Lecture 17

Penrose diagrams

A useful tool to understand the causal structure of various spacetimes is the Penrose diagram.

To explain the idea, we first exhibit some interesting new coordinates for Minkowski spacetime and then extract a general definition.

Minkowski spacetime is (in t, r, θ, φ coords)

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$$

Define $u = t - r$, ~~vstan~~ $v = t + r$, then

$$ds^2 = -du\,dv + \frac{1}{4}(u-v)^2 d\Omega_2^2$$

(note that $r \geq 0 \Rightarrow u \leq v$)

Notice that null geodesics in the radial direction travel along $u =$ constant or $v =$ constant.

Next, define

$$U = \tan^{-1} u$$
$$V = \tan^{-1} v.$$

then

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V}\left[-dU\,dV + \frac{\sin^2(V-U)}{4} d\Omega_2^2\right]$$

Note that $-\frac{\pi}{2} < U, V < \frac{\pi}{2}$ and also

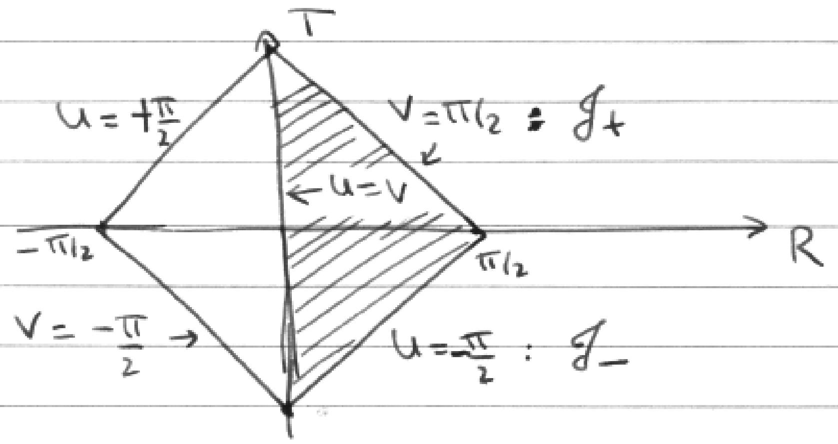
$$v \geq u \Rightarrow V \geq U.$$

Finally, define new coordinates T, R by:

$$V = T + R$$
$$U = T - R$$

Now ~~the~~ $V \geq U \Rightarrow R \geq 0$.

Thus we finally have:



Thus all of Minkowski spacetime is mapped into the <u>finite</u> region shaded above.

Let us now consider the boundaries. Clearly they represent regions "at infinity" in the original coordinates. They are characterised by various kinds of geodesics in the problem.

"Outgoing" Null geodesics with U = constant travel in the direction ↗. Thus they all terminate at $V = \pi/2$. This line is called $\mathcal{J}_+$ and denoted "future null infinity".

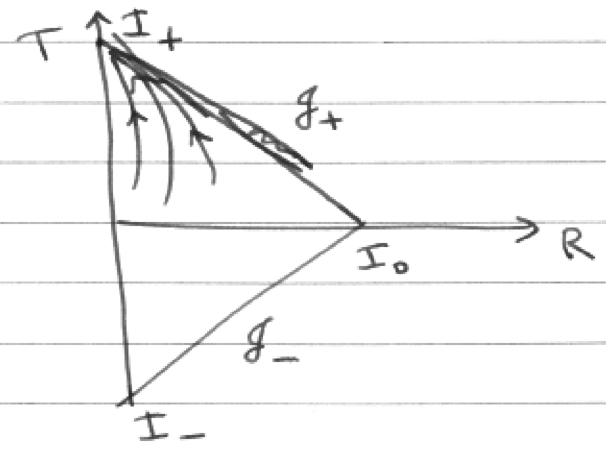
(Here "outgoing" means r increases with t in the original coordinates).

Similarly, ingoing null geodesics (r decreases with increasing t) are V equals constant, directed like this:

Such lines originate from $U = \frac{\pi}{2}$, called

$\mathcal{I}_-$ or "past null infinity".

Now consider particle trajectories for massive particles. These are timelike. A timelike geodesic is not a straight line in the new coordinates. However, being timelike, all such geodesics must curve to avoid $\mathcal{I}_+$ in the future:



Thus they all terminate at the single point I_+. Similarly they all curve to avoid $\mathcal{I}_-$ in the past and end up at I_-.

I_+, I_- are called future & past timelike infinity respectively.

Let's look at the timelike geodesics more carefully.

In u, v coords, $ds^2 = -du\,dv$

So: timelike condition is $\dot{u}\dot{v} = 1$

and geodesic is $\ddot{u} = \ddot{v} = 0$

Thus
$$u = a\tau + b$$
$$v = \frac{1}{a}\tau + c$$

or $(u-b)(v-c) = \tau^2$

or $(\tan U - b)(\tan V - c) = \tau^2$

As $\tau \to 0$, $\tan U = b$ or $\tan V = c$.
These are basically arbitrary points.

But as $\tau \to \infty$, $\tan U$ or $\tan V \to \pm\infty$. (~~same sign for both~~)

Moreover $\dfrac{\tan U - b}{\tan V - c} = a^2 = \text{const}$.

