Lecture 15 15/10/09

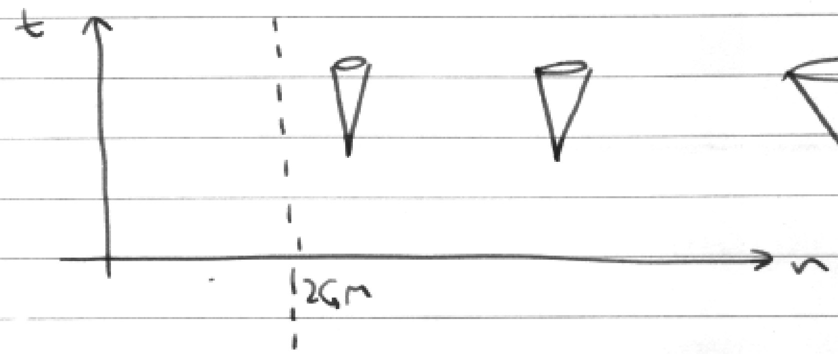
Properties of the Schwarzschild black hole.

(i) the exterior $r > 2GM$ is a spacetime that, for each fixed value of r, has a time direction (t) and two space directions (θ, ϕ). We cannot claim this to be true at $r = 2GM$ because this locus is singular in Schwarzschild coordinates. However in Kruskal coordinates we see that $r = 2GM$ lies at $X = \pm T$ which is <u>null</u>. (A null hypersurface is one whose normal vector is null, or equivalently which has one out of its three tangent vectors null)

(ii) Let us see how this arises. In Schwarzschild coordinates we have seen that

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}$$

This tells us that in these coordinates, the "slope" of the light cone tends to 1 as $r \to \infty$ and tends to ∞ as $r \to 2GM$:



Thus the light cones (in these coords) "close up" as we approach $r \to 2GM$.

(iii) This is related to the increasing <u>redshift</u> as we approach $r = 2GM$. An infalling observer measures time intervals in her proper time, $\Delta\tau_{in}$. A stationary observer measures intervals in coordinate time ΔT_∞. The relation is

$$\cancel{\Delta T_{\infty}(r) = \Delta T_\infty}$$

$$\Delta t = \Delta T_\infty = \frac{1}{\sqrt{1 - \frac{2GM}{r}}} \Delta\tau_{in}(r)$$

(this follows from the relation $\Delta\tau = \sqrt{-g_{00}}\, \Delta t$ that we derived in an earlier lecture)
Thus if the infalling observer emits signals at regular intervals $\Delta\tau_{in}(r)$, the observer at infinity sees them spaced by larger and larger gaps, tending to ∞ as $r \to 2GM$. In this sense the observer at infinity never sees the infalling observer <u>reach</u> $r = 2GM$.

(iv) ~~Now~~ ~~some~~ From the infalling observer's point of view nothing unusual happens until $r = 2GM$. What happens after that cannot be asked in (r, t) coordinates. But it <u>can</u> be asked in Kruskal coordinates. And here we see that $r = 2GM$ (ie $T = \pm X$) is regular in the geometry.

(v) This fact can be better understood in a hybrid system of coordinates called Eddington–Finkelstein coordinates.

The idea here is to retain r as a coordinate but replace the time by $v = t + r^*$. We do not introduce $u = t - r^*$, but just use the pair (r, v) as coordinates. Clearly this gives ingoing geodesics ($v =$ const) a privileged role.

