Lecture 13. 8/10/09

Massless particles

Let us revisit the action for a relativistic particle moving in a spacetime metric $g_{\mu\nu}(x)$.

We have previously seen that the action

$$S = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x)}$$

works for massive particles. But clearly the $m \to 0$ limit of this action is rather strange! $S = 0$ is not a useful action.

The key here is that the above action is invariant under reparametrisations $\tau \to \tau'(\tau)$. In this sense we have a "miniature version" of general coordinate invariance, along the particle world line.

We can find a more general version of the above action by introducing a "miniature version" of a ~~■~~ metric, only along the world-line. Thus, define the world-line distance:

$$ds_{wl}^2 = G(\tau)\, d\tau^2$$

$G(\tau)$ is a 1-component object, the world-line metric, and under $\tau \to \tau'(\tau)$ we have:

$$d\tau \to d\tau' = \frac{\partial \tau'}{\partial \tau} d\tau \quad , \quad G'(\tau') = \left(\frac{\partial \tau}{\partial \tau'}\right)^2 G(\tau)$$

So ds^2 is invariant.

With this metric, a simple natural action for $X^\mu(\tau)$ is:

$$S_1 = \frac{1}{2}\int d\tau \sqrt{G} \, . \, G^{-1} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} g_{\mu\nu}(x)$$

$$= \frac{1}{2}\int \frac{d\tau}{\sqrt{G}} \, g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu$$

Under $d\tau \to d\tau' = \frac{d\tau'}{d\tau} d\tau$

$$\frac{dx^\mu}{d\tau} \to \frac{dx^\mu}{d\tau'} = \frac{dx^\mu}{d\tau}\frac{d\tau}{d\tau'}$$

$$G(\tau) \to G'(\tau') = \left(\frac{d\tau}{d\tau'}\right)^2 G(\tau)$$

the action above is invariant:

$$\frac{1}{2}\int \frac{d\tau'}{\sqrt{G'}} \frac{dx^\mu}{d\tau'}\frac{dx^\nu}{d\tau'} g_{\mu\nu}(x) = \frac{1}{2}\int d\tau \cancel{\frac{d\tau'}{d\tau}} \cdot \frac{1}{\sqrt{G}} \cancel{\frac{d\tau'}{d\tau}} \frac{dx^\mu}{d\tau}\cancel{\frac{d\tau}{d\tau'}} \times \frac{dx^\nu}{d\tau}\cancel{\frac{d\tau}{d\tau'}} g_{\mu\nu}(x)$$

$$= \frac{1}{2}\int d\tau \, g_{\mu\nu}(x) \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

Note that $x^\mu(\tau)$ is itself "scalar" under the worldline reparametrisations:

$$x^{\mu\prime}(\tau') = x^\mu(\tau) .$$

Now to the above action we could think of adding a "miniature Einstein term" $\sqrt{G}R$, but in one dimension the Riemann tensor $R_{\tau\tau\tau\tau} = 0$ by antisymmetry.

However a "cosmological term" $\sim \int \sqrt{G}\, d\tau$ is allow

So consider $S_2 = -\frac{m^2}{2}\int d\tau \sqrt{G}$

where $\frac{m^2}{2}$ plays the role of the worldline cosmological constant.

Thus: $S = S_1 + S_2$

$$= \frac{1}{2}\int \frac{d\tau}{\sqrt{G}}\, g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu - \frac{m^2}{2}\int d\tau \sqrt{G}$$

The equation of motion of $G(\tau)$ is:

$$\frac{\delta S}{\delta G(\tau)} = 0 \Rightarrow \frac{1}{G(\tau)}\, g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu + m^2 = 0$$

while the equation of motion of x^μ is:

$$\frac{d}{d\tau}\left(\frac{1}{\sqrt{G}} g_{\mu\nu}\dot{x}^\nu\right) = \frac{1}{2\sqrt{G}}\, g_{\alpha\beta,\mu}\, \dot{x}^\alpha \dot{x}^\beta$$

$$\Rightarrow \quad -\frac{1}{2} G^{-3/2}\, \dot{G}\, g_{\mu\nu}\dot{x}^\nu + \frac{1}{\sqrt{G}} g_{\mu\nu,\alpha}\, \dot{x}^\nu \dot{x}^\alpha$$

$$+ \frac{1}{\sqrt{G}} g_{\mu\nu}\ddot{x}^\nu = \frac{1}{2\sqrt{G}}\, g_{\alpha\beta,\mu}\, \dot{x}^\alpha \dot{x}^\beta$$

