Finite-temperature Field Theory

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Initial Conditions in Heavy Ion Collisions
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Outline

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The thermal propagator and self energy

- At leading order, observe

$$D_0(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + p^2 + m^2} = \beta^2 \langle \varphi_n(\mathbf{p}) \varphi_{-n}(-\mathbf{p}) \rangle |_{\lambda=0}$$

- Natural generalization:

$$\begin{aligned} D(\omega_n, \mathbf{p}) &= \beta^2 \langle \varphi_n(\mathbf{p}) \varphi_{-n}(-\mathbf{p}) \rangle \quad \Leftrightarrow \\ D(\tau_1, \mathbf{x}_1; \tau_2, \mathbf{x}_2) &= \langle \varphi(\tau_1, \mathbf{x}_1) \varphi(\tau_2, \mathbf{x}_2) \rangle \end{aligned}$$

- Define scalar self energy Π as correction term to inverse propagator

$$\begin{aligned} D(\omega_n, \mathbf{p})^{-1} &= \omega_n^2 + p^2 + m^2 + \Pi(\omega_n, \mathbf{p}) \\ &= D_0^{-1} (1 + D_0 \Pi) \end{aligned}$$

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  - The masses and dispersion relations of quasiparticles
  - The interaction potential (possible screening)
- Self energy obtainable through computation of all connected 1PI two-point graphs
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Introducing finite chemical potentials

- So far in all field theory examples $\mu = 0$
  - Reason: No conserved charge associated with real scalar field
- Consider now Dirac fermions at chemical potential μ

$$\hat{H} \rightarrow \hat{H} - \mu\hat{N}, \qquad \hat{N} = \int d^3x\, \psi^\dagger \psi$$

  - Fermion number conserved due to global U(1) symmetry $\psi \rightarrow e^{i\alpha}\psi$
- Action changes now to

$$S_E = \int_0^\beta d\tau \int d^d x \bar{\psi} \Big\{ \gamma_0(\partial_\tau - \mu) - i\gamma_i\partial_i + m \Big\} \psi \Big]$$

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- Conclusion: With finite chemical potentials, Matsubara frequencies shift by iμ

$$\omega_n \quad \rightarrow \quad \omega_n + i\mu \quad = \quad (2n+1)\pi T + i\mu$$

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    - Obvious instability if $|\mu| > m$!
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How to compute sum-integrals?

- Perturbative calculations at $T \neq 0$ require performing sum-integrals

$$S = \sum_{\omega_n} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f(\omega_n, \mathbf{p}),$$
$$\omega_n = 2n\pi T \ \text{ or } \ (2n+1)\pi T + i\mu$$

- Two generic tricks for evaluating the sums: Contour integrals and 3d Fourier transforms
  - Optimal choice depends on whether fields massive or massless
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Contour integral trick

- Generic observation: May convert Matsubara sum into a contour integral via Residue theorem

$$T \sum_{n=-\infty}^{\infty} f(p_0 = i \times 2n\pi T) = \frac{1}{4\pi i} \int_C \mathrm{d}p_0 f(p_0) \coth\left(\frac{p_0}{2T}\right)$$

with C circulating poles of the coth function ($p_0 = i \times 2n\pi T$) in a counterclockwise direction

- Separating from coth a piece that vanishes at $T = 0$:

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$$= \frac{1}{2\pi T} \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} \mathrm{d}p_0 \left[f(p_0) + f(-p_0)\right] \left\{\frac{1}{2} + \frac{1}{e^{\beta p_0} - 1}\right\}$$

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- Advantage: Performing contour integral trick for each loop momentum, separate vacuum ($T = 0$) contribution from each diagram
  - $T = 0$ piece easy to evaluate with standard methods
  - Finite-T piece obviously UV safe, and can (usually) be evaluated by closing integration contour on the R.S. of complex plane
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3*d* Fourier transforms

- Assume now important simplification: All $T = 0$ masses zero. Then...

