Lecture 22 12/11/09

Let us now consider cosmological evolution in the presence of a scalar field. Thus instead of taking the stress-energy tensor $T_{\mu\nu}$ to be that of a perfect fluid, we consider a (real) scalar field $\varphi(x,t)$ with Lagrangian (or rather, action)

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right)$$

Earlier we defined

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$$

NOTE!

$$= + \left(\partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \right)$$

$$- g_{\mu\nu} V(\varphi)$$

$$= + \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + V(\varphi) \right)$$

Now let us consider the Einstein eqns with this $T_{\mu\nu}$ as the source:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

with $T_{\mu\nu}$ as above.

$$\boxed{S = \frac{1}{16\pi G} \int \sqrt{-g} R + \int \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right)}$$

Notice that $T_{\mu\nu}(\varphi)$ has the perfect fluid form if we choose as velocity:

$$u_\mu \simeq \partial_\mu \varphi .$$

We see that $T_{\mu\nu}$ has a term proportional to $g_{\mu\nu}$ and one prop. to $\partial_\mu \varphi \partial_\nu \varphi \sim u_\mu u_\nu$.

We may normalise

$$u_\mu = - \frac{1}{\sqrt{-g^{\alpha\beta}\partial_\alpha \varphi \partial_\beta \varphi}} \partial_\mu \varphi$$

(the overall sign is not fixed by requiring $u_\mu u^\mu = -1$)

Then
$$T_{\mu\nu} = - g_{\mu\nu}\left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + V(\varphi) \right) + \partial_\mu \varphi \partial_\nu \varphi$$

$$= -\left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + V(\varphi) \right) g_{\mu\nu} - \left(g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \right) u_\mu u_\nu$$

Identifying this with $p g_{\mu\nu} + (\rho + p) u_\mu u_\nu$, we have:

$$p = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - V(\varphi)$$

$$\rho = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + V(\varphi)$$

It follows that the rest frame of the fluid, in which $u_\mu = \pm(1,0,0,0)$ corresponds to taking φ independent of x^i, $i = 1, 2, 3$.

~~Press p~~ Choosing co-moving coordinates ($g_{00} = -1$) we have:

$$p = +\frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

$$\rho = +\frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

With these we can write down the Friedman eqns directly:

$$\frac{\dot{\rho}}{3(\rho+p)} = -\frac{\dot{a}}{a} = -H(t)$$

$$\Rightarrow \quad \frac{+\dot{\varphi}(\ddot{\varphi} + V')}{+3\dot{\varphi}^2} = -H$$

$$\rightarrow \quad \ddot{\varphi} + 3\dot{\varphi}H + V' = 0$$

Thus the scalar field behaves like a non-relativistic particle in a potential V and with a velocity-dependent drag force (friction) determined by $H(t)$.

The other Friedmann eqn is:

$$\cancel{\ddot{a} = -\frac{4\pi G}{3}(\rho+3p)a} \qquad \dot{a}^2 = \frac{8\pi G}{3}a^2\left(\frac{\dot{\varphi}^2}{2} + V(\varphi)\right) - K.$$

$$\cancel{= -\frac{8\pi G}{3}\left(\dot{\varphi}^2 + V(\varphi)\right)a}$$

Orthonormal frames

To conclude this course, I will switch gear and discuss the important concept of orthonormal frames.

The basic notion is to ~~ask if one can find a set of 1-forms that~~ properly choose a set of d covariant vector fields e^α_i defined as follows: (first we do this for 3-geometry and then extend it to 3+1 d spacetime).

i) e^α_i, for each α, is a 3-vector. So under general coordinate transformations,

$$e^\alpha_i \rightarrow e'^\alpha_i(x') = \frac{\partial x^j}{\partial x'^i} e^\alpha_j(x)$$

ii) The set e^α_i for $\alpha = 1,2,3$ are three lin indep 3-vectors. We can make an orthogonal rotation among them that preserves their properties:

$$e^\alpha_i(x) \rightarrow e'^\alpha_i(x) = M^\alpha{}_\beta e^\beta_i(x)$$

This is just a rotation of basis & doesn't involve general coordinate transformations.

iii) We would like to make these vectors orthogonal to each other:

$$e^\alpha_i e^\beta_j g^{ij}(x) = 0 \quad , \quad \alpha \neq \beta$$

and normalised: $\quad e^\alpha_i e^\beta_j g^{ij}(x) = \delta^{\alpha\beta}$

Of course the normalisation condition is preserved by the rotation:

$$M^\alpha{}_\gamma e^\gamma{}_i \, . \, M^\beta{}_\delta e^\delta{}_j \, g^{ij}$$

$$= M^\alpha{}_\gamma M^\beta{}_\delta \delta^{\gamma\delta} = (MM^T)^{\alpha\beta} = \delta^{\alpha\beta}$$

A set of 3 3-vectors satisfying the above conditions (and moreover undetermined upto $O(3)$ rotations) are called orthonormal frames.

