi_{\theta\varphi} = \cot\theta$$

Now we can write the geodesic eqns:

$$\ddot{t} + 2\Gamma^t_{tr}\dot{t}\dot{r} = 0$$

$$\ddot{r} + \Gamma^r_{rr}\dot{r}^2 + \Gamma^r_{tt}\dot{t}^2 + \Gamma^r_{\theta\theta}\dot{\theta}^2 + \Gamma^r_{\varphi\varphi}\dot{\varphi}^2 = 0$$

$$\ddot{\theta} + 2\Gamma^\theta_{r\theta}\dot{r}\dot{\theta} + \Gamma^\theta_{\varphi\varphi}\dot{\varphi}^2 = 0$$

$$\ddot{\varphi} + 2\Gamma^\varphi_{r\varphi}\dot{r}\dot{\varphi} + 2\Gamma^\varphi_{\theta\varphi}\dot{\theta}\dot{\varphi} = 0$$

$$\Rightarrow \quad \ddot{t} + \frac{2GM}{r(r-2GM)}\dot{r}\dot{t} = 0$$

$$\ddot{r} + \frac{GM}{r^3}(r-2GM)\dot{t}^2 - \frac{GM}{r(r-2GM)}\dot{r}^2 - (r-2GM)(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) = 0$$

$$\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \sin\theta\cos\theta\,\dot{\varphi}^2 = 0$$

$$\ddot{\varphi} + \frac{2}{r}\dot{\varphi}\dot{r} + 2\cot\theta\,\dot{\theta}\dot{\varphi} = 0$$

So far the parameter along the geodesic is general. For timelike trajectories (massive particles) we choose this parameter to be τ, the proper time, and then

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = U^\mu U_\mu = -1$$

For massless particles, we can again choose the parameter to be τ and then $U^\mu U_\mu = 0$. But this does not determine the parameter uniquely so we have to choose some arbitrary normalisation.

Finally, spacelike geodesics satisfy $U^\mu U_\mu = +1$ even though they don't correspond to any physical trajectory. To summarise, $\frac{dx^\mu}{ds} \frac{dx^\nu}{ds} g_{\mu\nu} = -\underset{\pm 1, 0}{\underset{\downarrow}{\epsilon}}$

Note also that if K^μ is a Killing vector then

$$K_\mu p^\mu \equiv K^\mu g_{\mu\nu} p^\nu \text{ is constant.}$$

For Schwarzschild we have four Killing vectors: three rotations and one time translation.

The time translation is simple: $K^\mu = (1,0,0,0)$

(it is useful to think of a Killing vector as generating an infinitesimal change of coordinates, as we have seen. Now $f(x) \to f(x^\mu + K^\mu)$ so $\delta f = K^\mu \partial_\mu f(x)$. Thus it is naturally represented by the first order differential operator $K^\mu \partial_\mu$. For $K^\mu = (1,0,0,0)$ this is just $\frac{\partial}{\partial t}$.)

Similarly $R^\mu = (0,0,0,1)$ corresponding to $R^\mu \partial_\mu = \frac{\partial}{\partial\varphi}$ is manifestly one of the three rotational Killing vectors. ~~Because we rotation~~ We don't need the other two KV's for a well-known reason. In a spherically symmetric background a particle will move in a plane (normal to the angular momentum vector).

Choosing coord's such that this coincides with the equatorial plane $\theta = \frac{\pi}{2}$, we effectively use up two of the three K.V's and the one left over is $\frac{\partial}{\partial\varphi}$.

Note that $K^\mu = (1,0,0,0) \Rightarrow K_\mu = \left(-\left(1-\frac{2GM}{r}\right), 0,0,0\right)$

$$R^\mu = (0,0,0,1) \Rightarrow R_\mu = (0,0,0, r^2 \sin^2\theta).$$
$$= (0,0,0, r^2) \text{ for } \theta = \frac{\pi}{2}$$

Let us work with massive particles & timelike geodesics then

$$E = -K_\mu p^\mu = -m\left(1-\frac{2GM}{r}\right)U^t = \text{energy}$$

and $L = R_\mu p^\mu = m r^2 U^\varphi$ = angular mom.

are the two conserved quantities.

