Lectures on hydrodynamics - Part I: Ideal (Euler) hydrodynamics

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Outline

- **Why hydro**
- **Ideal hydro equations of motion, Bjorken hydrodynamics**
- **Freezeout and initial conditions**
- **Equation of state**
- **Successes of ideal hydro and open problems**

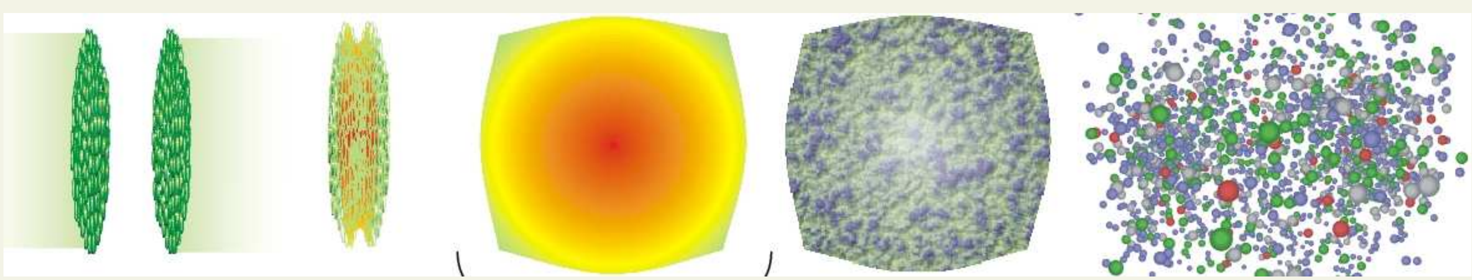
Hydrodynamics

- **describes a system near local equilibrium**

- **long-wavelength, long-timescale dynamics, driven by conservation laws**

- **in heavy-ion physics: mainly used for the plasma stage of the collision**

initial nuclei parton plasma hadronization hadron gas



$\uparrow$ $\sim 10^{-14}$ **m** **= 10 fermis** $\downarrow$

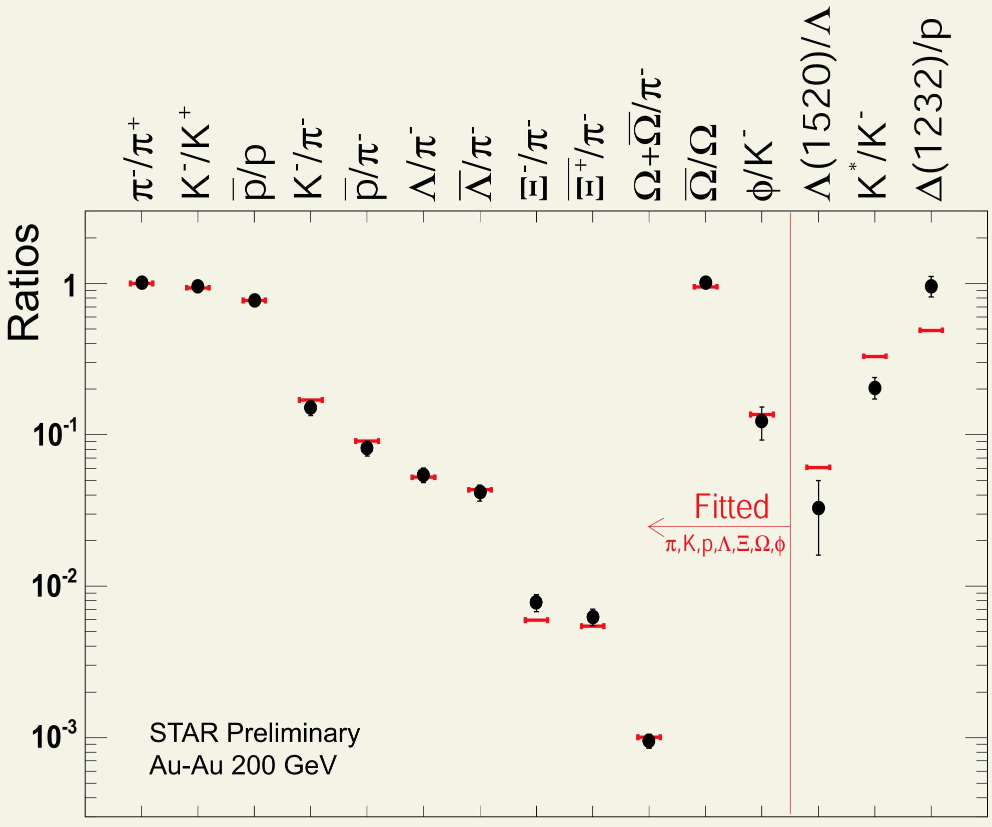
<—— $\sim 10^{-23}$ **sec = 10 yochto(!)-secs** ——>

nontrivial how hydrodynamics can be applicable at such microscopic scales

Particle ratios

particle ratios explained well from a simple statistical model

$$\frac{N_i}{V} = \frac{g_i}{2\pi^2} \int \frac{dp\ p^2}{\exp[E_i(p) - \mu_i]/T \pm 1}$$