As 3×3 matrices, e^α_i are non-singular since

$$|e|^2 |g|^{-1} = 1 \quad , \quad \text{so} \quad |e| = \sqrt{|g|} \, .$$

Consider now $\sum_\alpha e^\alpha_i e^\alpha_j$. This is a 2nd rank sym tensor under general coordinate transformation. Moreover, it is $O(3)$ invariant. Call it π_{ij}.

~~Mult~~ Multiplying by $e^\beta_k g^{jk}$ we have:

$$\pi_{ij} e^\beta_k g^{jk} = \sum_\alpha e^\alpha_i e^\alpha_j e^\beta_k g^{jk} = e^\beta_i$$

Thus $\pi_{ij} g^{jk} = \delta_i{}^k$

So π_{ij} is the inverse of g^{ij}, therefore it is equal to g_{ij}:

$$\boxed{e^\alpha_i e^\alpha_j = g_{ij}}$$

Thus e_i^α behaves like the "square root" of the metric g_{ij}.

We can also define inverse frames $E^{i\alpha}$ by

$$E^{i\alpha} e_i{}^\beta = \delta^{\alpha\beta}.$$

It is easy to show that $E^{i\alpha} E^{j\alpha} = g^{ij}$, and

$$E^{i\alpha} E^{j\beta} g_{ij} = \delta^{\alpha\beta}.$$

All the above is easily extended to spacetime except for one twist.

The group of rotations $O(3)$ (or $O(d)$) in space gets replaced by the Lorentz group $O(3,1)$ (or $O(d,1)$ in spacetime).

Thus:
$$e_\mu^a e_\nu^b g^{\mu\nu} = \eta^{ab}$$

$$e_\mu^a e_\nu^b \eta_{ab} = g_{\mu\nu}$$

We must distinguish between e_μ^a and $e_{a\mu} = \eta_{ab} e_\mu^b$

Similarly the inverse of e_μ^a is E_a^μ:

$$E_a^\mu e_\mu^b = \delta_a{}^b$$

So $E^{a\mu} e_\mu^b = \eta^{ab}$ and

$$E^{a\mu} E^{b\nu} g_{\mu\nu} = \eta^{ab}, \quad E^{a\mu} E^{b\nu} \eta_{ab} = g^{\mu\nu}.$$

The advantage of doing all this is that we can use e and E to convert any spacetime tensor (under general coord. transformations) to a scalar. The price we pay is that the resulting scalar will transform under local Lorentz transformations.

As an example, consider a contravariant vector V^μ. Define

$$V^a = e^a{}_\mu V^\mu$$

Under GCT, $V'^a(x') = e'^a{}_\mu(x') V'^\mu(x')$

$$= e^a{}_\lambda \frac{\partial x^\lambda}{\partial x'^\mu} V^\rho \frac{\partial x'^\mu}{\partial x^\rho}$$
$$= e^a{}_\lambda V^\lambda = V^a(x).$$

However under LLT,

$$V'^a(x) = e'^a{}_\mu(x) V^\mu(x)$$
$$= \Lambda^a{}_b\, e^b{}_\mu V^\mu = \Lambda^a{}_b(x) V^b(x)$$

where $\Lambda \eta \Lambda^T = \eta$, $\quad \eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

This effectively replaces the group "GCT" of general relativity, with the group "LLT" largely familiar from special relativity!

However the fact that the Lorentz transformations are local introduces new complications.

For example, $\partial_\mu V^a$ (where $V^a = e^a{}_\mu V^\mu$) is <u>not</u> an LLT vector:

$$V^a \rightarrow \Lambda^a{}_b V^b$$

$$\partial_\mu V^a \rightarrow \partial_\mu (\Lambda^a{}_b(x) V^b(x))$$

$$= \underbrace{\partial_\mu \Lambda^a{}_b V^b}_{\text{spoils the vector nature!}} + \Lambda^a{}_b \partial_\mu V^b$$

Thus we again need a covariant derivative, but this time on LLT tensors.

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b$$

ω has to be tailored so that $D_\mu V^a$ transforms as a GCT cov. vector as well as an LLT contravariant vector.

For this, consider

$$\partial_\mu (V^a W_a) = D_\mu (V^a W_a)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$(\partial_\mu V^a) W_a + V^a \partial_\mu W_a \qquad (D_\mu V^a) W_a + V^a D_\mu W_a$$

$$\downarrow$$

$$(\partial_\mu V^a) W_a + \partial_\mu W_a V^a$$
$$+ \omega_\mu{}^a{}_b V^b W_a + V^a \omega_{\mu a}{}^b W_b$$

$$RHS = \partial_\mu (V^a W_a) + \omega_\mu{}^{ab} (V_b W_a + V_a W_b)$$

It follows that $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$:

antisymmetric in (a,b).