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}}}{q^2 + (2n\pi T)^2} = \frac{\mathrm{e}^{-2|n|\pi T r}}{4\pi T},$$

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  - Simple Matsubara sums: Harmonic series
  - (Hyper)trigonometric integrals in coordinate space
- Result: (Almost) analytic results up to 4-loop order!

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$$\frac{p(T)}{T^4} = \frac{\pi^2 N}{90}\sum_{i=0}^{6} p_i \left(\frac{g}{4\pi}\right)^i , \tag{4.2}$$

where $g \equiv [g^2(\bar\Lambda)]^{1/2}$, and the coefficients read

$$p_0 = 1 , \tag{4.3}$$

$$p_1 = 0 , \tag{4.4}$$

$$p_2 = -\frac{5}{12}(N+2) , \tag{4.5}$$

$$p_3 = \frac{5\sqrt{2}}{9}(N+2)^{\frac{3}{2}} , \tag{4.6}$$

$$\begin{aligned} p_4 = {}& \frac{5}{36}(N+2)\left\{N\left[\ln\frac{\bar\Lambda}{4\pi T} + \gamma_E - 6\right] + \right. \\ & \left. +8\left[\ln\frac{\bar\Lambda}{4\pi T} - \frac{29}{40} + \frac{\gamma_E}{4} + \frac{3}{2}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{3}{4}\frac{\zeta'(-3)}{\zeta(-3)}\right]\right\} , \end{aligned} \tag{4.7}$$

$$\begin{aligned} p_5 = {}& -\frac{5}{9\sqrt{2}}(N+2)^{\frac{3}{2}}\left\{-12\ln\left(\frac{g}{\pi}\sqrt{\frac{N+2}{72}}\right) + \right. \\ & \left. +N\left[\ln\frac{\bar\Lambda}{4\pi T} + \gamma_E - \frac{3}{2}\right] + 8\left[\ln\frac{\bar\Lambda}{4\pi T} + \frac{9}{8} + \frac{\gamma_E}{4} - \frac{3}{4}\frac{\zeta'(-1)}{\zeta(-1)}\right]\right\} , \end{aligned} \tag{4.8}$$

$$\begin{aligned} p_6 = {}& -\frac{5}{108}(N+2)\left\{\left[72(N+2) - 6(N+8)\pi^2\right]\ln\left(\frac{g}{\pi}\sqrt{\frac{N+2}{72}}\right) + \right. \\ & +(N+8)^2\left(\ln\frac{\bar\Lambda}{4\pi T}\right)^2 + \\ & +N^2\left[\left(2\gamma_E - 12\right)\ln\frac{\bar\Lambda}{4\pi T} + 6 - 12\gamma_E + \gamma_E^2 + \frac{\zeta(3)}{36}\right] + \\ & +16N\left[\left(-\frac{493}{80} + \frac{5\gamma_E}{4} + \frac{3}{2}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{3}{4}\frac{\zeta'(-3)}{\zeta(-3)}\right)\ln\frac{\bar\Lambda}{4\pi T} - 0.9991160242(2)\right] + \\ & \left. +64\left[\left(-\frac{127}{160} + \frac{\gamma_E}{2} + 3\frac{\zeta'(-1)}{\zeta(-1)} - \frac{3}{2}\frac{\zeta'(-3)}{\zeta(-3)}\right)\ln\frac{\bar\Lambda}{4\pi T} - 9.0905637831(3)\right]\right\} . \end{aligned} \tag{4.9}$$

Renormalization of the theory

- As always in quantum field theories, in order to obtain finite results from perturbative calculations, we must renormalize the theory
  - Fields and parameters appearing in Lagrangian not physical, measurable quantities
  - Need to define parameters with renormalization corrections: $\varphi \;\rightarrow\; \mathcal{Z}_\varphi^{1/2}\varphi_\mathrm{R}$
- Simplification: Finite temperature does not generate any new divergences
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- In practice, need to introduce energy scale ($\Lambda^{2\varepsilon}$ from yesterday) at which renormalization is performed
    - Physical results independent of Λ — however, in practice useful to choose $\Lambda \sim T$ to minimize errors in finite-order calculations

