Lecture 14 13/10/09

Schwarzschild black holes.

As we have seen,

$$ds^2 = -\left(1 - \frac{GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{GM}{r}} + r^2 d\Omega_2^2$$

describes the spacetime outside any spherically symmetric body as long as $r > \tilde{r}$, ~~we~~ where $\tilde{r}$ is the radius of the matter distribution.

Normally $\tilde{r} \gg GM$, but now we will assume that $\tilde{r} < GM$. Then we are entitled to study the behaviour of this metric for $\tilde{r} < r < GM$.

Earlier in this course, we saw (lecture 5) that Rindler spacetime

$$ds^2 = -x^2 dt^2 + dx^2$$

appears singular at $x = 0$ because the metric develops a zero eigenvalue there. However we exhibited a coordinate transformation that maps Rindler space to a region of ordinary Minkowski space:

$$t = \tanh^{-1} \frac{t'}{x'}, \qquad x = \sqrt{x'^2 - t'^2}$$

Thus there is no singularity at $x = 0$. The only problem is that x, t coordinates describe the $x'^2 > t'^2$ region of Minkowski space i.e. $-x' \leq t' \leq x'$.

Evidently the apparent singularity in (x, t) coordinates arose only because these fail to cover all of Minkowski space.

There is a coordinate-invariant diagnostic of whether a singularity in certain coordinates is only apparent (ie due to the coordinates not covering the whole space) or is more genuine. Consider computing R for Rindler space. If we did this in (x, t) coordinates we would, after some work, discover that $R = 0$. In the (x', t') coordinates that fact is obvious! Since R is a scalar, $R = 0$ (or even just finite) in any one coord. system implies it is 0 (or finite) in any coord system.

In 2d, R determines $R_{\mu\nu\lambda\rho} = \frac{R}{2}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})$

However in 4d, an appropriate scalar is $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$. If this <u>does</u> diverge then the spacetime is singular and the singularity cannot be removed by a coordinate choice. However when curvature invariants are finite, it is less clear that the singularity <u>can</u> be removed by a coordinate choice and if so, how to find the right coordinates.

In 2d there <u>is</u> a general method. Notice that space is 1-d. In this case null geodesics can be readily found by solving

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

Digression: For a static metric in 2d, the null condition implies the geodesic equations.

Proof: $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \rightarrow g_{tt} \dot{t}^2 + g_{xx} \dot{x}^2 = 0$

Now $E = -g_{tt} \dot{t}$

So $g_{tt,\tau} \dot{t} + g_{tt} \ddot{t} = 0$

$g_{tt,x} \dot{x} \dot{t} + g_{tt} \ddot{t} = 0$

$\rightarrow \ddot{t} + \underbrace{g^{tt} g_{tt,x}}_{= 2\Gamma^t_{tx}} \dot{x} \dot{t} = 0$

$\ddot{t} + 2\Gamma^t_{tx} \dot{t} \dot{x} = 0$

Note that $\Gamma^t_{tt} = 0$ & and $\Gamma^t_{xx} = 0$ (due to static condition)

$\therefore$ The t geodesic eqn is satisfied.

Now $D_\tau (g_{tt} \dot{t}^2 + g_{xx} \dot{x}^2) = 0$

$= 2 g_{tt} \dot{t} (D_\tau \dot{t}) + 2 g_{xx} \dot{x} D_\tau \dot{x}$

Since $D_\tau \dot{t}$ is zero, we have $D_\tau \dot{x} = 0$ as well.

~~That~~ The null condition allows us to eliminate τ and find $\frac{dx}{dt}$ where $(t, x) = x^\mu$.

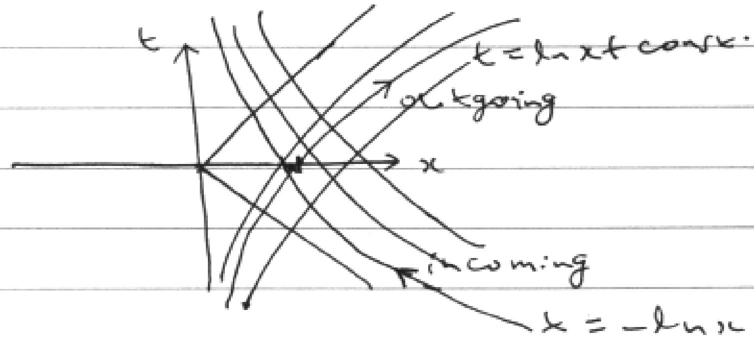
eg in Rindler spacetime,

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -x^2 \dot{t}^2 + \dot{x}^2 = 0$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = x^2 \Rightarrow \frac{dx}{dt} = \pm x$$

$$\Rightarrow \therefore t = \pm \ln x + \text{const} \quad \text{or} \quad x = e^{\pm(t - \text{const})}$$

In one of the solns, x increases with t (+ sign → outgoing geodesic) and in the other it decreases (− sign → incoming geodesic).