Thus, take $ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}}$

and let $v = t + r_*$ where

$$r_* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

Then $dr_* = \frac{dr}{1 - \frac{2GM}{r}}$

and $t = v - r_*$, so

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\left(dv - \frac{dr}{1 - \frac{2GM}{r}}\right)^2 + \frac{dr^2}{1 - \frac{2GM}{r}}$$

$$= -\left(1 - \frac{2GM}{r}\right) dv^2 + 2\,dv\,dr$$

$\rightarrow$ v is the new "time"

In this form of the metric, the null geodesics are:

$$\frac{dv}{dr} = 0 \qquad \text{(ingoing)}$$

$$\frac{dv}{dr} = 2\left(1 - \frac{2GM}{r}\right)^{-1} \qquad \text{(outgoing)}$$

So, the ingoing null geodesics are trivial: $v =$ const. (as expected)

As with Kruskal coordinates, the E-F coords are non-singular at $r = 2GM$. However they exhibit a dramatic property of the spacetime:

Outgoing null geodesics flip sign as $r < 2GM$, and $\frac{dv}{dr}$ becomes < 0. This means that

for $r < 2GM$, as v ~~[illegible]~~ increases, r decreases !

Thus we see that no light ray (and in fact no test particle) can ever escape ~~from~~ once it is inside $r = 2GM$. Hence this (null) surface is called the event horizon and the spacetime is called a black hole.

(vi) If we are safely in the region $r < 2GM$, then r is a timelike coordinate. We actually know this in the original Schwarzschild coordinates:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}}$$

$$\overset{(\text{for } r<2GM)}{=} \left|1 - \frac{2GM}{r}\right| dt^2 - \frac{dr^2}{\left|1 - \frac{2GM}{r}\right|}$$

In EF coordinates we see this as:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2\,dv\,dr$$

$$\overset{r<2GM}{=} \left|1 - \frac{2GM}{r}\right| dv^2 + 2\,dv\,dr \qquad \longrightarrow$$

$$= \left|1-\frac{2GM}{r}\right| \Big(\underbrace{dv + \frac{dr}{\left|1-\frac{2GM}{r}\right|}}_{dv'} \Big)^2 - \frac{dr^2}{\left|1-\frac{2GM}{r}\right|^2}$$

So again r is timelike.

(vi) The proper time for a particle at $r < 2GM$ to fall into $r = 0$ is finite:

Consider a massive particle: then $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$

$$\Rightarrow \quad -\left(1-\frac{2GM}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{2GM}{r}} = -1$$

For $r < 2GM$ this is:

$$\left|1-\frac{2GM}{r}\right| \dot{t}^2 - \frac{\dot{r}^2}{\left|1-\frac{2GM}{r}\right|} = -1$$

or $$\dot{r}^2 = \left|1-\frac{2GM}{r}\right| + \left|1-\frac{2GM}{r}\right|^2 \dot{t}^2$$

$$> \left|1-\frac{2GM}{r}\right|$$

$$\therefore \quad \left|\frac{dr}{d\tau}\right| > \sqrt{\frac{2GM}{r}-1}$$

So $$\tau_{max} = \int_0^{2GM} \frac{dr}{\sqrt{\frac{2GM}{r}-1}} = \pi GM$$

This means that any object falls to the centre in proper time πGM or less.
At the centre the falling object encounters infinite curvature, so arbitrarily strong gravitational field!

(vii) ~~Pro~~ Our study of null geodesics basically shows that $r = 2GM$ is non-singular while suggesting that $r = 0$ is singular (that singularity is present in all coordinate systems we've used).
We have computed all components of $R_{\mu\nu\lambda\rho}$ in terms of α and β where

$$e^{2\alpha} = e^{-2\beta} = \left(1 - \frac{2GM}{r}\right)$$

Therefore it is straightforward, though tedious, to show that

$$R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = \frac{48(GM)^2}{r^6}$$

which confirms the presence of a singularity at $r=0$ while nothing happens for $r > 0$. In fact at $r = 2GM$,

$$R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = \frac{3}{4(GM)^4}$$

which is of order 1 in units of $\frac{1}{(GM)^4}$ (length^4)

(viii) The horizon has an <u>area</u> associated to it. This is just the area of the 2-sphere spanned by the θ and φ coordinates. Now the metric for these coordinates is the usual one $ds^2 = r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$ and we know such a sphere has area $4\pi r^2$. On the horizon, $r = 2GM$ so

$$A = 16\pi (GM)^2 .$$

(ix) One might worry about the following. In Eddington-Finkelstein coordinates (v, r) we found that the null geodesics are:

$$\frac{dv}{dr} = 0 \quad \text{(ingoing)}$$

$v = t + r_* \Rightarrow$ as t increases, r decreases

$$\frac{dv}{dr} = 2\left(1 - \frac{2GM}{r}\right)^{-1} \quad \text{(outgoing).}$$

Then we realized that for $r > 2GM$, $\frac{dv}{dr}$ turns over and r decreases with increasing v.