Multiplying by $\sqrt{G}$ and $g^{\lambda\mu}$ we have:

$$\ddot{x}^\lambda + g^{\lambda\mu}\left(g_{\mu\alpha,\beta} - \frac{1}{2} g_{\alpha\beta,\mu}\right)\dot{x}^\alpha \dot{x}^\beta$$

$$= \frac{1}{2}\frac{\dot{G}}{G}\, \dot{x}^\lambda$$

ie $$\ddot{x}^\lambda + \Gamma^\lambda_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta = \frac{1}{2}\frac{\dot{G}}{G}\, \dot{x}^\lambda$$

Notice that $\Gamma^{\tau}_{\tau\tau} = \frac{1}{2} G^{-1} (G_{,\tau} + G_{,\tau} - G_{,\tau})$

$$= \frac{1}{2} \frac{\dot{G}}{G}$$

So

$$\ddot{x}^{\lambda} + \Gamma^{\lambda}_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta} - \Gamma \dot{x}^{\lambda} = 0$$

ie $D_{\tau}(\dot{x}^{\lambda}) = 0$ in the presence of a worldline metric.

Now we can physically interpret the system:

$$\frac{1}{G(\tau)} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} + m^2 = 0$$

$$\ddot{x}^{\lambda} + \Gamma^{\lambda}_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta} - \Gamma(\tau) \dot{x}^{\lambda} = 0$$

Note that the canonical momentum is

$$p_{\mu} = \frac{\partial L}{\partial \dot{x}^{\mu}} = \frac{1}{\sqrt{G}} g_{\mu\nu} \dot{x}^{\nu}$$

Thus $\frac{1}{G} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} + m^2 = g^{\mu\nu} p_{\mu} p_{\nu} + m^2 = 0$

So we have recovered $p^2 = -m^2$.

Alternatively we can solve the eqn for G and eliminate it:

$$G = -\frac{g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}}{m^2}$$

Substituting back in the action, we get:

$$S = \frac{m}{2}\int \frac{d\tau\, g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}} - \frac{m^2}{2}\int d\tau \frac{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}{m}$$

$$= -m\int d\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

which was our original action!

But we now see that <u>this</u> works only for $m \neq 0$

For $m = 0$ we have the equations:

$$\frac{1}{G} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

$$\ddot{x}^\lambda + \Gamma^\lambda_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta - \Gamma(\tau)\dot{x}^\lambda = 0$$

Since $G(\tau) \neq 0$ (non-singular nature of metric) we have:

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$$
$$\ddot{x}^\lambda + \Gamma^\lambda_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta - \Gamma(\tau)\dot{x}^\lambda = 0$$

We still have $p_\mu = \frac{1}{\sqrt{G}} g_{\mu\nu}\dot{x}^\nu$ so $g^{\mu\nu}p_\mu p_\nu = 0$.

Also, in terms of p_μ the geodesic equation becomes

$$\frac{dp^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu p^\lambda = 0$$

The presence of G in the relation between p^μ and $\dot{x}^\mu$ is inconvenient, but we can eliminate it by using the $\tau \to \tau'$ ~~sym~~ invariance to fix $G(\tau) = 1$. Alternatively it could be any constant, say $\frac{1}{\alpha^2}$, then $p^\mu = \alpha \dot{x}^\mu$ for any α.

To summarise, the motion of a massless particle is governed by the equations:

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$
$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$$

$\Big\}$ τ is a parameter chosen such that $G(\tau) = 1$ (it is not the proper time!)

Then $p^\mu = \dot{x}^\mu$ and $p^\mu p_\mu = 0$.

