Random Matrix Models of String Theory
Part I of 2

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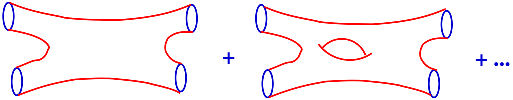
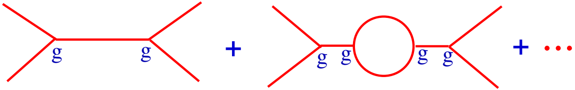
Outline

Introduction

- String theory was originally defined as a sum over worldsheets of ever-increasing genus.



- This is analogous to defining field theory by its expansion over Feynman diagrams.



- The number of handles in the surfaces, like the number of loops in the diagrams, count the order in perturbation theory.

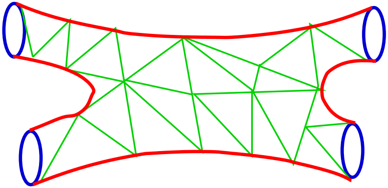
- In field theory, there is a non-perturbative formulation (e.g. Lagrangian path integral) that contains information about such things as solitons, tunnelling and confinement.
- There exists a non-perturbative formulation of string theory too – but so far, it is known only about rather specific spacetime backgrounds.
- This is the random matrix formulation describes strings propagating in very low dimensional spacetimes, such as two.
- Hence, strings propagating in two spacetime dimensions (one space, one time) will be the subject of these lectures.

- The road to the nonperturbative formulation is rather long. We will start with a theory that almost achieves this, but fails. This is called the $c = 1$ bosonic string.
- The theory is still rather interesting, in that we knows its partition function and scattering amplitudes to all orders in perturbation theory.
- Then we will turn our attention to the more recently understood noncritical type 0A and 0B strings. In perturbation theory these are very much like the bosonic string, but they are also non-perturbatively well-defined.

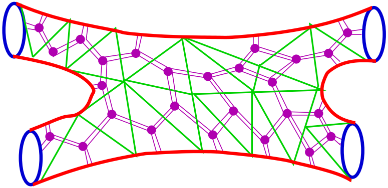
Outline

Random Matrices - Generalities

- There are two different ways to motivate the random matrix approach. Let us first start with the traditional motivation.
- The idea is to start with an action principle which generates, not Riemann surfaces but discrete (lattice-like) versions of them.
- This is quite easy to achieve. A discrete Riemann surface can be made by gluing together triangles:



- The next step would be to write a function that, on expanding, generates these triangles.
- This is achieved via a trick called lattice duality. Put a vertex at the centre of every triangle, and connect every pair of vertices by a line that cuts the common boundary of the triangles.



- In fact it's natural to thicken these new lines to double lines. One sees now that the Riemann surface is covered by polygons glued together at their common edges.

- The polygons can have different numbers of sides. But the dual diagram always has three lines meeting at a point, precisely because we did lattice duality on triangles.
- Now we are almost done. Double lines are generated by matrices because they have two indices.
- And three-point vertices are generated by cubic couplings among the matrices.
- This suggests a random matrix integral will do the job::

$$\mathcal{Z} = \int [dM]\, e^{-N\,\mathrm{tr}\left(\frac{1}{2}M^2 + gM^3\right)}$$

where M are $N \times N$ Hermitian random matrices.

- This is still a little vague. What do we mean “do the job”? And is this the unique action for the purpose? Please be patient...
- The random matrix integral we wrote should be thought of as a field theory path integral, except that instead of fields we have matrices. Instead of an integral over space and time, we have a trace.
- The integral can be evaluated using the very same technique we learn in field theory: solve the quadratic (Gaussian) part explicitly and treat the rest in perturbation theory.

