Lecture 21. 5/11/09

We can compare such redshift observations conveniently to theory if we assume a slow variation for $a(t)$ and compute energies in a non-relativistic approximation. The

$$E = \text{Kinetic} + \text{potential energy}$$

$$= \frac{1}{2} m \dot{X}^{i\,2} - \frac{GmM(X^i)}{|\vec{X}|}$$

where m is the mass of a particle outside a ball of matter of mass M, and X^i are Cartesian (not co-moving) coordinates.

Now $\dot{X}^i = \frac{\dot{a}(t)}{a(t_0)} X^i(t_0)$ (since $X^i(t) = \frac{a(t)}{a(t_0)} X^i(t_0)$)

and $\frac{M}{|\vec{X}|} = \frac{4\pi\rho\,|\vec{X}|^2}{3} = \frac{4\pi\rho}{3}\frac{a^2(t)}{a^2(t_0)}|X^i(t_0)|^2$

So $E = \frac{1}{2} m \frac{\dot{a}(t)^2}{a(t_0)^2}|X^i(t_0)|^2 - \frac{4\pi\rho G m}{3}\frac{a^2(t)}{a(t_0)^2}|X^i(t_0)|^2$

$$= \frac{m|X^i(t_0)|^2}{2a^2(t_0)}\left[\dot{a}^2 - \frac{8\pi G}{3}a(t)^2\rho(t)\right]$$

We have seen that the quantity in square brackets is just $-K$ (the curvature parameter), so

$$E = -\frac{1}{2}\frac{m|X^i(t_0)|^2}{a^2(t_0)}K$$

Thus if $K<0$ then $E>0$ and a particle can escape to infinity in the gravitational potential. For $K=0$ there is still escape but asymptotically ($\dot{r}\to 0$ at ∞) and for $K>0$ the expansion stops & the universe collapses back.

Also note that $\rho + 3p > 0$ for dust ($p = 0$) or radiation ($p = \frac{1}{3}\rho$), therefore in the absence of a cosm. constant (which gives $p = -\rho$ and $\rho + 3p = -2\rho$ which is $-$ve for $\rho > 0$) one has:

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a$$

so $\ddot{a} < 0$.

In fact we find if $q_0 = -\frac{\ddot{a}(t_0)}{H_0^2 a(t_0)}$ (q_0 = "deceleration parameter")

$$q_0 = \frac{4\pi G}{3}\frac{(\rho + 3p)}{H_0^2}\Big|_{t=t_0}$$

$$= \frac{\rho_0 + 3p_0}{2\rho_{crit,0}} \qquad \left(\text{recall } \rho_{crit} = \frac{3H^2}{8\pi G}\right)$$

Now $\Omega - \frac{k}{\dot{a}^2} = 1$ evaluated at $t = t_0$ $\Rightarrow$

$$\frac{\rho_0}{\rho_{crit,0}} - \frac{k}{\dot{a}^2(t_0)} = 1$$

For pressureless dust, $p_0 \ll \rho_0$ so

$$2q_0 - \frac{k}{\dot{a}^2} = 1$$

So
$q_0 > \frac{1}{2} \Rightarrow k > 0$ (scaled $\rightarrow k = 1$)
$q_0 = \frac{1}{2} \Rightarrow k = 0$
$q_0 < \frac{1}{2} \Rightarrow k < 0$.

For radiation, $p = \frac{1}{3}\rho$ so $\rho_0 + 3p_0 = 2\rho_0$

then $q_0 - \frac{k}{\dot{a}^2|_{t_0}} = 1$

so $\left.\begin{array}{l} q_0 > 1 \\ \quad = 1 \\ \quad < 1 \end{array}\right\}$ $\begin{array}{l} k > 0 \\ \;\; = 0 \\ \;\; < 0 \end{array}$

~~[illegible]~~ Age of the universe

i) For a matter-dominated flat universe,

$$a(t) \sim t^{2/3} \Rightarrow \frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t}$$

So $t_0 = \frac{2}{3H_0}$ = age of universe

Using $H_0 = 100\,h$ km/sec/Mpc
(this defines h as a dimensionless number in convenient units, it is of order 1 and currently thought to be ~ 0.7)

$$\rightarrow t_0 = \frac{6.52 \times 10^{+9}}{h} \text{ years.}$$

$$\sim 10^{+9} \text{ years.}$$

Some stars are known to be older than this, which essentially rules out this model.

A purely radiation-dominated universe gives instead

$$t_0 = \frac{1}{2H_0}$$

which is not much better.

Finally the vacuum-dominated universe has

$$a(t) \sim e^{H_0 t}$$

so the age of the universe is infinite ("toy" model) in this

We see that inclusion of vacuum energy makes the age of the universe <u>longer</u>.

ii) We can write a general formula for the age of the universe with a mixture of matter, radiation, vacuum energy and curvature.