~~Now consider D~~ Now under LLT we have:

$$D_\mu V'^a = D_\mu (\Lambda^a{}_b V^b) = \partial_\mu V'^a + \omega'_\mu{}^a{}_b V'^b$$

$$= \partial_\mu (\Lambda^a{}_b V^b) + \omega'_\mu{}^a{}_c \Lambda^c{}_b V^b$$

$\cancel{\partial_\mu V'^a + \omega_\mu{}^a{}_b V'^b}$

$= \cancel{\partial_\mu (\Lambda^a{}_b V^b) + \omega_\mu{}^a{}_b}$

$$\Lambda^a{}_b D_\mu V^b$$

$$= \Lambda^a{}_b \partial_\mu V^b + \Lambda^a{}_b \omega_\mu{}^b{}_c V^c$$

Comparing: $\partial_\mu \Lambda^a{}_b V^b + \omega'_\mu{}^a{}_c \Lambda^c{}_b V^b$

$$= \Lambda^a{}_b \omega_\mu{}^b{}_c V^c$$

Thus $\omega'_\mu{}^a{}_c \Lambda^c{}_b V^b = \Lambda^a{}_c \omega_\mu{}^c{}_b V^c - \partial_\mu \Lambda^a{}_b V^b$

Since this is true for arbitrary V^b we have:

$$\omega'_\mu{}^a{}_c \Lambda^c{}_b = \Lambda^a{}_c \omega_\mu{}^c{}_b - \partial_\mu \Lambda^a{}_b$$

or, on inverting and using matrix language,

$$\omega'_\mu = \Lambda \omega_\mu \Lambda^{-1} - \partial_\mu \Lambda \Lambda^{-1}$$

ie $\omega'_\mu{}^a{}_b = \Lambda^a{}_c \omega_\mu{}^c{}_d (\Lambda^{-1})^d{}_b - (\partial_\mu \Lambda)^a{}_c (\Lambda^{-1})^c{}_b$

For those familiar with Yang-Mills fields, note that this is the same transformation law as that of the YM vector potential under gauge transformations! (but with gauge group $SO(2,1)$.

How do we determine ω? From

$$D_\mu (e^a_\nu e_{a\lambda}) = D_\mu g_{\nu\lambda} = 0$$

$$\| $$

$$(D_\mu e^a{}_\nu) e_{a\lambda} + e^a_\nu D_\mu e_{a\lambda}$$

we see that $D_\mu e^a{}_\nu$ is a <u>sufficient</u> condition to have $D_\mu g_{\nu\lambda} = 0$. Accordingly we impose it as a requirement:

$$D_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma^\lambda_{\mu\nu} e^a{}_\lambda = 0.$$

$$\Rightarrow \quad \omega_\mu{}^a{}_b = - E_b{}^\nu \partial_\mu e^a_\nu + \Gamma^\lambda_{\mu\nu} e^a{}_\lambda E_b{}^\nu$$

On the RHS everything depends on $e_{a\mu}$ (even $\Gamma(g)$, using $g = ee^T$). Thus $\omega_\mu{}^a{}_b$ is determined in terms of the frames. And they are now covariantly constant. We find (exercise)

$$\omega_\mu{}^{ab} = \frac{1}{2} E^{a\nu} \left[\partial_\mu e^b_\nu - \partial_\nu e^b_\mu - e^c_\mu E^{b\sigma} (\partial_\nu e_{c\sigma} - \partial_\sigma e_{c\nu}) \right] - (a \leftrightarrow b)$$

ω is called the Levi-Civita spin connection.

Finally, we note that

$$D_\mu D_\nu V^a - D_\nu D_\mu V^a$$

$$= \left(\partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c \omega_\mu{}^c{}_b\right) V^b$$

$$\equiv R_{\mu\nu}{}^a{}_b V^b$$

In matrices: $\underset{\sim}{R}_{\mu\nu} = \partial_\mu \underset{\sim}{\omega}_\nu - \partial_\nu \underset{\sim}{\omega}_\mu + [\underset{\sim}{\omega}_\mu, \underset{\sim}{\omega}_\nu]$

We can define a tensor

$$R^\lambda{}_{\rho\mu\nu} = E_a{}^\lambda e^b{}_\rho R_{\mu\nu}{}^a{}_b$$

and remarkably, this is the very <u>same</u> Riemann tensor that we encountered at the beginning of our course!

Note again that the formula for $\underset{\sim}{R}_{\mu\nu}$ in terms of $\underset{\sim}{\omega}_\mu$ is the same as the formula for $\underset{\sim}{F}_{\mu\nu}$ in terms of $\underset{\sim}{A}_\mu$ in Yang-Mills theory.