- For this we need to develop some rules. First, let M be an $N \times N$ Hermitian matrix.
- The measure in the integral is then:

$$[dM] \equiv \prod_{i=1}^{N} dM_{ii} \prod_{i<j=1}^{N} dM_{ij} dM_{ij}^{*}$$

- Now we evaluate the Gaussian matrix integral in the presence of a source:

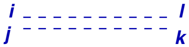
$$\int [dM]\, e^{-N\,\mathrm{tr}(\frac{1}{2}M^2 + JM)} = \left(\frac{2\pi}{N}\right)^{\frac{N^2}{2}} e^{N\,\mathrm{tr}\frac{J^2}{2}}$$

- Next we use this to compute the propagator:

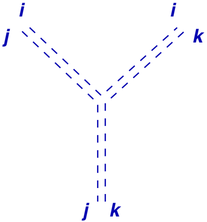
$$\langle M_{ij} M_{kl} \rangle \equiv \frac{\int [dM]\, M_{ij} M_{kl}\, e^{-N\,\mathrm{tr}\frac{1}{2}M^2}}{\int [dM]\, e^{-N\,\mathrm{tr}\frac{1}{2}M^2}} = \frac{1}{N}\delta_{il}\delta_{jk}$$

- By virtue of its structure, the propagator is naturally represented in terms of double lines:

$$\langle M_{ij} M_{kl} \rangle =$$



- Next, consider the cubic term. This can be used to generate a cubic vertex, as in field theory:



- Combining these elements we see that the perturbation expansion of our matrix model is a dual triangulated surface.

- The matrix integral generates all possible closed diagrams. Therefore it will produce all types of Riemann surfaces. The topology of the surface is defined by the particular diagram.
- Indeed we know that if:

$$\begin{aligned} \text{number of vertices} &= V \\ \text{number of edges} &= E \\ \text{number of faces} &= F \end{aligned}$$

  one has the relation:

$$V - E + F = 2 - 2h$$

  where h is the genus of the surface.
- The same relation is true on the dual graph, with

$$V \leftrightarrow F$$

- Now each vertex has a factor of gN, each propagator has $\frac{1}{N}$ and each face has a factor of N from the sum over matrix indices.
- Therefore a given graph in the expansion will be of order:

$$(gN)^V N^{-E} N^F = g^V N^{2-2h}$$

  We learn that $\frac{1}{N^2}$ is the genus expansion parameter, and g is an additional coupling constant to be held fixed.
- Thus the partition function can be written:

$$\mathcal{Z}(g, N) = \sum_{h=0}^{\infty} \mathcal{Z}_h(g)\, N^{2-2h}$$

Eigenvalue Reduction and Vandermonde determinant

- A Hermitian matrix can always be diagonalised:

$$M = U \Lambda U^\dagger$$

  where U is a unitary matrix, and

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$$

  is a diagonal matrix of eigenvalues.
- The unitary matrix decouples from the action, which we can write as:

$$\mathrm{tr}(\tfrac{1}{2}M^2 + gM^3) = \sum_{i=1}^{N} (\tfrac{1}{2}\lambda_i^2 + g\lambda_i^3)$$

- Next we reduce the integration measure to eigenvalues:

$$[dM] = \prod_{i=1}^{N} d\lambda_i \; \Delta(\lambda)^2$$

where we see the appearance of the Vandermonde determinant:

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

- This arises as follows. We have:

$$\begin{aligned} dM &= dU\,\Lambda\,U^\dagger + U\,d\Lambda\,U^\dagger + U\,\Lambda\,dU^\dagger \implies \\ U^\dagger\,dM\,U &= d\Lambda + [U^\dagger\,dU, \Lambda] \end{aligned}$$

- Next we use two facts:
  (i) $d\alpha = U^\dagger\, dU$ is the infinitesimal element in the Lie algebra (tangent space to the unitary group).
  (ii) the measures $[dM]$ and $[dM'] = [U^\dagger\, dM\, U]$ are the same.
- Then we have:

$$dM'_{ij} = d\lambda_i \delta_{ij} + d\alpha_{ij}(\lambda_i - \lambda_j)$$

Geometrically, this means that the identity metric on the N^2-dimensional space with coordinates dM'_{ij} transforms to a nontrivial metric:

$$G_{AB} = \mathrm{diag}(1, 1, \cdots, 1, (\lambda_1 - \lambda_2)^2, (\lambda_1 - \lambda_3)^2, \cdots)$$

in the coordinates (λ_i, α_{ij}).