(For massless particles $p^\mu = \alpha U^\mu$ with arbitrary α, so $E = -\alpha\left(1-\frac{2GM}{r}\right)U^t$, $L = \alpha r^2 U^\varphi$

where α is arbitrary and reflects the arbitrary scale of U^μ.)

We can use the conserved charges to find the particle orbits in the Schwarzschild geometry.

Consider: $u^\mu u_\mu = -1 \Rightarrow$

$$g_{tt}(u^t)^2 + g_{rr}(u^r)^2 + \underbrace{g_{\theta\theta}(u^\theta)^2}_{\substack{=0 \\ \text{as we have} \\ \text{fixed } \theta = \frac{\pi}{2}}} + g_{\varphi\varphi}(u^\varphi)^2 = -1$$

$$\Rightarrow \quad -\left(1-\frac{2GM}{r}\right)(u^t)^2 + \frac{1}{1-\frac{2GM}{r}}(u^r)^2 + r^2(u^\varphi)^2 = -1$$

$$\Rightarrow \quad -\left(1-\frac{2GM}{r}\right)^2(u^t)^2 + (u^r)^2 + \left(1-\frac{2GM}{r}\right)\left(r^2(u^\varphi)^2+1\right) = 0$$

$$\Rightarrow \quad -\frac{E^2}{m^2} + (\dot{r})^2 + \left(1-\frac{2GM}{r}\right)\left(\frac{L^2}{m^2r^2}+1\right) = 0$$

Multiply by $\frac{m}{2}$ ~~and~~ re-arrange to get:

$$\frac{1}{2}m\dot{r}^2 + \frac{m}{2}\left(1-2\frac{GM}{r}\right)\left(\frac{L^2}{m^2r^2}+1\right) = \frac{E^2}{2m}$$

$$\Rightarrow \quad \frac{1}{2}m\dot{r}^2 \cancel{*} - \frac{GmM}{r} + \frac{L^2}{2mr^2} - \frac{G\cancel{m}ML^2}{mr^3} = \frac{E^2}{2m} - \frac{m}{2}$$

Define $V(r) = -\dfrac{GmM}{r} + \dfrac{L^2}{2mr^2} - \dfrac{G\cancel{m}ML^2}{mr^3}$

$$\mathcal{E} = \frac{E^2}{2m} - \frac{m}{2}$$

Then we have mapped the problem of Schwarzschild orbits to the equivalent radial problem:

$$\frac{1}{2}m\dot{r}^2 + V(r) = \mathcal{E} \qquad \text{(with } V(r) \text{ as above).}$$

The resemblance to a non-relativistic radial problem with some "effective" potential $V(r)$ and "effective" energy $\mathcal{E}$ looks like a trick. However consider the Newtonian orbits of a particle of mass m and angular momentum L about an object of mass $M \gg m$. Then we have

$$\frac{1}{2} m \dot{r}^2 + V_N(r) = E_N$$

where $V_N(r) = -\frac{GmM}{r} + \underbrace{\frac{L^2}{2mr^2}}_{\text{centrifugal barrier}}$.

Moreover $E_{GR} = m + E_N$ and $E_N \ll m$ in the Newtonian limit. In this approx. we have:

$$\mathcal{E} \equiv \frac{E^2}{2m} - \frac{m}{2} = \frac{(E_N + m)^2}{2m} - \frac{m}{2} \sim E_N \quad !$$

Thus indeed everything reduces correctly to the Newtonian case if the term $\frac{GM}{mr^3} L^2$ can be neglected. But this term constitutes a precise, measurable departure from Newtonian gravity.

We can now continue to work out the orbits. Note that we are strictly working with massive particles. Some aspects of the derivation will need to be re-worked in the case of photons (which is experimentally very important!) We will return to that later.

