Finite-temperature Field Theory

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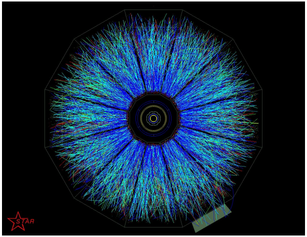
Initial Conditions in Heavy Ion Collisions
Goa, India, September 2008

Why is finite T/μ physics ($\approx$ QCD) interesting?

High energy physics applications

- Heavy ion experiments $\Rightarrow$ Need quantitive understanding of non-Abelian plasmas at
  - High T and small/moderate μ
  - Moderately large couplings
  - In and (especially) out of equilibrium
- Early universe thermodynamics
  - Signatures of phase transitions
  - EW baryogenesis
  - Cosmic relics
- Neutron star cores



In these lectures we will, however, only cover the very basics of finite-T formalism, restricting ourselves to

- Equilibrium physics — imaginary-time formalism
- Purely perturbative tools

For other approaches/setups, see in particular Guy Moore's lectures...

Plan of the lectures

- Lecture 1: Basic tools in finite-temperature field theory
  - Reminder of statistical physics
  - Scalar field theories and equilibrium thermodynamics
  - Interacting scalar fields at $T \neq 0$: $\lambda\varphi^4$ theory
- Lecture 2: Gauge field theories and linear response
  - More tools for finite-temperature calculations
  - Gauge symmetry, QED and QCD
  - Linear response theory
- Lecture 3: Special applications in QCD
  - Phases of hot and dense QCD
  - The IR problem
  - Dimensional reduction and high-T effective theories

Some useful reading

Material used in preparing these lectures:

- **J. Kapusta, Finite-temperature field theory**
- M. Le Bellac, Thermal field theory
- A. Rebhan, Thermal gauge field theories, hep-ph/0105183
- D. Rischke, Quark-gluon plasma in equilibrium, nucl-th/0305030
- ...

Outline

Basics of Statistical Physics and Thermodynamics
- Statistical QM
- Simple examples

Scalar fields at finite temperature
- Partition function for scalar field theory
- Non-interacting examples

Interacting scalar fields
- Feynman rules at finite temperature
- Thermodynamics of φ^4 theory

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Grand canonical ensemble

- Assume system that is free to exchange both energy and particles with a reservoir
  - "Grand canonical ensemble"
  - System described by temperature T and chemical potentials μ_i
- Basic operator *statistical density matrix* $\hat{\rho}$

$$\hat{\rho} = Z^{-1} \exp\Big[-\beta\underbrace{(\hat{H} - \mu_i \hat{N}_i)}_{\equiv \bar{H}}\Big],$$

$$\langle \hat{A} \rangle = \mathrm{Tr}[\hat{\rho}\hat{A}]$$

  - $\beta = 1/T$ and μ Lagrangian multipliers ensuring conservation of E and N_i

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- Most important quantity in equilibrium thermodynamics: The partition function Z

$$Z(V,T,\mu_i) = \mathrm{Tr}\, e^{-\beta\bar{H}} = \sum_i \langle i | e^{-\beta\bar{H}} | i \rangle = \sum e^{-\beta E_i}$$

- Several thermodynamic quantities available through derivatives of Z (due to above relation for expectation values)

$$\begin{aligned} p &= -\Omega/V = \frac{1}{\beta V}\ln Z, \\ N_i &= \frac{1}{V}\langle \hat{N}_i \rangle = \frac{\partial P}{\partial \mu_i} \\ \varepsilon &= \frac{1}{V}\langle \hat{H} \rangle = -\frac{1}{V}\frac{\partial \ln Z}{\partial \beta} \end{aligned}$$

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$$s = S/V = \frac{1}{V}\langle -\ln\hat{\rho}\rangle = \frac{1}{V}\ln Z + \frac{\beta}{V}\langle \hat{H} - \mu_i \hat{N}_i\rangle$$
$$= \frac{\partial P}{\partial T} = \beta(P + \varepsilon - \mu_i N_i)$$