- To transform the measure, we compute

$$\sqrt{G} = \prod_{i \neq j} (\lambda_i - \lambda_j) = \Delta(\lambda)^2$$

  and therefore

$$[dM] = [dU] \prod_{i=1}^{N} d\lambda_i \, \Delta(\lambda)^2$$

- The integral over dU is just a numerical factor since the integrand is independent of it. That completes the proof.

Outline

- Let us now return to our goal of extracting a string theory from the matrix integral.
- Recall that the expansion of the integral is:

$$\mathcal{Z}(g, N) = \sum_{h=0}^{\infty} \mathcal{Z}_h(g)\, N^{2-2h}$$

- We notice that the large-N limit picks out the genus-0 contribution. In string theory, this would be tree level.
- But this is still not string theory. The genus-0 partition function, $\mathcal{Z}_0(g)$, describes discrete surfaces with all possible numbers of vertices.

- We would like to take a continuum limit where $\mathcal{Z}_0(g)$ is dominated by graphs with very many vertices (the dual graph then has many triangles).
- Defining the area of a triangulation as the number of triangles (or in the dual graph, the number of vertices), we are looking for infinite-area surfaces.
- To achieve this we exploit the constant parameter g. As g is increased, the partition function undergoes a phase transition:

$$\mathcal{Z}_0(g) \sim (g - g_c)^{2-\gamma}$$

for some critical exponent γ.

- We have:

$$\mathcal{Z}_0(g) \sim (g-g_c)^{2-\gamma} \sim \sum_{n=1}^{\infty} n^{\gamma-3} \left(\frac{g}{g_c}\right)^n$$

and therefore

$$\langle n \rangle \sim \frac{1}{\mathcal{Z}_0(g)} \sum_{n=1}^{\infty} n \cdot n^{\gamma-3} \left(\frac{g}{g_c}\right)^n \sim \frac{\partial}{\partial g} \log \mathcal{Z}_0 \sim \frac{1}{g-g_c}$$

- Therefore, the average area diverges as $g \to g_c$.

- We see that to recover a continuum, tree-level theory we need to take the limit:

$$N \longrightarrow \infty, \qquad g \longrightarrow g_c$$

- Remarkably, by changing this limit slightly, we can get a continuum theory that includes all genus contributions.
- First of all we expect that the divergence as $g \longrightarrow g_c$ is a local phenomenon on the worldsheet. Therefore the value of g_c is the same in all genus.
- Next we claim that in genus h, the divergence goes as:

$$\mathcal{Z}_h(g) \sim (g - g_c)^{(2-\gamma)(1-h)}$$

- Thus the full partition function behaves near $g \to g_c$ as:

$$\mathcal{Z}(g,N) \sim \sum_h F_h \left[N(g-g_c)^{(2-\gamma)/2}\right]^{2-2h} = \sum_h F_h\, g_s^{2h-2}$$

where

$$g_s \equiv \left[N(g-g_c)^{(2-\gamma)/2}\right]^{-1}$$

- Thus, to obtain a continuum theory that includes all genus we simply take the limit:

$$N \to \infty,\ \ g \to g_c,\ \ g_s \equiv \left[N(g-g_c)^{(2-\gamma)/2}\right]^{-1} \ \text{fixed}$$

and it is g_s that will be the new genus expansion parameter, or string coupling.

- The above limit is called the double scaling limit.

- The next step is to carry out the genus expansion of this matrix model in the double-scaling limit and see if it has the properties expected of a string theory.
- In fact by varying the matrix potential, one finds a series of string theories. These can be identified by their susceptibility χ to be the $(q = 2, p)$ minimal CFT's coupled to worldsheet gravity (a Liouville field theory).
- Instead of pursuing this direction, I would like to introduce a somewhat different matrix model that leads to a more interesting string theory.

Outline

Matrix Quantum Mechanics

- Consider a Hermitian matrix $M(t)$ that depends on a parameter t. Let's write a matrix model:

$$\mathcal{Z} = \int [dM(t)]\, e^{-N \int dt\ \mathrm{tr}(\frac{1}{2} D_t M^2 + \frac{1}{2} M^2 - \frac{g}{3!} M^3)}$$

  where

$$D_t M \equiv \dot{M} + [A_t, M]$$

  This is a path integral for gauged matrix quantum mechanics.
- In terms of the genus expansion, this model has the same properties as the simpler model of constant matrices.
- However, it also has a parameter t that will endow the string theory with a time direction.