For a massive particle we have two basic types of choices:

(i) Set $G(\tau) = 1$ again, then

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + m^2 = 0$$
$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$$

Then $p^\mu = \dot{x}^\mu$ again, and $p_\mu p^\mu = -m^2$.

or (ii) eliminate $G(\tau) = -\dfrac{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{m^2}$

Now $\dot{x}^\mu$ has no constraint on it, and
$p^\mu = \dfrac{m\dot{x}^\mu}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}$ so $p^\mu p_\mu = -m^2$.

In this picture, $u^\mu \equiv \dfrac{p^\mu}{m} = \dfrac{\dot{x}^\mu}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}$

Note that (i) and (ii) agree for a special choice: $G(\tau) = \dfrac{1}{m^2}$. In this case (i) gives

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + 1 = 0$$

so τ is indeed the proper time. In (ii), just choose τ to be the proper time, then $p^\mu = m\dot{x}^\mu$.

Now we can return to the study of (massless) particle orbits in the Schwarzschild metric.

We have $p^\mu = \dot{x}^\mu$ and $\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} = 0$.

Then E (conserved energy) $= -K_\mu p^\mu = -K_\mu \dot{x}^\mu$

$$= \left(1 - \frac{2GM}{r}\right) u^t$$

And L (conserved angular momentum) $= R_\mu \frac{dx^\mu}{d\tau}$

$$= r^2 u^\varphi$$

The equation $\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} = 0$ now gives:

$$-\left(1 - \frac{2GM}{r}\right)(u^t)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}(u^r)^2 + r^2(u^\varphi)^2 = 0$$

Multiply by $1 - \frac{2GM}{r}$ $\rightarrow$

$$-E^2 + \dot{r}^2 + \frac{L^2}{r^2}\left(1 - \frac{2GM}{r}\right) = 0$$

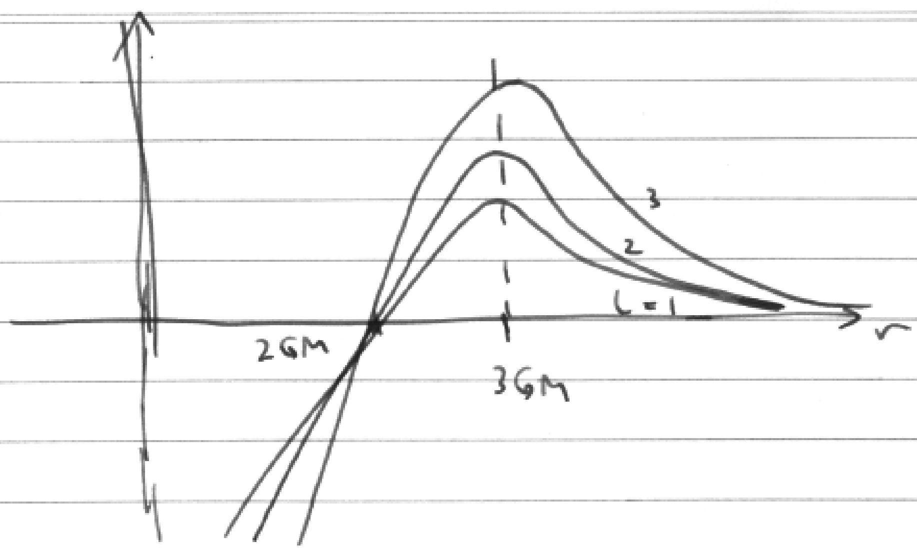
$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{E^2}{2} = \mathcal{E}$$

where $V(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$ so $V(r) = 0$ at $r = 2GM$

Circular orbits occur when $\frac{dV}{dr} = 0$

so $$-\frac{L^2}{r^3} + 3\frac{GML^2}{r^4} = 0$$

$\rightarrow$ $r_c = 3GM$ independent of L !



We see that the circular orbits are <u>unstable</u>.