- Finally obtain energy through other quantities as

$$E = -PV + TS + \mu_i N_i$$

- Important to note: Thermodynamic pressure = hydrodynamics pressure = $\frac{1}{3}\langle T^{ii}\rangle$

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QM harmonic oscillator

Use energy basis to obtain (in canonical ensemble):

$$
\begin{aligned}
\mathcal{Z} &= \sum_{n=0}^{\infty} \langle n | e^{-\beta \hat{H}} | n \rangle = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(\frac{1}{2}+n)} = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} = \frac{1}{2\sinh\left(\frac{\hbar\omega}{2T}\right)} \\
F &= T\ln\left(e^{\frac{\hbar\omega}{2T}} - e^{-\frac{\hbar\omega}{2T}}\right) = \frac{\hbar\omega}{2} + T\ln\left(1-e^{-\beta\hbar\omega}\right) \\
&\overset{T\ll\hbar\omega}{\approx} \frac{\hbar\omega}{2} \\
&\overset{T\gg\hbar\omega}{\approx} -T\ln\left(\frac{T}{\hbar\omega}\right), \\
S &= -\ln\left(1-e^{-\beta\hbar\omega}\right) + \frac{\hbar\omega}{T}\frac{1}{e^{\beta\hbar\omega}-1} \\
&\overset{T\ll\hbar\omega}{\approx} \frac{\hbar\omega}{T}e^{-\frac{\hbar\omega}{T}} \\
&\overset{T\gg\hbar\omega}{\approx} 1+\ln\frac{T}{\hbar\omega}, \\
E &= F+TS = \hbar\omega\left[\frac{1}{2}+\frac{1}{e^{\beta\hbar\omega}-1}\right] \\
&\overset{T\ll\hbar\omega}{\approx} \frac{\hbar\omega}{2} \\
&\overset{T\gg\hbar\omega}{\approx} T\,.
\end{aligned}
$$

One degree of freedom

- Consider one state system with $E = \omega$, and ignore zero point energy

$$\hat{H} \mid n\rangle \;=\; \omega \hat{N} \mid n\rangle \;=\; n\omega \mid n\rangle$$

- For bosons/fermions with chemical potential μ, obtain

$$Z_b = \mathrm{Tr}\, e^{-\beta(\omega-\mu)\hat{N}} = \sum_{n=1}^{\infty} \langle n \mid e^{-\beta(\omega-\mu)\hat{N}} \mid n\rangle = \frac{1}{1-e^{-\beta(\omega-\mu)}},$$

$$Z_f = \mathrm{Tr}\, e^{-\beta(\omega-\mu)\hat{N}} = \sum_{n=1}^{1} \langle n \mid e^{-\beta(\omega-\mu)\hat{N}} \mid n\rangle = 1 + e^{-\beta(\omega-\mu)}$$

- Taking derivatives...

$$N_b = \frac{1}{e^{\beta(\omega-\mu)}-1}, \quad E_b = \omega N_b$$

$$N_f = \frac{1}{e^{\beta(\omega-\mu)}+1}, \quad E_f = \omega N_f$$

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Partition function

- Consider real scalar field theory with no chemical potentials

$$S = \int \mathrm{d}t \int \mathrm{d}^3x \mathcal{L},$$

$$\mathcal{L} = \frac{1}{2}\left\{(\partial_t\phi)^2 - (\partial_i\phi)^2\right\} - V(\phi)$$

- To obtain equilibrium thermodynamics, want to compute partition function

$$Z = \int \mathcal{D}\phi \langle \phi \,|\, \mathrm{e}^{-\beta\hat{H}} \,|\, \phi \rangle$$