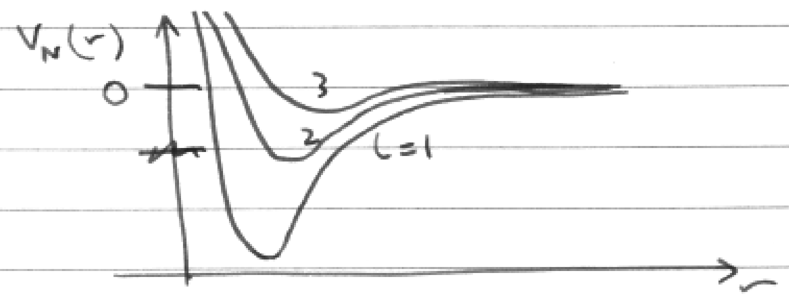
Physical insight is gained by plotting $V(r)$ with and without the $\frac{1}{r^3}$ correction.

$$V_N(r) = -\frac{GmM}{r} + \frac{L^2}{2mr^2}$$

This has a stationary point $\frac{\partial V}{\partial r} = 0$ at

$$r_c = \left(\frac{L}{m}\right)^2 \frac{1}{GM}$$

which increases with increasing $\frac{L}{m}$, while $V(r_c) = -\frac{m}{2}\left(\frac{GM}{L/m}\right)^2$ becomes less -ve for increasing $\frac{L}{m}$



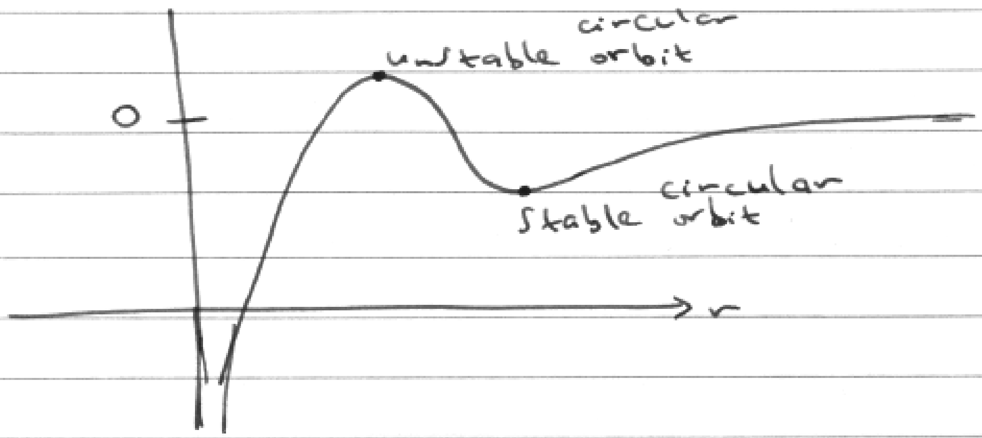
Contrast this with

$$V_{GR}(r) = -\frac{GmM}{r} + \frac{L^2}{2mr^2} - \frac{GML^2}{mr^3}$$

$$\frac{\partial V}{\partial r} = 0 \Rightarrow r = \frac{\left(\frac{L}{m}\right)^2 \pm \left(\frac{L}{m}\right)^2\sqrt{1 - \frac{12(GM)^2}{(L/m)^2}}}{2GM}$$

So there are two turning points for V as long as $\left(\frac{L}{m}\right)^2 \geq 12\, GM$. As $r \to 0$

(formally! we are not yet allowed to consider this because $r > \tilde{r}$ where $\tilde{r}$ is the size of the matter distribution in the star or whatever object it is) $V(r) \to -\infty$, so:



$$\frac{\left(\frac{L}{m}\right)^2}{GM} > \sqrt{12}$$

Note that $r_c^+ = \dfrac{\left(\frac{L}{m}\right)^2 + \left(\frac{L}{m}\right)^2\sqrt{1 - \frac{12(GM)^2}{(L/m)^2}}}{2GM}$

