Lecture 16 20/10/09

Having studied the Schwarzschild solution as a vacuum solution of the Einstein eqns, we would now like to consider solutions in the presence of matter. Thus we will be studying solns of

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

For this we need some preliminaries.

We have seen that for a scalar field,

$$T_{\mu\nu} \sim \partial_\mu \varphi \partial_\nu \varphi - \ldots$$

So in general it is made out of field operators. In this way we get coupled equations among the gravitational field and other matter fields.

However, we may instead choose a given and particularly simple $T_{\mu\nu}$ corresponding to some matter distribution.

We saw earlier that $T_{00} \sim \rho$, the matter density. Now let us make this more precise.

In flat spacetime we have $\partial_\mu T^{\mu\nu} = 0$

$$\rightarrow \quad \partial_0 T^{0\nu} + \partial_i T^{i\nu} = 0$$

$\rightarrow P^\nu = \int T^{0\nu} d^3x$ is constant in time:

$$\frac{\partial}{\partial t} P^\nu = \partial_0 \int T^{0\nu} d^3x = -\int \partial_i T^{i\nu} d^3x = 0$$

$P^\nu = \int T^{0\nu} d^3x$ is the 4-momentum, so $T^{0\nu}$ is the 4-momentum density.

How did we single out the 0 component? We have a timelike Killing vector $\frac{\partial}{\partial t}$ in flat spacetime. The corresponding density is in a spatial volume.
More generally a 3-volume is specified by a timelike "normal" vector N_μ. Then

$N_\mu T^{\mu\nu}$ is interpreted as the 4-momentum density in that 3-volume.

In GR we can similarly interpret $N_\mu T^{\mu\nu}$ for any given timelike vector N_μ. However now $\partial_\mu T^{\mu\nu} \neq 0$, instead $D_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\alpha} T^{\alpha\nu} + \Gamma^\nu_{\mu\alpha} T^{\mu\alpha} = 0$. So $T^{\mu\nu}$ is not "conserved" but rather evolves in response to the spacetime metric $g_{\mu\nu}$.

In flat spacetime an important special case of $T^{\mu\nu}$ is a "perfect fluid." In such a fluid, heat conduction, viscosity, dissipation etc are negligible. They are characterised by their rest energy density ρ and isotropic pressure p, and ~~th~~ in the rest frame.:

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

Note that in such a case $\partial_\mu T^{\mu\nu} = 0$ is just

$$\partial_0 T^{00} = 0 \quad \text{(constant energy density)}$$

and $\partial_i T^{ij} = 0$ (homogeneous & isotropic pressure)

$$\rightarrow \partial_i p = 0$$

In a frame moving with velocity U^μ we must have (by Lorentz covariance)

$$T^{\mu\nu} = \alpha U^\mu U^\nu + \beta \eta^{\mu\nu}$$ where α, β are scalars.

Now if $U^\mu = (1,0,0,0)$ (rest frame) then

$$\cancel{T^{\mu\nu}}\ T^{00} = \alpha - \beta$$
$$\cancel{T^{0i}}\ T^{0i} = 0$$
$$T^{ij} = \beta$$

$\therefore$ $\beta = p$ and $\alpha = \rho + p$.

Therefore $T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p \eta^{\mu\nu}$

Now this is easily generalized to curved spacetime:

$$T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p g^{\mu\nu}$$

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Physical way of thinking about $T^{\mu\nu}$:

$T^{\mu\nu}$

$\mu = 0$: density

$\mu = i$: flux

$\nu = 0$: energy

$\nu = i$: momentum

Thus $T^{\mu\nu}$:

$$\begin{bmatrix} \text{energy density} & \text{energy flux} \\ \text{momentum density} & \text{momentum flux} \end{bmatrix}$$

Aside:

Energy-momentum of the vacuum.