We have $\Omega = \frac{\rho}{\rho_{crit}}$ for $\rho = (\rho_M, \rho_R, \rho_V)$

and $\Omega_K = \frac{-K}{\dot{a}^2} = -\frac{K}{a^2 H(t)^2}$ for curvature,

with $\Omega_M + \Omega_R + \Omega_V + \Omega_K = 0$

~~Now~~ $\cancel{\dot{a}^2 - \frac{8\pi G \rho a^2}{3} = -K \Rightarrow}$

~~Working at the present time $t = t_0$,~~

For $i = (M, R, V)$ we have

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_{crit}} = \left(\frac{3H(t)^2}{8\pi G}\right)^{-1} \rho_i(t)$$

Now $\Omega_i(t) = \frac{\rho_i(t)}{\rho_{crit}(t)} = \frac{\rho_{i,0}}{\rho_{crit,0}} \frac{\rho_i}{\rho_{i,0}} \frac{\rho_{crit,0}}{\rho_{crit}(t)}$

and $\frac{\rho_{i,0}}{\rho_{crit,0}} = \Omega_{i,0}$ (ie Ω_i at $t = t_0$)

$$\frac{\rho_i}{\rho_{i,0}} = \left(\frac{a_0}{a}\right)^3 \quad \text{for } i = M$$

$$= \left(\frac{a_0}{a}\right)^4 \quad \text{for } i = R$$

$$= \left(\frac{a_0}{a}\right)^2 \quad \text{for } i = K$$

$$= 1 \quad \text{for } i = V.$$

Finally, $\frac{\rho_{crit,0}}{\rho_{crit}(t)} = \frac{H_0^2}{H(t)^2}$.

Thus $\Omega_M + \Omega_R + \Omega_V + \Omega_K = 1$

$$\Rightarrow \Omega_{M,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{R,0}\left(\frac{a_0}{a}\right)^4 + \Omega_{V,0} + \Omega_{K,0}\left(\frac{a_0}{a}\right)^2 = \left(\frac{H(t)}{H_0}\right)^2$$

Recall that $\frac{a(t_0)}{a(t)} = 1+z$, so $\frac{\dot{a}}{a_0} = \frac{-\dot{z}}{(1+z)^2}$

$$\frac{\dot{a}}{a} = \frac{\dot{a}}{a_0}\frac{a_0}{a} = \frac{-\dot{z}}{(1+z)^2} \cdot 1+z = \frac{-\dot{z}}{1+z}$$

Thus

$$H_0\sqrt{\Omega_{M,0}(1+z)^3 + \Omega_{R,0}(1+z)^4 + \Omega_{V,0} + \Omega_{K,0}(1+z)^2} = \frac{-\dot{z}}{1+z}$$

which can now be integrated to give:

$$\int_0^{t_0} dt = \frac{1}{H_0}\int_\infty^0 \frac{-dz}{(1+z)\sqrt{\left(\Omega_{M,0}(1+z)^3 + \Omega_{R,0}(1+z)^4 + \Omega_{K,0}(1+z)^2 + \Omega_{V,0}\right)}}$$

Here we used that at $t=0$, $a(t)=0$ so $z=\infty$.

Defining $x = \frac{1}{1+z}$, we get

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{dx}{x\sqrt{\Omega_{M,0}x^{-3} + \Omega_{R,0}x^{-4} + \Omega_{V,0} + \Omega_{K,0}x^{-2}}}$$

The special "pure" cases derived earlier are easily derived from this.

Note one consequence of the eqn that we have derived above:

$$\Omega_{M,0}x^{-3} + \Omega_{R,0}x^{-4} + \Omega_{V,0} + \Omega_{K,0}x^{-2} = \frac{H(t)}{H_0}$$

For the universe to stop expanding, $H(t)=0$ at some t. Then

$$\Omega_{M,0}x^{-3} + \Omega_{R,0}x^{-4} + \Omega_{V,0} + \Omega_{K,0}x^{-2} = 0$$