The height of the barrier is

$$V(r = 3GM) = \frac{1}{54}\frac{L^2}{(GM)^2}$$

Thus classically a photon will penetrate the barrier if

$$\mathcal{E} \gtrsim \frac{L^2}{(GM)^2}$$

ie $E \gtrsim \frac{L}{GM}$. $\left(\text{precisely: } \frac{E^2}{2} > \frac{1}{54}\frac{L^2}{(GM)^2} \Rightarrow b < 3\sqrt{3}\,GM\right)$

Note that $b^{-1} = \frac{E}{L}$ has dimensions of (<u>length</u>)$^{-1}$.

b, having dimensions of length, is called the <u>impact</u> <u>parameter</u>. Thus small b ($b \lesssim GM$) is sufficient to cause the photon to fall in. In other words overcoming the potential barrier is done by <u>aiming</u> the particle correctly. If b is too large, merely increasing E just increases L and does not cause radial plunge.
Capture cross-section for light: $\sigma = \pi b^2 = 27\pi M^2$.

Let us now work out the bending of light in the Schwarzschild metric. Suppose we have solved for a radial trajectory $r(\tau)$ from

$$\frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E}, \quad V(r) = \frac{L^2}{2r^3}(r-2GM)$$

in the case where $b > 3\sqrt{3}\,GM$.

Now $L = r^2\dot{\varphi}$

so $\dot{\varphi} = \dfrac{L}{r^2}$

Now $$\frac{\partial\varphi}{\partial r} = \frac{\dot{\varphi}}{\dot{r}} = \frac{L}{r^2}\cdot\frac{1}{\sqrt{2(\mathcal{E}-V(r))}}$$

$$= \frac{L}{r^2}\frac{1}{\left(E^2 - \frac{L^2}{r^3}(r-2GM)\right)^{1/2}} = \frac{1}{r^2\left(\frac{1}{b^2} - \frac{1}{r^3}(r-2GM)\right)^{1/2}}$$

$$= \frac{1}{\left(r^4/b^2 - r(r-2GM)\right)^{1/2}}$$

The turning point of the radial function $r(\tau)$ occurs when $V(r) = \mathcal{E}$, ie

$$\frac{L^2}{2r_0^3}(r_0 - 2GM) = \frac{E^2}{2}$$

ie $$b^2(r_0 - 2GM) = r_0^3$$

Now the deviation $$\Delta\varphi = \int_{-\infty}^{+\infty} dr\,\frac{\partial\varphi}{\partial r}$$

$$= 2\int_{r_0}^{\infty} dr\,\frac{\partial\varphi}{\partial r} = 2\int_{r_0}^{\infty}\frac{dr}{\left(\frac{r^4}{b^2} - r(r-2GM)\right)^{1/2}}$$

So $$\Delta\varphi = 2\int_0^{\frac{1}{r_0}} \frac{du}{\left(\frac{1}{b^2} - u^2 + 2GMu^3\right)^{1/2}} \qquad \left(u = \frac{1}{r}\right)$$

Note that $u = \frac{1}{r_0}$ is precisely a root of the denominator.

As a check, if $M = 0$ then $r_0 = b$ and

$$\Delta\varphi = 2\int_0^{\frac{1}{b}} \frac{du}{\sqrt{\frac{1}{b} - u^2}} = 2\int \frac{dv}{\sqrt{1-v^2}} = \pi \text{ (indep of } b)$$

Now let's evaluate $\Delta\varphi$ for small M.
Sending $u = \frac{1}{b} v$,

$$\Delta\varphi = 2\int_0^{\frac{b}{r_0}} \frac{dv}{\left(1 - \frac{2GM}{b}v\right)^{1/2}} \frac{1}{\left(\frac{1}{1-\frac{2GM}{b}v} - v^2\right)^{1/2}}$$

$$\simeq 2\int_0^{\frac{b}{r_0}} dv \left(1 + \frac{GM}{b}v\right) \frac{1}{\left(1 + \frac{2GM}{b}v - v^2\right)^{1/2}}$$

and r_0 is defined so that $v = \frac{b}{r_0}$ is still a root of the denominator.

The integral is $\sim \frac{\pi}{2} + \frac{2GM}{b}$ so $\Delta\varphi - \pi = \frac{4GM}{b}$

For M = mass of the sun and b = diameter of the sun, $\Delta\varphi - \pi = 1.75$ seconds of arc.