Outline

Constructing the partition function

- Consider SU(N) YM coupled to $m = 0$ fundam. fermions

$$\begin{aligned}\mathcal{L}_{\text{QCD}} &= \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi} \not{D} \psi, \\ F^a_{\mu\nu} &\equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \\ D_\mu &\equiv \partial_\mu - i g A^a_\mu T^a, \quad \text{Tr}\, T^a T^b = \frac{1}{2} \delta^{ab}\end{aligned}$$

  - Easy to restrict to pure Yang-Mills or QED later

- Theory invariant under gauge transformation

$$\begin{aligned}A_\mu \equiv A^a_\mu T^a &\to \Omega^{-1} A_\mu \Omega + \frac{i}{g} \left(\partial_\mu \Omega^{-1} \right) \Omega, \\ \psi &\to \Omega^{-1} \psi, \\ \Omega &= \exp\left[i g T^a \alpha^a \right]\end{aligned}$$

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- Obvious issue in evaluating Z the overcounting of degrees of freedom due to gauge symmetry
  - Famous example: Free photons in QED give twice the usual black body pressure!
  - How to restrict to physical Hilbert space?
- Usual choice: Temporal $A_0 = 0$ gauge
  - Coordinates and momenta now A_i and

$$\Pi_i^a = \frac{\delta L}{\delta \dot{A}_i^a} = \dot{A}_i^a,$$

- Resulting Hamiltonian

$$H_{\text{temp}} = \int \mathrm{d}^3x \left\{ \frac{1}{2}\Pi_i^a \Pi_i^a - \frac{1}{4} F_{ij}^a F_{ij}^a - \bar{\psi}\gamma_i D_i \psi - \psi^\dagger \partial_0 \psi \right\}$$

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- Gauss' law not part of Hamiltonian equations of motion $\Rightarrow$ Must include it separately
  - Introduce into path integral *projection operator onto the space of physical states*

$$P = \int_{\Lambda(\infty)=0} \mathcal{D}\Lambda \exp\Big[i\beta \int \mathrm{d}^3x \Lambda^a G^a\Big],$$
$$G^a \equiv \partial_i F^a_{i0} + g f^{abc} A^b_i F^c_{i0} + T^a \psi^\dagger \psi$$

- Result: After renaming $\Lambda \Rightarrow A_0$, obtain expected expression

$$Z_{\text{QCD}} = \int_{\substack{A_\mu \text{ per.} \\ \psi \text{ antip.}}} \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\Big[-\int_0^\beta \mathrm{d}x_0 \int \mathrm{d}^3x \left(\mathcal{L}_{\text{QCD}} - \psi^\dagger \mu \psi\right)\Big]$$

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Z_{\rm QCD} = \int_{\substack{A_\mu \ per. \\ \psi \ antip.}} \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\Big[-\int_0^\beta \mathrm{d}x_0 \int \mathrm{d}^3x \left(\mathcal{L}_{\rm QCD} - \psi^\dagger \boldsymbol{\mu} \psi\right)\Big]
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Removing the residual gauge freedom

- Standard choice for fixing residual gauge freedom: Covariant gauge condition

$$F^a[A] \equiv \partial_\mu A^a_\mu - f^a = 0,$$

with f^a undetermined

- Insert now into the path integral $1 = \Delta\Delta^{-1}$ with

$$\Delta[A] \equiv \int_{\substack{\Omega \; per. \\ \in \mathrm{SU}(N)}} \mathcal{D}\Omega \, \delta[F^a[A^\Omega]]$$