It is possible (and quite natural) to add a term to the $\int \sqrt{g} R$ action which has zero-derivatives of the metric, namely

$$= -\frac{\Lambda}{8\pi G} \int d^4x \sqrt{g}$$

This is proportional to the "volume of spacetime"! The constant coefficient Λ is called the "cosmological constant". The corresponding modified Einstein eqns are:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

This can be thought of as modifying whatever energy-momentum $T_{\mu\nu}$ is present to:

$$T'_{\mu\nu} \to T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}$$

~~Suppose~~ Could we think of the extra term as corresponding to a perfect fluid? Compare

$$-\frac{\Lambda}{8\pi G} g_{\mu\nu} \text{ with } (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$$

Thus $p_{vac} = -\dfrac{\Lambda}{8\pi G}$ and $\rho_{vac} = -p_{vac} = \dfrac{\Lambda}{8\pi G}$

In this sense the cosmological constant Λ corresponds to the vacuum energy density of the universe. Experimentally, $\rho_{vac} \sim 10^{-8}$ erg/cc or $\rho_{vac} \sim (10^{-12}\ \text{GeV})^4$.

Now we turn to the study of a simple non-vacuum solution. Consider

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

where $T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$

Again we assume a spherically symmetric solution:

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega_2^2$$

With some calculation (just taking linear combs of what we did before) we find:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$\Rightarrow \quad G_{tt} = \frac{1}{r^2} e^{2\alpha - 2\beta} \left(2r\beta' - 1 + e^{2\beta} \right)$$

$$G_{rr} = \frac{1}{r^2} \left(2r\alpha' + 1 - e^{2\beta} \right)$$

$$G_{\theta\theta} = r^2 e^{-2\beta} \left(\alpha'' + (\alpha')^2 - \alpha'\beta' + \frac{1}{r}(\alpha' - \beta') \right)$$

$$G_{\varphi\varphi} = \sin^2\theta \, G_{\theta\theta}$$

For the RHS, we choose the 4-velocity U_μ to have only a timelike component:

$$U_\mu = (U_t, 0, 0, 0) \qquad \text{(rest frame of fluid)}$$

Since $u^\mu u_\mu = -1$, $g^{tt}(u_t)^2 = -1$

$\Rightarrow u_t = e^\alpha$.

Thus $T_{\mu\nu} = \begin{bmatrix} \rho e^{2\alpha} & & & \\ & p e^{2\beta} & & \\ & & p r^2 & \\ & & & p r^2 \sin^2\theta \end{bmatrix}$

Note that spherical symmetry $\Rightarrow p = p(r)$, $\rho = \rho(r)$.

Thus: Einstein's eqns $\Rightarrow$

(i) $\frac{1}{r^2} e^{-2\beta(r)}(2r\beta' - 1 + e^{2\beta(r)}) = 8\pi G \rho(r)$

(ii) $\frac{1}{r^2} e^{-2\beta(r)} \cancel{(2r\beta' - 1 + e^{2\beta})}$
$\times (2r\alpha' + 1 - e^{2\beta(r)}) = 8\pi G p(r)$

(iii) $e^{-2\beta(r)}\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{1}{r}(\alpha' - \beta')\right) = 8\pi G p(r)$

(the $G_{\phi\phi}$ eqn is the same as (iii))

Note that (i) depends only on $\beta(r)$ and $\rho(r)$, and we know that for $\rho = 0$, the soln is

$$e^{2\beta(r)} = \frac{1}{1 - \frac{2GM}{r}}.$$

~~[illegible]~~ Thus we define a fn $m(r)$ such that $e^{2\beta(r)} = \frac{1}{1 - \frac{2Gm(r)}{r}}$ solves (i).