- Here, A_t is a $U(N)$ gauge field, due to which the matrix model has a local (in time) gauge symmetry:

$$M(t) \to U^\dagger(t)\, M(t)\, U(t)$$

- We can gauge fix $A_t = 0$, but must remember to impose its equation of motion ("Gauss Law"):

$$[M, \dot{M}] = 0$$

 on physical states.

- The eigenvalue reduction comes about by diagonalising the matrix:

$$M(t) = U(t)\,\Lambda(t)\,U(t)^\dagger$$

- We appear to have a problem. The matrix model action does not reduce only to eigenvalues:

$$\begin{aligned}\mathrm{tr}(\dot{M}^2) &= \mathrm{tr}(\dot{\Lambda} + [U^\dagger \dot{U}, \Lambda])^2 = \mathrm{tr}(\dot{\Lambda}^2 + [U^\dagger \dot{U}, \Lambda]^2) \\ &= \sum_{i=1}^{N} \dot{\lambda}_i^2 + \sum_{i<j} (\lambda_i - \lambda_j)^2\, \dot{\alpha}_{ij}\, \dot{\alpha}_{ji}\end{aligned}$$

where $\dot{\alpha}_{ij} = (U^\dagger \dot{U})_{ij}$.

- Moreover, the Vandermonde determinant will now appear in the measure at every time t.

- To avoid these two inconveniences, it is convenient to pass to the Hamiltonian, which acts on a Hilbert space of wave functions: $\Psi(M_{ij})$ or $\Psi(\lambda_i, \alpha_{ij})$.
- In terms of M, the Hamiltonian is just:

$$\begin{aligned} H &= -\frac{1}{2}\sum_i \frac{\partial^2}{\partial M_{ii}^2} - \sum_{i<j} \frac{\partial}{\partial M_{ij}}\frac{\partial}{\partial M_{ji}} - \frac{1}{2}\mathrm{tr}M^2 + \frac{g}{3!\sqrt{N}}\mathrm{tr}M^3 \\ &= H_{kin} + H_{int} \end{aligned}$$

  where we first scaled the matrix M by $\frac{1}{\sqrt{N}}$.
- However, because of the metric that we saw earlier, the kinetic term H_{kin} is nontrivial in the λ_i, α_{ij} coordinates.

- Indeed, the correct answer is:

$$\begin{aligned} H_{kin} &= -\tfrac{1}{2}\frac{1}{\sqrt{G}}\frac{\partial}{\partial\lambda_i}\sqrt{G}\,\frac{\partial}{\partial\lambda_i} + \sum_{i<j}\frac{1}{(\lambda_i-\lambda_j)^2}\,\frac{1}{\sqrt{G}}\,\Pi_{ij}\sqrt{G}\,\Pi_{ji} \\ &= -\tfrac{1}{2}\frac{1}{\Delta(\lambda)^2}\frac{\partial}{\partial\lambda_i}\Delta(\lambda)^2\frac{\partial}{\partial\lambda_i} + \sum_{i<j}\frac{1}{(\lambda_i-\lambda_j)^2}\Pi_{ij}\Pi_{ji} \end{aligned}$$

where

$$\Pi_{ij} = [\Lambda,[\Lambda,\dot{\alpha}]]_{ij}$$

is the canonical momentum conjugate to α_{ji}.

- However, the Gauss law constraint $[M,\dot{M}] = 0$ precisely implies that:

$$[\Lambda,[\Lambda,\dot{\alpha}]] = 0$$

on physical states. Thus the second term in H vanishes.