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- Recall path integral for transition amplitudes in QFT:

$$\langle \phi_b(\mathbf{x}) \mid e^{-i\hat{H}(t_1-t_0)} \mid \phi_a(\mathbf{x})\rangle = \mathcal{N} \int_{\substack{\varphi(t_0,\mathbf{x})=\phi_a(\mathbf{x}) \\ \varphi(t_1,\mathbf{x})=\phi_b(\mathbf{x})}} \mathcal{D}\varphi \, \exp\left[i \int_{t_0}^{t_1} dt \int d^3x \, \mathcal{L} \right]$$

- For thermodynamic purposes, set $t_0 = t_1$, $t \to -i\tau$:

$$iS \to -\int_{-it_0}^{\beta - it_0} \mathrm{d}\tau \int \mathrm{d}^d x \mathcal{L}_E,$$

$$\mathcal{L}_E = \frac{1}{2}\left\{ (\partial_t \varphi)^2 + (\partial_i \varphi)^2 \right\} + V(\varphi)$$

- Finally, taking the trace...

$$Z = \mathcal{N} \int_{periodic} \mathcal{D}\varphi \, \exp\left[-\int_{-it_0}^{\beta - it_0} \mathrm{d}\tau \int \mathrm{d}^d x \mathcal{L}_E \right]$$

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A few things to note:

- Periodicity of fields: Integrate over all φ satisfying $\varphi(-it_0, \mathbf{x}) = \varphi(\beta - it_0, \mathbf{x})$
  - Simple consequence of taking the trace
- Compactness of time direction
  - For equilibrium thermodynamics, may set $t_0 = 0$
- Real Gaussian path integral
  - No issues of existence/convergence
- Inclusion of finite chemical potentials straightforward (coming up in detail...)

Side remark: Direct proof of periodicity

- Define thermal (time-ordered) Green's function

$$G_B(\mathbf{x},\mathbf{y};\tau,0) \quad = \quad \mathrm{Tr}\left\{\hat{\rho}\, T_\tau\left[\hat{\phi}(\mathbf{x},\tau)\hat{\phi}(\mathbf{y},0)\right]\right\}$$

- Using cyclic property of the trace...

$$\begin{aligned}
G_B(\mathbf{x},\mathbf{y};\tau,0) &= Z^{-1}\mathrm{Tr}\left\{\mathrm{e}^{-\beta\hat{H}}\mathrm{e}^{\beta\hat{H}}\hat{\phi}(\mathbf{y},0)\mathrm{e}^{-\beta\hat{H}}\hat{\phi}(\mathbf{x},\tau)\right\} \\
&= Z^{-1}\mathrm{Tr}\left\{\mathrm{e}^{-\beta\hat{H}}\hat{\phi}(\mathbf{y},\beta)\hat{\phi}(\mathbf{x},\tau)\right\} \\
&= Z^{-1}\mathrm{Tr}\left\{\mathrm{e}^{-\beta\hat{H}}T_\tau\left[\hat{\phi}(\mathbf{x},\tau)\hat{\phi}(\mathbf{y},\beta)\right]\right\} \\
&= G_B(\mathbf{x},\mathbf{y};\tau,\beta)
\end{aligned}$$

Massive scalar fields

- Specialize now to $V(\varphi) = \frac{1}{2}m^2\varphi^2$
  - One free bosonic dof
- Following rules of Gaussian integration

$$\begin{aligned}\ln Z &= \ln\left\{\mathcal{N}\left(\det\left(-\partial_\tau^2 - \partial_i^2 + m^2\right)\right)^{-1/2}\right\} \\ &= -\frac{1}{2}\mathrm{Tr}\ln\left(-\partial_\tau^2 - \partial_i^2 + m^2\right) + \mathrm{const.}\end{aligned}$$

- How to take a trace of an operator containing derivatives?
  - Go to momentum space!