- And use gauge invariance of action to obtain

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- Upon change of variables $A^\Omega \;\longrightarrow\; A$, the Ω-integral obviously factorizes $\Rightarrow$

$$Z_{\rm QCD} \;=\; \int_{\substack{A_\mu\, per.\\ \psi\, antip.}} \mathcal{D}A_\mu \mathcal{D}\bar\psi \mathcal{D}\psi\, \Delta^{-1}[A]\delta[F^a[A]]\exp\Big[-S[A]\Big]$$

- Finally, write gauge condition

$$\Delta^{-1}[A] \;=\; \det\left(\frac{\delta F^a(x)}{\delta\alpha^b(x')}\right)\bigg|_{F^a=0} \;\equiv\; \det M^{ab},$$

$$M^{ab}(x,y) \;=\; \partial_\mu\Big\{\Big(\partial_\mu\delta^{ab}+gf^{abc}A^c_\mu\Big)\delta(x-y)\Big\}$$

in terms of anticommuting, but periodic 'ghost' fields η, $\bar\eta$:

$$\det M^{ab} \;=\; \int_{\eta\, per.} \mathcal{D}\bar\eta\mathcal{D}\eta\exp\bigg[-\int_0^\beta {\rm d}x_0\int {\rm d}^3x\int_0^\beta {\rm d}y_0\int {\rm d}^3y\,\bar\eta^a(x)M^{ab}(x,y)\eta^b(y)\bigg]$$

- Upon change of variables $A^\Omega \rightarrow A$, the Ω-integral obviously factorizes $\Rightarrow$

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- Multiplying the functional integral by

$$\exp\left[-\frac{1}{2\xi}\int_0^\beta \mathrm{d}x_0 \int \mathrm{d}^3x (f^a(x))^2\right]$$

and integrating over f^a, we obtain the final result

$$Z_{\mathrm{QCD}} = \int_{\substack{A_\mu\ per.\\ \psi\ antip.}} \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{\eta}\mathcal{D}\eta \exp\left[-\int_0^\beta \mathrm{d}x_0 \int \mathrm{d}^3x\, \mathcal{L}_{\mathrm{eff}}\right],$$

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{QCD}} + \frac{1}{2\xi}(\partial_\mu A_\mu^a)^2 - \psi^\dagger \boldsymbol{\mu}\psi + \bar{\eta}^a\left(\partial^2\delta^{ab} + gf^{abc}A_\mu^c\partial_\mu\right)\eta^b$$

- Feynman rules again obtained from $T=0$ ones taking into account discreteness of p_0
  - Gluons and ghosts (despite anticommutativity!) periodic in τ ($\omega_n = 2n\pi T$)
  - Quarks antiperiodic ($\omega_n = (2n+1)\pi T$)

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- Note difference to $T = 0$: Even when ghosts decouple (Abelian theories), they still contribute to grand potential and other thermodynamic quantities!

Outline

Linear response

- So far, only equilibrium systems considered
  - What if system disturbed by small perturbation: $\hat{H} \to \hat{H}_0 + \delta\hat{H}(t)$, with $\delta\hat{H}$ turned on at $t = t_0$?
- Goal: Want to compute shifts in expectation values $\langle\hat{A}(\mathbf{x},t)\rangle$ up to linear order in $\delta\hat{H}$

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\begin{aligned}
\delta\langle\hat{A}(\mathbf{x},t)\rangle &= \int_{t_0}^{t} \mathrm{d}t' \, \mathrm{Tr}\Big[\hat{\rho}\,\partial_{t'}\hat{A}(\mathbf{x},t')\Big] \\
&= i\int_{t_0}^{t} \mathrm{d}t' \, \mathrm{Tr}\Big[\hat{\rho}\,[\delta\hat{H}(t'),\hat{A}(\mathbf{x},t')]\Big] \\
&\approx i\int_{t_0}^{t} \mathrm{d}t' \, \mathrm{Tr}\Big[\hat{\rho}\,[\delta\hat{H}(t'),\hat{A}(\mathbf{x},t)]\Big] + \mathcal{O}(\delta^2)
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- Consider scalar field coupled to external source J:

$$\delta\hat{H}(t') = \int \mathrm{d}^3x'\, J(\mathbf{x}',t')\hat{\varphi}(\mathbf{x}',t')$$