- We are left with the kinetic Hamiltonian

$$H_{kin} = -\tfrac{1}{2}\sum_{i=1}^{N} \frac{1}{\Delta(\lambda)^2} \frac{\partial}{\partial \lambda_i} \Delta(\lambda)^2 \frac{\partial}{\partial \lambda_i}$$

- Using the identity:

$$\sum_{i=1}^{N} \frac{\partial^2}{\partial \lambda_i^2} \Delta(\lambda) = 0$$

  we can re-write this Hamiltonian as:

$$H_{kin} = -\tfrac{1}{2}\sum_{i=1}^{N} \frac{1}{\Delta(\lambda)} \frac{\partial^2}{\partial \lambda_i^2} \Delta(\lambda)$$

- This acts on wave functions $\Psi(\lambda)$ that are symmetric under interchange of all the eigenvalues.

- The Schrödinger equation:

$$H\Psi(\lambda) = E\Psi(\lambda)$$

can now be re-written

$$\tilde{H}\tilde{\Psi}(\lambda) = E\tilde{\Psi}(\lambda)$$

where

$$\begin{aligned} \tilde{H} &= \Delta(\lambda)\, H \frac{1}{\Delta(\lambda)} = \sum_{i=1}^{N} \left(-\frac{1}{2}\frac{\partial^2}{\partial \lambda_i^2} - \frac{1}{2}\lambda_i^2 + \frac{g}{3!\sqrt{N}}\lambda_i^3 \right) \\ \tilde{\Psi}(\lambda) &= \Delta(\lambda)\Psi(\lambda) \end{aligned} \tag{1}$$

- Thus we are left with a system of mutually noninteracting particles with coordinates λ_i moving in a common potential. The extra Δ factor makes the wave functions fermionic.

Outline

Free Fermions and the $c = 1$ String

- We have reduced the Hamiltonian of Matrix Quantum Mechanics to a sum of one-particle Hamiltonians:

$$H = \sum_{i=1}^{N} h(\lambda_i)$$

where

$$h(\lambda) = -\frac{1}{2}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{2}\lambda^2 + \frac{1}{3!\sqrt{\beta}}\lambda^3, \qquad \beta = \frac{N}{g^2}$$

- We now wish to study this free fermion system in a large-N, double-scaled limit.

- What do we want to know about the system?
- We would like to compute the partition function of the matrix model. In Hamiltonian formulation, this can be written:

$$\mathcal{Z} = {}_{out}\langle 0| e^{-HT} |0\rangle_{in}$$

- For large times T, it is the ground state energy that contributes:

$$\lim_{T\to\infty} \frac{\ln Z}{T} = -E_{gr}$$

- Therefore we will try to compute the ground state energy of the free fermions.
- First, it is convenient to redefine variables in a way that provides us some physical intuition.

- If we send $\lambda \to \sqrt{\beta}\,\lambda$ then the single-particle Schrödinger equation becomes:

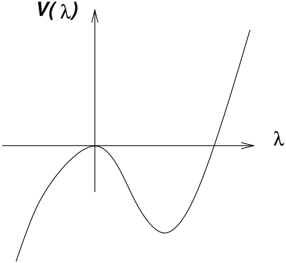
$$\left(-\frac{1}{2\beta^2}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{2}\lambda^2 + \frac{1}{3!}\lambda^3\right)\Psi(\lambda) = \frac{1}{\beta}\,E\Psi(\lambda)$$

- The advantage of this is that we can interpret β^{-1} as $\hbar$, Planck's constant. The equation is then written:

$$\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{2}\lambda^2 + \frac{1}{3!}\lambda^3\right)\Psi(\lambda) = \hbar\,E\,\Psi(\lambda) = \varepsilon\,\Psi(\lambda)$$

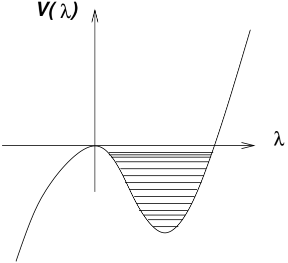
- The kinetic term has the usual form for quantum mechanics, and E on the RHS is the energy measured in units of Planck's constant.

- Now we can start to understand the double scaling limit. The potential looks like this:



- The Hamiltonian is actually unbounded below. However, eigenvalues localised on the right will tunnel through the barrier at a rate of order $e^{-\beta} = e^{-\frac{N}{g^2}}$.