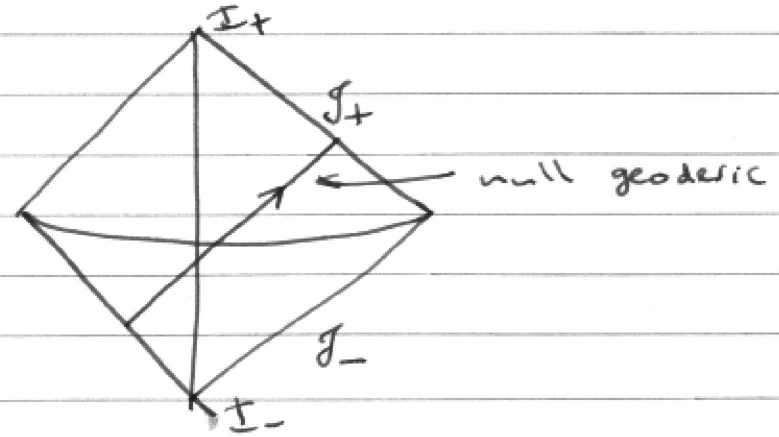
So both $\tan U$, $\tan V \to \pm\infty$ together.
(same sign for both)

Thus $T+R$, $T-R \to \pm\frac{\pi}{2}$ together.

Thus $R \to 0$, $T \to \pm\frac{\pi}{2}$ as promised!

These are the points I_-, I_+.

Now it may look strange that null geodesics originate at $\mathcal{J}_-$ and end at $\mathcal{J}_+$. However, recall that we have suppressed the θ, φ directions. If we bring back one of them, we have the structure:



Notice that a null geodesic always passes through $R=0$ but it's perfectly smooth!

Finally it's easy to see that spacelike geodesics (which don't represent the world-line of any particle) start and end at I_0 (called "spacelike infinity").

Note that all the above applies only to geodesics! Non-geodesic curves need not go from I_- to I_+ even for massive particles.

The Penrose diagram is not very useful for flat Minkowski space, but it achieves two things simultaneously:

(i) all of spacetime is mapped into a finite region, with "infinities" all visible,

ii) light-cones are always at $45°$ everywhere.

Let us now define the Penrose diagram for the Schwarzschild solution. The latter is given in Kruskal coordinates (which we now rename as U', V') by:

$$ds^2 = -32(GM)^3 \frac{e^{-r/2GM}}{r} dU' dV' + r^2 d\Omega_2^2$$

where $\left(\frac{r}{2GM} - 1\right) e^{r/2GM} = -U'V'$

Recall that $r = 2GM$ (horizon) $\rightarrow U'V' = 0$

$\rightarrow U' = 0$ or $V' = 0$

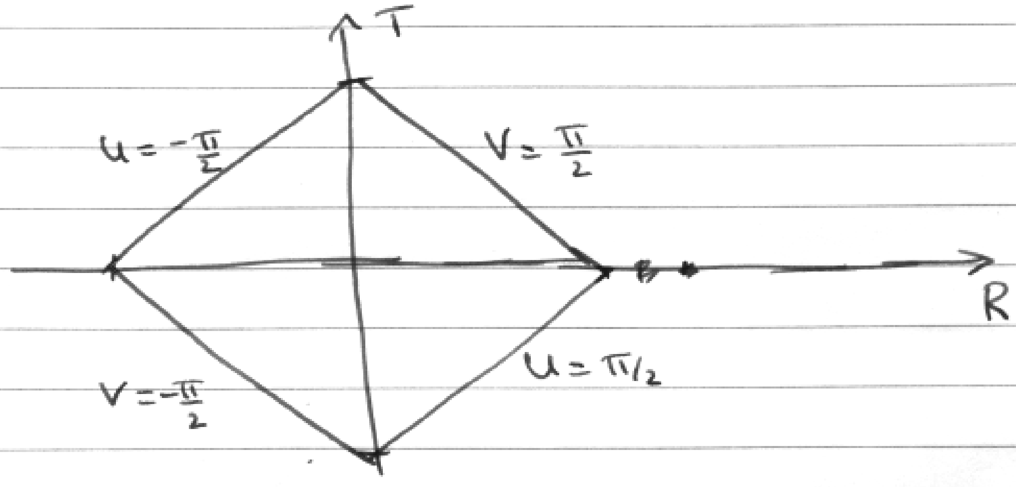
and $r \rightarrow 0$ (singularity) $\rightarrow U'V' = 1$.

The physical region $r \geq 0$ is $U'V' \leq 1$.

The Penrose coordinates are defined by:

$$U = \tan^{-1} U', \quad V = \tan^{-1} V'; \quad \begin{array}{l} U = T - R \\ V = T + R \end{array}$$

Thus the $U'V'$ plane $-\infty < U', V' < \infty$ is mapped to $-\frac{\pi}{2} < U, V < \frac{\pi}{2}$:



However there is a restriction. We have $U'V' \leq 1 \Rightarrow \tan U \tan V \leq 1$.