- For change in $\langle\hat{\varphi}\rangle$, obtain integral of source coupled to *retarded* Green's function

$$\delta\langle\hat{\varphi}(\mathbf{x},t)\rangle = \int_{-\infty}^{\infty}\mathrm{d}t' \int \mathrm{d}^3x' J(\mathbf{x}',t')D^R(\mathbf{x},t;\mathbf{x}',t'),$$

$$D^R(\mathbf{x},t;\mathbf{x}',t') \equiv \mathrm{Tr}\left[\hat{\rho}\,[\hat{\varphi}(\mathbf{x},t),\hat{\varphi}(\mathbf{x}',t')]\right]\theta(t-t')$$

or in Fourier space...

$$\delta\langle\hat{\varphi}(\omega,\mathbf{p})\rangle = J(\omega,\mathbf{p})D^R(\omega,\mathbf{p})$$

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Coupling of QED plasma to static electric field

- Consider coupling QED plasma to static classical background field $\mathbf{E}_{cl}$

$$\begin{aligned} H &= \frac{1}{2}\left\{(\mathbf{E}+\mathbf{E}_{cl})^2+\mathbf{B}^2\right\} \\ &= \frac{1}{2}\left\{\mathbf{E}^2+\mathbf{B}^2\right\}+\mathbf{E}\cdot\mathbf{E}_{cl}+\frac{1}{2}\mathbf{E}_{cl}^2 \\ &\equiv H_0+\delta H \end{aligned}$$

- To first order in E^i_{cl}, obtain now for the shift in $\langle \mathbf{E}\rangle$:

$$\delta\langle E_i(\mathbf{x},t)\rangle = -i\int_{-\infty}^{\infty}\mathrm{d}t'\int \mathrm{d}^3x'\,E^j_{cl}(\mathbf{x},t)\,\mathrm{Tr}\left[\hat{\rho}\left[E_i(\mathbf{x},t),E_j(\mathbf{x}',t')\right]\right]\theta(t-t')$$

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- Writing E_i's in terms of derivatives of A_μ and going to Fourier space...

$$E^i_{net}(\mathbf{x}) = -\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}} p_i p_j E^j_{cl}(\mathbf{p}) D^R_{00}(\omega = 0, \mathbf{p})$$

where we have assumed a covariant gauge.

- With rotational invariance ($\mathbf{E}_{cl} \sim \mathbf{p}$), finally obtain

$$\mathbf{E}_{net}(\mathbf{p}) = \frac{\mathbf{E}_{cl}(\mathbf{p})}{\epsilon(\mathbf{p})},$$

$$\epsilon(\mathbf{p}) \equiv 1 + \frac{\Pi_{00}(\omega = 0, \mathbf{p})}{p^2}$$

- In IR limit ($p \ll T$), obtain $\Pi_{00}(\omega = 0, \mathbf{p}) \rightarrow m_D^2$
  - Potential between charges $V(r) = q_1 q_2 \frac{\mathrm{e}^{-m_D r}}{4\pi r}$ — screening!

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Outline

Summary (so far)

So far, we've reviewed some basic formalism of FTFT's:

- Path integral formulation of partition function and other equilibrium thermodynamic quantities
  - Bosons and fermions, zero and finite density
  - Interacting theories at weak coupling: Feynman rules at finite T and evaluation of sum integrals
- Special issues with gauge symmetry
  - Ghosts and gauge choices
- Response of system to small external perturbations
  - Possible screening of charge due to interactions

Next, we'll start applying this machinery to QCD...