- Therefore at this stage we have to bid farewell to our hopes of the theory being nonperturbatively well-defined.
- However, as long as we are only interested in perturbation theory in $\frac{1}{N^2}$, we can ignore tunneling.
- In this approximation, the Hamiltonian has discretely spaced levels on the right of the barrier, with typical spacing of order $\hbar = \beta^{-1}$.



- Very qualitatively, we see that the depth of the well is of order 1, and the level spacing is roughly of order

$$\frac{1}{\beta} = \frac{g^2}{N}$$

- We have to fill up the well with N fermions. Because of the Pauli principle, in the ground state they will fill the first N levels.
- Thus the topmost level ("Fermi level") will be at a height of order g^2 above the bottom of the well.
- And g is precisely the parameter in our control.
- For small g, the Fermi level can be below the barrier. But for large enough g, this level will rise above the barrier and eigenvalues will spill out to the other side.
- This is precisely the phase transition that makes continuum Riemann surfaces!

- To do better than this crude approximation, we use the WKB method to find the eigenvalues of this potential.
- This tells us that the n'th energy eigenvalue ε_n is given by:

$$\oint p_n(\lambda) d\lambda_n = \frac{2\pi}{\beta} n$$

where:

$$p_n(\lambda) = \sqrt{2(\varepsilon_n + \frac{1}{2}\lambda^2 - \frac{1}{3!}\lambda^3)}$$

and the integral is over a closed classical orbit.
- If the topmost orbit has turning points λ_+, λ_-, the Fermi level μ_F satisfies:

$$\int_{\lambda_-}^{\lambda_+} \sqrt{2(\mu_F + \frac{1}{2}\lambda^2 - \frac{1}{3!}\lambda^3)}\, d\lambda = \pi \frac{N}{\beta} = \pi g^2$$

- This confirms our qualitative guess that tuning g is responsible for tuning the Fermi level.
- Since we are going to take the limit of large N, it is convenient to analyse this problem in terms of the density of states of the system:

$$\rho(\varepsilon) = \frac{1}{\beta} \sum_{i=1}^{N} \delta(\varepsilon - \varepsilon_i)$$

- Then we have:

$$E_{gr} = \beta \varepsilon_{gr} = \beta \sum_{i=1}^{N} \varepsilon_i = \beta^2 \int_{V_{min}}^{\mu_F} d\varepsilon\, \varepsilon\, \rho(\varepsilon)$$

$$g^2 = \frac{N}{\beta} = \int_{V_{min}}^{\mu_F} d\varepsilon\, \rho(\varepsilon)$$

- To compute the density of states, we equate the two expressions for g^2 to get:

$$g^2 = \int_{V_{min}}^{-\mu} d\varepsilon\, \rho(\varepsilon) = \frac{1}{\pi} \int_{\lambda_-}^{\lambda_+} \sqrt{2(-\mu + \frac{1}{2}\lambda^2 - \frac{1}{3!}\lambda^3)}\, d\lambda$$

  where we have defined the positive quantity $\mu = -\mu_F$.

- Differentiating in $-\mu$, we get:

$$\begin{aligned} -\frac{\partial g^2}{\partial \mu} &= \rho(-\mu) = \frac{1}{\pi} \int_{\lambda_-}^{\lambda_+} \frac{d\lambda}{\sqrt{2(-\mu + \frac{1}{2}\lambda^2 - \frac{1}{3!}\lambda^3)}} \\ &= -\frac{1}{\pi} \log \mu + \mathcal{O}(\beta^{-2}) \end{aligned}$$

- We are looking for a singularity at a critical value g_c, so we define:

$$\Delta = \pi(g_c^2 - g^2)$$

  and seek a relation between Δ and μ, given that both go to zero together.
- From the previous page we have:

$$\frac{\partial \Delta}{\partial \mu} = \pi\rho(-\mu) = -\log \mu$$

  which can be integrated to give:

$$\Delta(\mu) = -\mu \log \mu$$

- The last step is to differentiate the equation

$$E_{gr} = \beta^2 \int_{V_{min}}^{-\mu} d\varepsilon\, \varepsilon\, \rho(\varepsilon)$$

to get:

$$\frac{\partial E_{gr}}{\partial \mu} = -\beta^2\, \mu\, \rho(-\mu)$$

which on integrating gives:

$$E_{gr} = \frac{1}{2\pi} (\beta\mu)^2 \log \mu$$

- With this we have performed the single-scaled limit of this matrix model and found the free energy (log of the partition function) in genus 0.
- Note that the key result was the logarithmic behaviour of the density of states as a function of μ as $\mu \to 0$.
- To leading order in the WKB approximation, this depended only on the quadratic part of the potential. In fact, this is true to all orders in the WKB approximation.