Now consider instead the (u, r) E-F coords: $u = t - r_*$. Then:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - 2\, du\, dr$$

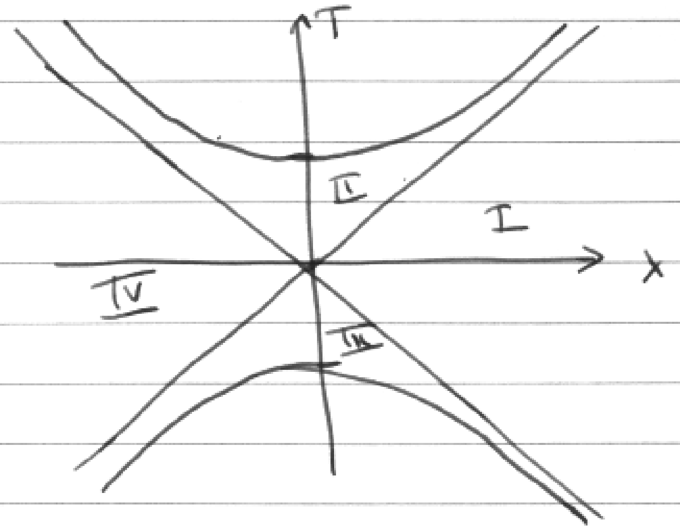
This time, $$\frac{du}{dr} = -2\left(1 - \frac{2GM}{r}\right) \quad \text{(ingoing)}$$

$$\frac{du}{dr} = 0 \quad \text{(outgoing)}$$

$u = t - r_*$ $\Rightarrow$

For $r > 2GM$, $\frac{du}{dr} < 0$ means r decreases as u increases. But for $r < 2GM$, evidently r increases with u. What can this mean?

To understand this, note the following. Recall the maximal Kruskal extension:

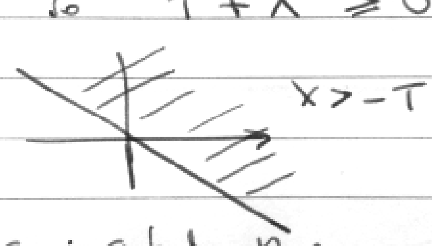


The Schwarzschild coordinates (r, t) cover only region I, while the Kruskal coordinates X, T cover all of I, II, III, IV.

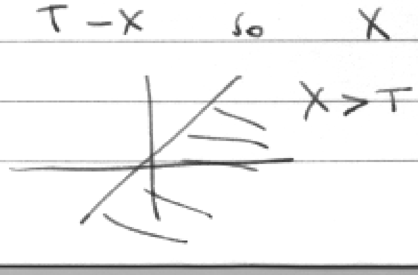
However the Eddington–Finkelstein coordinates, being intermediate, work as follows:

(v, r) coords cover I + II only
(u, r) coords cover I + III only

Proof. If v is meaningful then $V = e^{v/4GM} \geq 0$
But $V = T + X$, so $T + X \geq 0$ ie $X \geq -T$
which is this:



While if u is meaningful then $U = -e^{-u/4GM} \leq 0$
But $U = T - X$ so $X > T$:



Therefore the incoming/outgoing geodesics in (u, r) coordinates are really <u>past-directed</u> and communicating with region $\underline{\text{III}}$. That is also a region with $r < 2GM$ but distinct from region $\underline{\text{II}}$ because it has a singularity in the <u>past</u>.