Lecture 20. 3/11/09.

We have seen how to solve Einstein's equations for an evolving homogeneous, isotropic universe with scale factor $a(t)$. But what is the $a(t)$ we expect in nature & what observations does it relate to?

The FRW metric is:

$$ds^2 = -dt^2 + a(t)^2 \left(g^{(3)}_{ij}\, dx^i\, dx^j\right).$$

The radial geodesic eqn is:

$$\partial_\lambda^2 t + \Gamma^t_{\rho\rho} (\partial_\lambda \rho)^2 = 0$$

(anticipating a null geodesic, we have used an arbitrary **affine** parameter λ).

Now $g^{(3)}_{ij}\, dx^i\, dx^j = \dfrac{d\rho^2}{1-k\rho^2}$ (since $d\theta, d\phi = 0$ along a radial trajectory)

So $dt = \dfrac{a(t)\, d\rho}{\sqrt{1-k\rho^2}}$ for a null trajectory.

Also $\Gamma^t_{\rho\rho} = \dfrac{a\dot{a}}{1-k\rho^2}$, so

$$\partial_\lambda^2 t + \frac{a\dot{a}}{1-k\rho^2} (\partial_\lambda \rho)^2 \Rightarrow \partial_\lambda^2 t + \frac{\dot{a}}{a} (\partial_\lambda t)^2 = 0$$

Now $\dot{a} = \dfrac{da}{dt} = \dfrac{\partial_\lambda a}{\partial_\lambda t}$ so

$$\partial_\lambda^2 t + \frac{\partial_\lambda a}{a}\partial_\lambda t = 0$$

$$\rightarrow \quad \frac{\partial_\lambda^2 t}{\partial_\lambda t} = -\frac{\partial_\lambda a}{a}$$

$$\Rightarrow \quad \partial_\lambda t = \frac{\omega_0}{a}$$

where ω_0 has dimensions of inverse time (ie frequency).

Now recall that $E_{obs} = -g_{\mu\nu} \underbrace{U^\mu}_{\text{observer 4-vel.}} \underbrace{\frac{dx^\nu}{d\lambda}}_{\text{photon 4-vel.}}$

$= -g_{00}\,\partial_\lambda t$ (since $U^\mu = (1,0,0,0)$ for co-moving observer)

$= \partial_\lambda t = \dfrac{\omega_0}{a(t)}$

Thus $\dfrac{E_{obs}(t_1)}{E_{obs}(t_2)} = \dfrac{a(t_2)}{a(t_1)}$ → cosmological red-shift.

If $a(t_2) > a(t_1)$, ie the universe is expanding, then $E_{obs}(t_2) < E_{obs}(t_1)$ ie $\nu(t_2) < \nu(t_1)$ and the frequency is shifted downwards: → redshift.

If $a(t_2) < a(t_1)$ then we have a blueshift.

We see that the physical meaning of ω_0 is that it represents the ~~red~~ frequency as measured at the time when $a(t) = 1$.

We may define z by $1+z = \dfrac{a(t_2)}{a(t_1)}$

So $z > 0$ → redshift

$z < 0$ → blueshift

The cosmological redshift is sometimes interpreted as the Doppler shift due to points in an expanding universe receding from each other. However this can be slightly misleading (though it is OK for slow, uniform expansion). Doppler shift depends on the velocity of the emitter at the time of emission ($\dot{a}(t_1)$) and the absorber at absorption ($\dot{a}(t_2)$). However here the redshift depends on the <u>values</u> $a(t_1)$, $a(t_2)$.

Suppose t_2 = today, which we label t_0.

For slow variation, $a(t) = a(t_0)\left(1 + (t - t_0)H_0 + \dots\right)$

where $H_0 = \left.\frac{\dot{a}(t)}{a(t)}\right|_{t=t_0}$ = Hubble const.

Hence for small redshifts, $z \ll 1$, $z \sim H_0(t-t_0)$

Now if $t - t_0 = \Delta t$ is small then

$$\Delta t = \frac{\Delta\rho}{\sqrt{1-k\rho^2}}\, a(t) \simeq d$$

(instantaneous proper distance).

Then $\boxed{z \simeq H_0 d}$ Hubble's law.

The LHS can be thought of as v, the recession velocity of galaxies, but again only for $z \ll 1$.

Note that redshifts as large as $z \sim 7$ have been observed and here the approximations above are not applicable (so $a(t) \sim a(t_0)(1 + (t-t_0)H_0 + \dots)$ is not justified.