Next we choose these geodesics to be lines of constant u, v where u, v are new coordinates, i.e.

$$u = t - \ln x \qquad \rightarrow \qquad x = e^{\frac{v-u}{2}}$$
$$v = t + \ln x \qquad\qquad\quad t = \frac{u+v}{2}$$

Now the metric is $ds^2 = -e^{v-u}\,du\,dv$ reflecting the fact that the coordinates are "null" (no du^2 or dv^2 in the metric).

Next, the Rindler metric is static and has a timelike Killing vector $\frac{\partial}{\partial t}$. So the conserved energy is

$$E = -g_{\mu\mu} K_\mu \dot{x}^\mu = -g_{\mu\nu} K^\nu \dot{x}^\mu$$

$$= x^2 \frac{dt}{d\tau} \quad (\tau \text{ is the affine parameter})$$

Thus $\tau = \frac{1}{E} \int x^2 dt + \text{const}$

Along the u = constant geodesic (outgoing) we have

$$\tau = \frac{1}{E} \int e^{v-u} \frac{dv}{2}$$

$$= \frac{e^{v-u}}{2E} + \text{const.}$$

Since u is constant, this means $\tau_{in} \sim e^{v}$ is an affine parameter along the outgoing geodesic. Similarly $\tau_{out} \sim -e^{-u}$ is an affine parameter along the incoming geodesic.

This leads us to choose new coordinates

$$U = -e^{-u}, \quad V = e^{v} \quad \text{and} \quad \boxed{ds^2 = -dU dV}$$

(Finally, $U = T - X$, $V = T + X$ $\to$ Minkowski metric)

Now in the original Rindler spacetime,

$$-\infty < t < \infty, \quad 0 < x < \infty$$

$$\Rightarrow \quad U = -e^{-u} = -xe^{-t} < 0$$

$$V = e^{v} = xe^{t} > 0$$

At this stage, we ~~simply~~ have no singularity in ds^2 at the previously "bad" point $x = 0$ $(U = V = 0)$. Now we declare that $-\infty < U < \infty$, $-\infty < V < \infty$ and end up with Minkowski space time.

Let us now apply these ideas to the Schwarzschild geometry and try to understand if $r = 2GM$ is a "coordinate" singularity or whether this singularity can be removed.

~~Suppose~~ We work with the "2d part" of the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}}$$

(Note that this trivially solves $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$ in 2d!, since in 2d, $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$).

We repeat the procedure followed above for Rindler. Null geodesics are given by:

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 = -\left(1 - \frac{2GM}{r}\right)\dot{t}^2 + \frac{1}{1 - \frac{2GM}{r}} \dot{r}^2$$

So $$\frac{dt}{dr} = \pm \frac{1}{1 - \frac{2GM}{r}}$$

So $$t = \pm \int \frac{dr\, r}{r - 2GM} + \text{const}$$

$$= \pm \int dr \left(1 + \frac{2GM}{r - 2GM}\right) + \text{const}$$

$$= \pm \left(r + 2GM \ln\left(\frac{r}{2GM} - 1\right)\right) + \text{const}.$$

Define $r_* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$

this is called the "tortoise coordinate", because

$$\frac{dr_*}{dr} = \frac{1}{1 - \frac{2GM}{r}} \quad \text{so as } r \to 2GM, \quad \frac{dr_*}{dr} \to \infty$$

ie r_* varies rapidly for very small variation in r. Conversely for fixed small variation of r_* (when it is large and $-$ve), r appears to freeze in its approach to $2GM$.