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- Expand fields in terms of their Fourier modes

$$\begin{aligned}\varphi(\tau, \mathbf{x}) &= T \sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \mathrm{e}^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n \tau)} \varphi_n(\mathbf{p}), \\ \omega_n &= 2n\pi T, \; n \in \mathbb{Z}\end{aligned}$$

  - Fourier *series* in τ direction due to compactness of temporal direction

- Now, a short exercise produces (up to T-indep. constant)

$$\begin{aligned}\ln Z &= -\frac{V}{2} \sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \ln\left(n^2 + \frac{\beta^2(p^2+m^2)}{(2\pi)^2}\right) \\ &= -V \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left[\frac{\sqrt{p^2+m^2}}{2T} + \ln\left(1 - \mathrm{e}^{-\beta\sqrt{p^2+m^2}}\right)\right]\end{aligned}$$

- Reminiscent of point particle result for bosons!

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Brief note on fermions

- For free Dirac fermions, going from QFT transition amplitude to finite-T partition function proceeds as above

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \mathcal{L}_E\right],$$
$$\mathcal{L} = \bar{\psi}\left\{\gamma_0\partial_\tau - i\gamma_i\partial_i + m\right\}\psi$$

- What about periodicity?
  - Spinor fields satisfy canonical *anticommutation* relations and are represented by Grassmann variables
- Let's try to repeat our proof of periodicity for bosonic fields...

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- Define again thermal Green's function

$$G_F(\mathbf{x},\mathbf{y};\tau,0) \;=\; \mathrm{Tr}\left\{\hat{\rho}\, T_\tau\left[\hat{\psi}(\mathbf{x},\tau)\hat{\psi}(\mathbf{y},0)\right]\right\}$$

  - This time field operators anticommute!

- Using cyclic property of the trace...

$$\begin{aligned}
G_F(\mathbf{x},\mathbf{y};\tau,0) &= Z^{-1}\mathrm{Tr}\left\{e^{-\beta\hat{H}}e^{\beta\hat{H}}\hat{\psi}(\mathbf{y},0)e^{-\beta\hat{H}}\hat{\psi}(\mathbf{x},\tau)\right\} \\
&= Z^{-1}\mathrm{Tr}\left\{e^{-\beta\hat{H}}\hat{\psi}(\mathbf{y},\beta)\hat{\psi}(\mathbf{x},\tau)\right\} \\
&= -Z^{-1}\mathrm{Tr}\left\{e^{-\beta\hat{H}}T_\tau\left[\hat{\psi}(\mathbf{x},\tau)\hat{\psi}(\mathbf{y},\beta)\right]\right\} \\
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- Conclusion: Fermion fields antiperiodic in time inside the path integral!

$$
\begin{aligned}
\psi(\beta,\mathbf{x}) &= -\psi(0,\mathbf{x}) \\
&\Rightarrow \\
\psi(\tau,\mathbf{x}) &= T\sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\mathrm{e}^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\psi_n(\mathbf{p}), \\
\omega_n &= (2n+1)\pi T,\ n\in\mathbb{Z}
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Scalar field theory with self-interactions

- Consider a scalar field theory with interaction term proportional to small parameter λ
  - Action $S = S_0 + \lambda S_I$
- Now expand in coupling

$$\begin{aligned}
\ln Z &= \ln\left\{\mathcal{N}\int_{periodic}\mathcal{D}\varphi\, e^{-(S_0+\lambda S_I)}\right\} \\
&= \ln\left\{\mathcal{N}\int_{periodic}\mathcal{D}\varphi\, e^{-S_0}\sum_{k=0}^{\infty}\frac{(-\lambda S_I)^k}{k!}\right\} \\
&= \ln Z_0 + \ln\left\{1+\sum_{k=1}^{\infty}\frac{(-\lambda)^k}{k!}\frac{\int\mathcal{D}\varphi\, e^{-S_0}S_I^k}{\int\mathcal{D}\varphi\, e^{-S_0}}\right\}
\end{aligned}$$