From the identity

$$\tan(U+V) = \frac{\tan U + \tan V}{1 - \tan U \tan V}$$

we see that the end-point $\tan U \tan V = 1$ (ie the singularity at $r=0$) becomes

$$\frac{\tan U + \tan V}{\tan(U+V)} = 0.$$

For the numerator to vanish, $\tan U + \tan V = 0$, we have $U+V=0$ and the denominator also vanishes. The ratio can be checked to be finite.

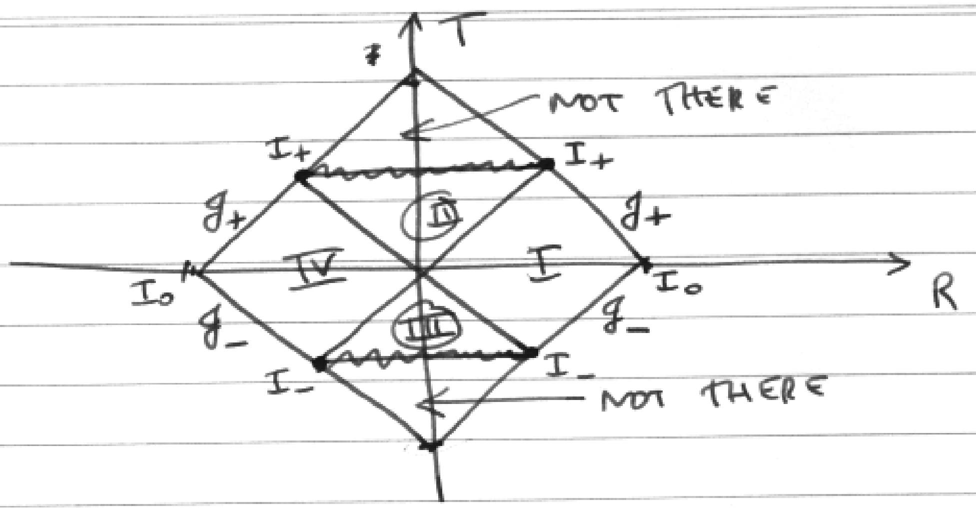
Thus we instead must have $\tan(U+V) = \pm\infty$, ie $U+V = \frac{\pi}{2}$ or $-\frac{\pi}{2}$.

In turn this means $T = \pm\frac{\pi}{4}$. Moreover

$-\frac{\pi}{4} \leq R \leq \frac{\pi}{4}$ (otherwise either ~~$U > \frac{\pi}{2}$ or $V < -\frac{\pi}{2}$~~ U or V falls outside the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$).

Thus the allowed region in the Penrose diagram is truncated to:

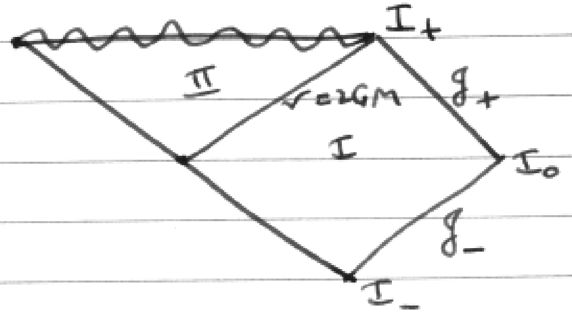
$\longrightarrow$



Now we easily identify regions I, II, III, IV of the Kruskal diagram, since

$U' > 0 \Rightarrow u > 0$ etc.

We see that in the maximally extended diagram there are two copies ~~one~~ of everything: $\mathcal{J}_+$, $\mathcal{J}_-$, I_+, I_-, I_0. However if we only keep regions I and II then we have:



The causal structure is now completely transparent.

Reissner – Nordstrom solution

A different type of spherically symmetric solution is found when $T_{\mu\nu}$ corresponds to an electric or magnetic field.

For electromagnetism one easily finds that

$$T_{\mu\nu} = F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}$$

We assume the field strength is of the form:

$$E_r = F_{rt}(r) = \frac{Q}{r^2}$$

$$B_r = \frac{F_{\theta\varphi}}{r^2 \sin\theta} = \frac{P}{r^2}$$

where the constants (Q, P) are the electric/magnetic charges.

Then, one finds

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2}\right)} + r^2 d\Omega_2^2$$

→ a nice and relatively simple generalisation of Schwarzschild!

This solution has <u>two horizons</u>, at

$$1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2} = 0$$

which is solved by

$$r = GM \pm \sqrt{(GM)^2 - G(Q^2+P^2)}$$

There can of course be no solns, if $Q^2+P^2 > GM^2$. In this case there is no horizon and the singularity at $r=0$ is timelike (rather than spacelike as for Schwarzschild. This situation is considered ~~...~~ "unphysical" as the energy of EM fields overcomes the total mass.

For $Q^2+P^2 < GM^2$ there are two horizons, $r_\pm$. They both turn out to be coordinate singularities, removable by a change of coordinates.

~~...~~ r is spacelike for $r > r_+$, then timelike for $r_- < r < r_+$, then spacelike again for $r < r_-$.

For $r_+ = r_-$ we have an "extremal" RN black hole.