- To see this, let us go back to the original form of the one-particle Hamiltonian:

$$h(\lambda) = -\frac{1}{2}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{2}\lambda^2 + \frac{1}{3!\sqrt{\beta}}\lambda^3$$

- We see that as $\beta \to \infty$, the cubic term disappears completely. The states we are considering in this limit have energy $-\beta\mu$ which is kept finite.
- Thus from now on our single-particle Hamiltonian is:

$$h(\lambda) = -\frac{1}{2}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{2}\lambda^2$$

- Now we look at the double-scaled theory. We will see that the genus expansion parameter is $\beta\mu$.
- For this, the density of states will prove particularly useful. This time we need to know $\rho(\mu)$ to all orders in $\beta\mu$.
- We can write:

$$\rho(\mu) = \mathrm{tr}\,\delta(h+\beta\mu) = \frac{1}{\pi}\,\mathrm{Im}\,\mathrm{tr}\left[\frac{1}{h+\beta\mu-i\epsilon}\right] \quad (2)$$

$$= \frac{1}{\pi}\,\mathrm{Im}\int_0^\infty dT\, e^{-(\beta\mu-i\epsilon)T}\,\mathrm{tr} e^{-hT} \quad (3)$$

- Now we use the fact that our Hamiltonian is the continuation of a simple harmonic oscillator:

$$\tilde{h}(\lambda) = -\frac{1}{2}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{2}\omega^2\lambda^2$$

to the case $\omega = -i$. We easily see that:

$$\begin{aligned}
\mathrm{tr} e^{-\tilde{h}T} &= e^{-\frac{\omega T}{2}} + e^{-\frac{3\omega T}{2}} + e^{-\frac{5\omega T}{2}} + \cdots \\
&= \frac{e^{-\frac{\omega T}{2}}}{1 - e^{-\omega T}} \\
&= \frac{1}{2\sinh \omega T/2}
\end{aligned}$$

- Now we set $\omega \to -i$ and simultaneously use the $i\epsilon$ prescription to rotate $T \to iT$. Thus:

$$\rho(\mu) = \frac{1}{\pi}\,\mathrm{Im}\int_0^\infty dT\, e^{-i\beta\mu T}\,\frac{1}{2\sinh T/2}$$

- A small problem: this is logarithmically divergent at the lower limit of integration. This can be removed by differentiating and integrating back in $\beta\mu$.

- The result is best expressed in terms of the dilogarithm function:

$$\Psi(x) \equiv \frac{\partial}{\partial x} \log \Gamma(x)$$

and we find:

$$\begin{aligned} \rho(\mu) &= -\frac{1}{\pi} \Psi(\tfrac{1}{2} + i\beta\mu) \\ &= \frac{1}{\pi} \left(-\log \mu + \sum_{n=1}^{\infty} \frac{2^{2n-1} - 1}{n} |B_{2n}| (2\beta\mu)^{-2n} \right) \end{aligned}$$

- We clearly see that the genus expansion parameter in the double scaling limit is:

$$g_s = (\beta\mu)^{-1}$$

and it is held fixed as $\beta \to \infty, \mu \to 0$.

- Finally we recall that $E_{gr}(\mu) = \beta^2 \int d\mu\, \mu\, \rho(\mu)$ to write:

$$E_{gr}(g_s) = -\frac{1}{8\pi}\Big(-4g_s^{-2}\log g_s + \tfrac{1}{3}\log g_s + \sum_{h=2}^{\infty} \frac{2^{2h-1}-1}{2^{2n}h(h-1)(2h-1)}\,|B_{2h}|\, g_s^{2h-2}\Big)$$

- This is precisely the all-genus free energy of a string theory, the bosonic $c = 1$ string theory.