  - Note underlying assumption here: $[S_0, S_I] = 0$

Scalar field theory with self-interactions

- Consider a scalar field theory with interaction term proportional to small parameter λ
  - Action $S = S_0 + \lambda S_I$
- Now expand in coupling

$$
\begin{aligned}
\ln Z &= \ln\left\{ \mathcal{N} \int_{periodic} \mathcal{D}\varphi \, \mathrm{e}^{-(S_0 + \lambda S_I)} \right\} \\
&= \ln\left\{ \mathcal{N} \int_{periodic} \mathcal{D}\varphi \, \mathrm{e}^{-S_0} \sum_{k=0}^{\infty} \frac{(-\lambda S_I)^k}{k!} \right\} \\
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$$

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- Conclusion: In evaluating Z, end up computing expectation values of powers of S_I within unperturbed theory

$$\langle S_I^k \rangle_0 = \frac{\int \mathcal{D}\varphi \, \mathrm{e}^{-S_0} S_I^k}{\int \mathcal{D}\varphi \, \mathrm{e}^{-S_0}}$$

- No external legs: Partition function available through the computation of *vacuum* diagrams
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Feynman rules for ϕ^4 theory

- Assume massive scalar theory with $L_I = \varphi^4$
  - Graphs composed of four-φ vertices and propagators

$$D_0(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + p^2 + m^2}$$

- Feynman rules almost the same as at $T = 0$:
  - As usual, only need to compute *connected* diagrams
  - Same symmetry factors, vertex functions, etc.
- Only changes due to discrete values of p_0:
  - In integration measure: $\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \rightarrow T \sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3}$
  - In vertices: $\delta^{(4)}(P_1 - P_2) \rightarrow \delta_{n_1, n_2} \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2)$
- Question: How to evaluate the necessary sum-integrals?
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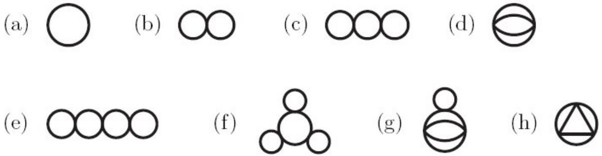
Example: Pressure of massless φ^4 theory

- To obtain pressure of φ^4 theory to 4 loops, must evaluate



- Up to two loops:

$$p(T) = p_0(T) - 3\lambda \left(\sum\!\!\!\!\!\!\int_P \frac{1}{\omega_n^2 + p^2} \right)^2$$

$$\equiv p_0(T) - 3\lambda \left(\sum\!\!\!\!\!\!\int_P \frac{1}{P^2} \right)^2$$

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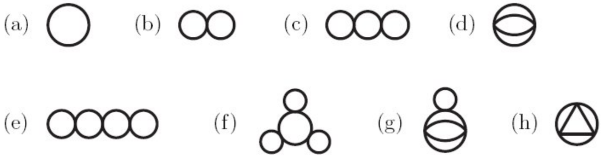

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where the sum-integral reads...

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\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} &= \Lambda^{2\varepsilon} T \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^{3-2\varepsilon} p}{(2\pi)^{3-2\varepsilon}} \frac{1}{p^2 + (2n\pi T)^2} \\
&= \frac{2\Gamma(-1/2+\varepsilon)}{(4\pi)^{3/2-\varepsilon}} \Lambda^{2\varepsilon} T (2\pi T)^{1-2\varepsilon} \sum_{n=1}^{\infty} n^{1-2\varepsilon} \\
&= \frac{2\Gamma(-1/2+\varepsilon)}{(4\pi)^{3/2-\varepsilon}} \Lambda^{2\varepsilon} T (2\pi T)^{1-2\varepsilon} \zeta(-1+2\varepsilon) \\
&= \frac{T^2}{12} + \mathcal{O}(\varepsilon)
\end{aligned}
$$

- Recalling the free theory result, finally obtain

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p(T) = \frac{\pi^2 T^4}{90} \left(1 - \frac{15}{8\pi^2} \lambda \right) + \mathcal{O}(\lambda^2)
$$

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