~~In other words,~~ With this substitution,

$$\text{LHS of (i)} = \frac{1}{r^2}\left(1-\frac{2Gm(r)}{r}\right) \times$$

$$\left(2r\left[\frac{Gm'}{r}\frac{1}{1-\frac{2Gm(r)}{r}} - \frac{Gm}{r(r-2Gm(r))}\right]\right.$$

$$\left. -1+\frac{1}{1-\frac{2Gm(r)}{r}}\right)$$

$$= \frac{2}{r^2}\left(1-\frac{2Gm}{r}\right)\frac{Gm'}{r-2Gm(r)} - \frac{2Gm}{r^3}$$

$$+ \frac{1}{r^2}\left(1-1+\frac{2Gm}{r}\right)$$

$$= \frac{2Gm'}{r^2}$$

while the RHS is $8\pi G\rho$, so

$$m' = 4\pi r^2\rho(r)$$

$$\Rightarrow m(r) = \int_0^r dr' \, 4\pi r'^2 \rho(r')$$

This is the total mass enclosed in a sphere of radius r. If $\tilde{r}_0$ is the matter radius, then

$$m(\tilde{r}_0) = \int_0^{\tilde{r}_0} dr \, 4\pi r^2 \rho(r) = M \text{, the mass of the star.}$$

However there is a subtlety. While it is true that

$$M = 4\pi \int dr\, r^2 \rho(r)$$

is the total mass of the star (from Gauss's law in Newtonian approx), the total integrated energy density is different:

$$\tilde{M} = \int d^3x \sqrt{\| g_{(3)} \|}\ \rho(r)$$

where $g_{(3)\,ij} = \begin{pmatrix} e^{2\beta(r)} & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix}$

so $\sqrt{\| g_{(3)} \|} = e^{\beta} r^2 \sin\theta$

Hence $\tilde{M} = 4\pi \int dr \dfrac{r^2 \rho(r)}{\sqrt{1 - \frac{2Gm(r)}{r}}}$

and clearly $\tilde{M} > M$. (because $e^{\beta} > 0$ for a soln that is <u>not</u> a black hole!)

We interpret $\tilde{M}$ as the energy of the constituents of the star if they were non-interacting; and it is lowered to M due to the binding energy ~~Ebt~~ $E = \tilde{M} - M$.

Next the G_{rr} eqn gives:

$$\frac{1}{r^2}\left(1 - \frac{2Gm(r)}{r}\right)\left(2r\alpha' + 1 - \frac{1}{1 - \frac{2Gm(r)}{r}}\right) = 8\pi G\, p(r)$$

ie $$\alpha' = \frac{G}{r}\,\frac{(m(r) + 4\pi p(r)\, r^3)}{r - 2Gm(r)}$$

Now the $G_{\theta\theta}$ eqn gives one more relation between $\alpha(r)$ and $p(r)$. However it is a 2nd order eqn involving α''. It is more convenient to use momentum conservation.

$$D_\mu T^{\mu\nu} = 0$$

Setting $\nu = r$,

$$D_r T^{rr} + D_t T^{tr} = 0$$ (θ, φ components don't contribute).

$$\Rightarrow \quad \partial_r T^{rr} + 2\Gamma^r_{rr} T^{rr}$$

$$+ \cancel{\partial_t T^{tr}} + \Gamma^t_{tr} T^{rr} + \Gamma^r_{tt} T^{tt} = 0$$

use $T^{rr} = e^{-2\beta} p$, $T^{tt} = e^{-2\alpha}\rho$

$$\Rightarrow \quad \partial_r(e^{-2\beta} p) + 2\beta'(e^{-2\beta} p)$$

$$+ \alpha'(e^{-2\beta} p) + \alpha' e^{2(\alpha-\beta)} e^{-2\alpha}\rho = 0$$

(use: $\Gamma^r_{rr} = \beta'$, $\Gamma^t_{tr} = \alpha'$, $\Gamma^r_{tt} = \alpha' e^{2(\alpha-\beta)}$)

$$\rightarrow \quad p' = -(p+\rho)\alpha'$$

$$\boxed{p' = -(p+\rho)\,\frac{G}{r}\,\frac{(m(r) + 4\pi p(r)\, r^3)}{r - 2Gm(r)}}$$ Tolman–Oppenheimer Volkov eqn

This equation determines (in principle) $p(r)$ in terms of $\rho(r)$. Therefore it cannot determine both. For that we need an equation of state connecting ρ and p, eg: $p = K\rho^{\gamma}$.