Now as before, define null coordinates

$$u = t - r_*$$
$$v = t + r_*$$

then, $ds^2 = -\left(1 - \frac{2GM}{r}\right) du\, dv$

where r solves $r + 2GM \ln\left(\frac{r}{2GM} - 1\right) = \frac{v-u}{2}$
ie $r = r(v,u)$

Now $\ln\left(\frac{r}{2GM} - 1\right) = \dfrac{\frac{v-u}{2} - r}{2GM}$

So $\frac{r}{2GM} - 1 = e^{\frac{v-u}{4GM}} e^{-\frac{r}{2GM}}$

$$1 - \frac{2GM}{r} = \frac{2GM}{r} e^{\frac{v-u}{4GM}} e^{-\frac{r}{2GM}}$$

So $ds^2 = -\frac{2GM}{r} e^{-\frac{r}{2GM}} e^{\frac{v-u}{4GM}} du\, dv$

now the ~~constant~~ factor $\frac{2GM}{r} e^{-\frac{r}{2GM}}$ is perfectly well-behaved as $r \to 2GM$.

The remaining part $-e^{\frac{v-u}{4GM}} du\,dv$ suggests, as in the Rindler case, the transformation

$$U = -e^{-u/4GM} \qquad V = e^{v/4GM}$$

and $$ds^2 = -32(GM)^3 \frac{e^{-r/2GM}}{r} dU\,dV$$

As $r \to 2GM$, $r^* \to -\infty$; $u \to +\infty$, $v \to -\infty$; $U \to 0$, $V \to 0$. So we can <u>extend</u> the spacetime by allowing U, V to take all values such that the original r is > 0.

Finally, ~~transform~~ let $U = T - X$
$V = T + X$

so $$ds^2 = \frac{32(GM)^3 e^{-\frac{r}{2GM}}}{r} \left(-dT^2 + dX^2\right)$$

and we can restore $+ r^2 d\Omega_2^2$.

Here $r = r(T, X)$ is defined by

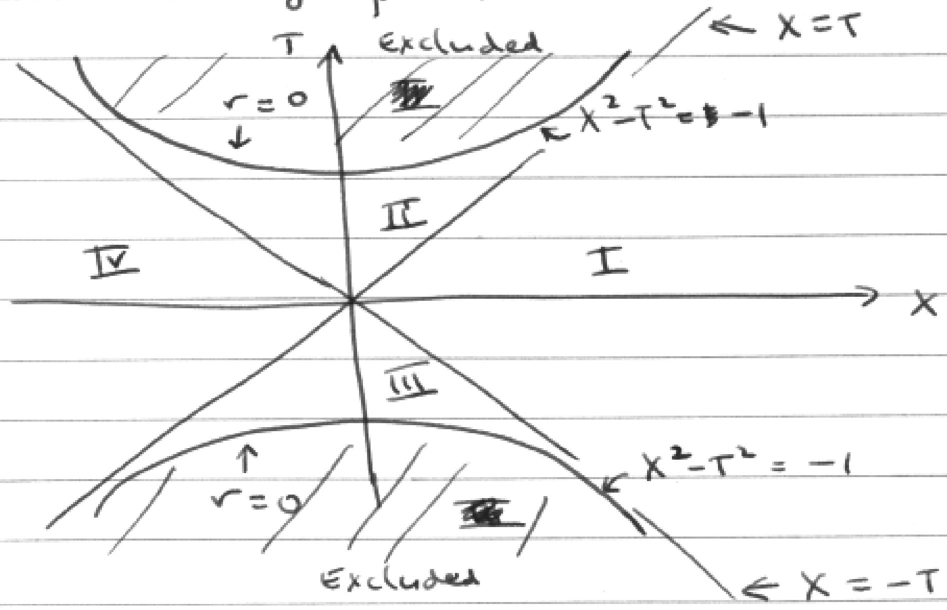
$$\left(\frac{r}{2GM} - 1\right) e^{r/2GM} = X^2 - T^2 \qquad \text{so } r = 2GM \Rightarrow X = \pm T$$

and $$t = \frac{u+v}{2} = -2GM \log(-U) + 2GM \log V$$

$$= 2GM \log \frac{X+T}{X-T} = 4GM \tanh^{-1} \frac{T}{X}$$

The allowed range of X, T (the Kruskal-Szekeres coordinates) is such that $r(X,T) > 0$.
Now $r = 0 \Rightarrow X^2 - T^2 = -1$ and LHS is monotonic so $r > 0 \Rightarrow X^2 - T^2 > -1$.

By allowing all X, T such that $r > 0$, we get the following picture:



The original region $r > 2GM$ is region I. One can see that this region one covers all of $r > 2GM$. Region IV is identical but is a causally disconnected region.

As we will see, region II is a "black hole" from which no signal or observer can escape. It has the singularity at $r = 0$ in its future. Region III is a time-reversed "white hole" from which every signal must leave in finite time.