$$> \frac{(L/m)^2}{2GM} \quad \text{for} \quad (L/m)^2 > \sqrt{12}\, GM$$

$$> 6\, GM \quad : \text{smallest possible radius of a stable circular orbit.}$$

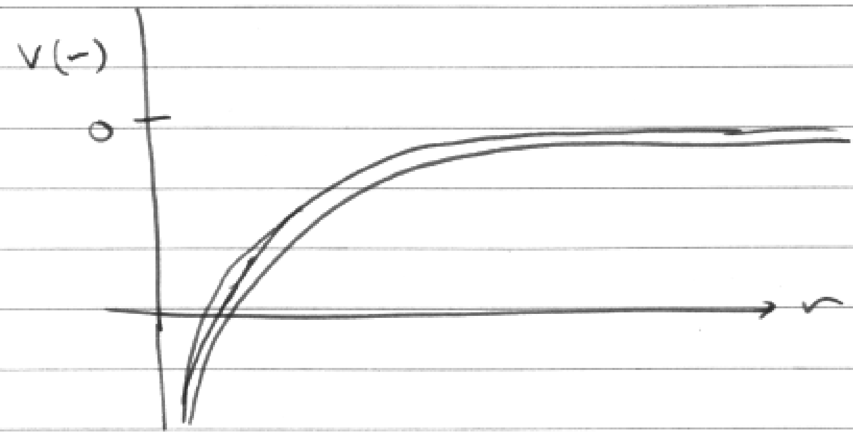
The bound is saturated when $\frac{L}{m} = \sqrt{12}G$ and we have a single circular orbit.

For very large $\frac{L}{m} \to \infty$, $r_c^{\pm} = \dfrac{\left(\frac{L}{m}\right)^2 \pm \left(\frac{L}{m}\right)^2\left(1 - \frac{6(GM)^2}{(L/m)^2}\right)}{2GM}$

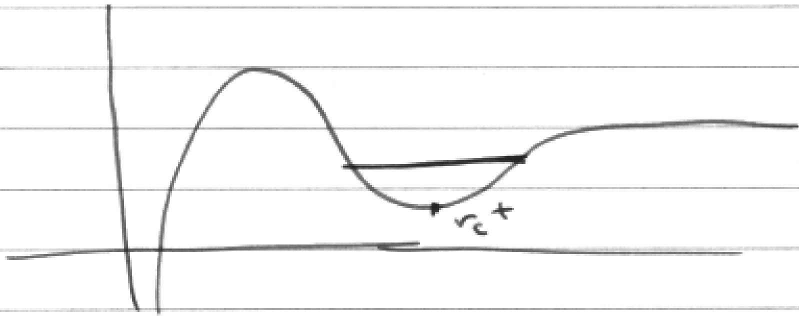
$$r_c^+ = \frac{(L/m)^2}{GM}, \quad r_c^- = 3\, GM.$$

Finally for $L/m < \sqrt{12}\, GM$ there is <u>no barrier</u>.

Therefore in this case we have:



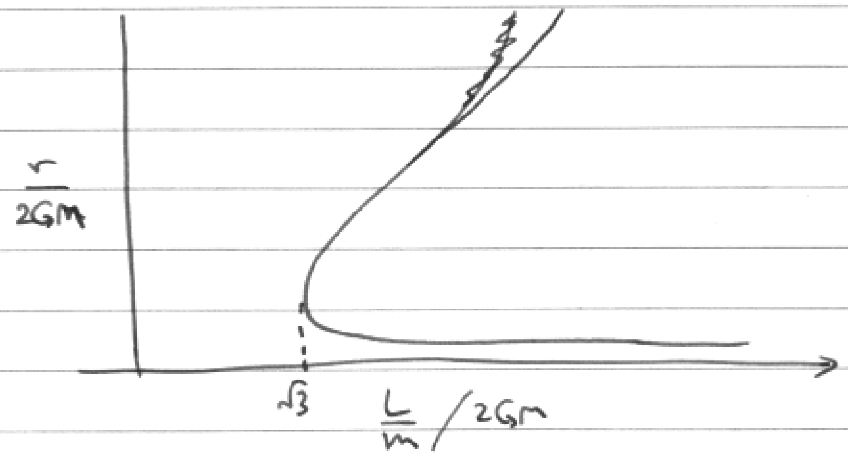
Whenever there is a stable circular orbit ($L/m > \sqrt{12}GM$
there can be bound, non-circular motion:



However because of the GR correction term, it will <u>not</u> be elliptical.

Also there are unbounded orbits that come in from infinity and go back out. And for high enough energy the barrier can be overcome and the particle falls in.

The condition for stable circular orbits can be expressed in the following way.



Most of this graph is uninteresting if we restrict to $r > \tilde{r} >> 2GM$.