must have a real root. ~~sufficiently large x (small redshifts) one~~

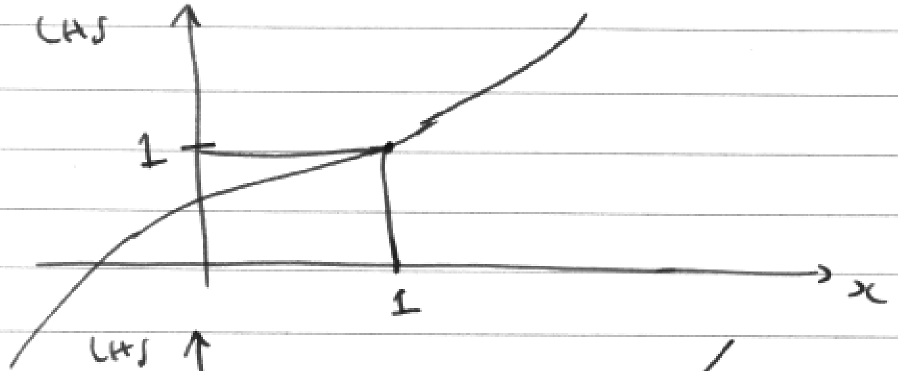
For this question we need $t > t_0$ (ie t is in the future) so $x = \frac{a(t)}{a(t_0)} > 1$.

For large enough x we can neglect $O(x^{-4})$ so we have the eqn:

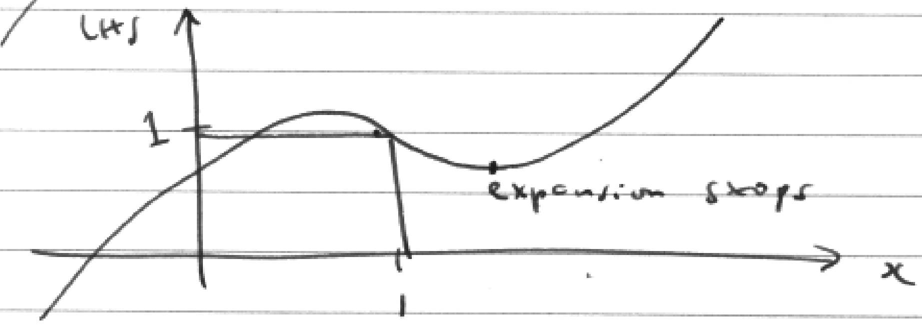
$$\Omega_{M,0} + \Omega_{K,0}\,x + \Omega_{V,0}\,x^3 = 0$$

which is a cubic.

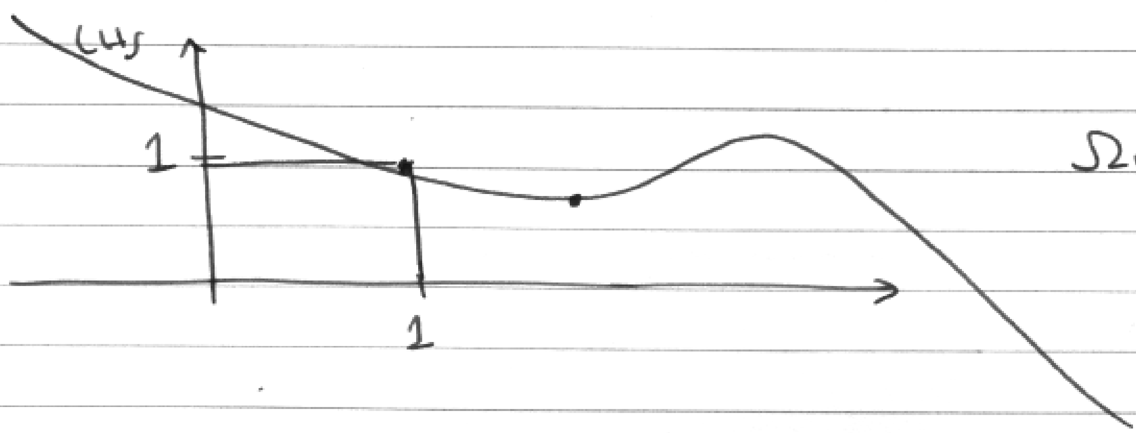
What we know about this expression is that at $x = 1$, LHS $= 1$. ~~Hence~~ So the question is whether it falls to 0 for some large x:



for $\Omega_{V,0} > 0$



for $\Omega_{V,0} > 0$
(if $\Omega_{K,0} < 0$)
ie $K > 0$



$\Omega_{V,0} < 0$

Comment: from the Friedman eqn

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

we note that the contribution of vacuum energy to the RHS is $\sim \rho + 3p = -2\rho$ so

$$\text{RHS} \sim \frac{8\pi G}{3}\rho$$

Now if $\rho > 0$ (ie $\Lambda > 0$) then $\ddot{a} > 0$ and this contributes an acceleration to the expansion of the universe.

If instead $\rho < 0$ (ie $\Lambda < 0$) then $\ddot{a} < 0$ and the universe is in a state of decelerated expansion.

Note the amusing fact that

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}$$

thus it is possible to have $\ddot{a} > 0$ (accelerating scale factor) and decreasing Hubble constant at the same time!