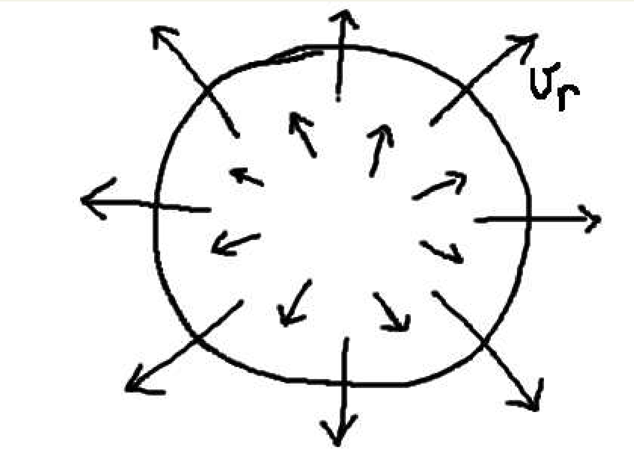
central Au+Au:

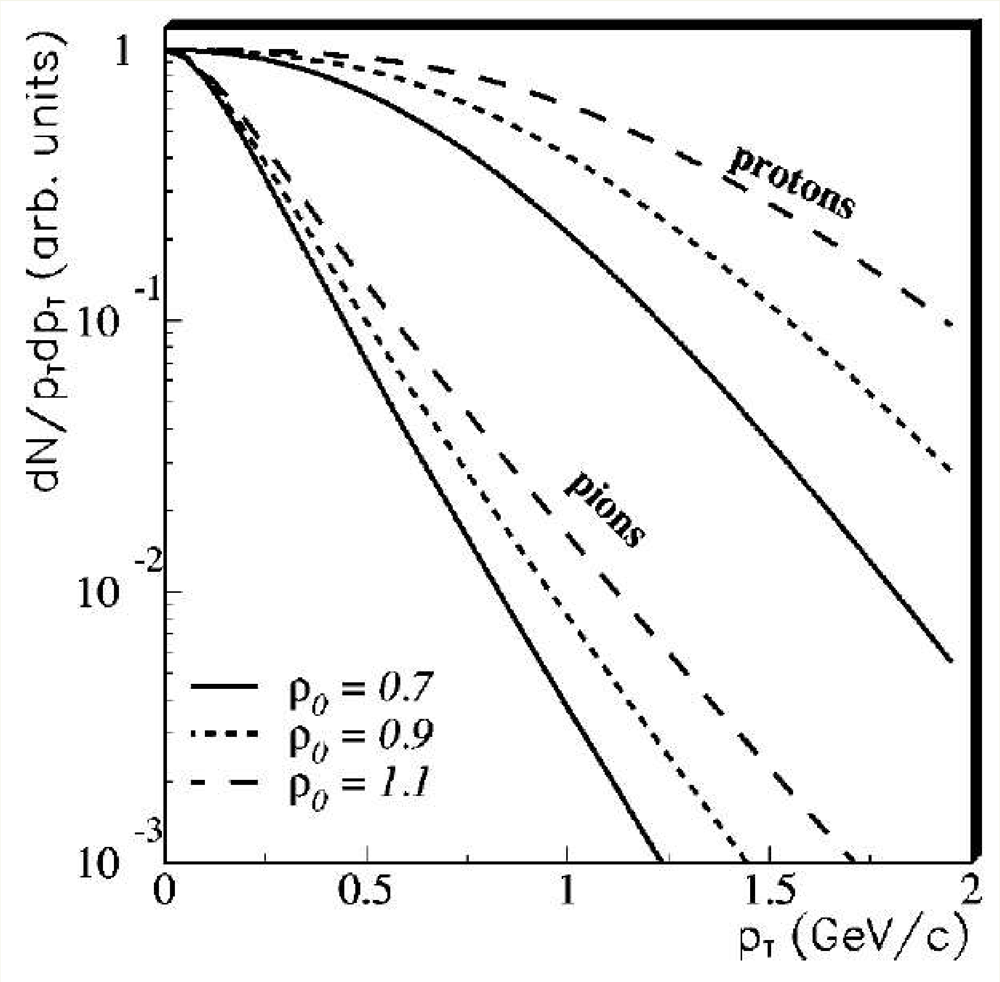
$T_{chem} = 160 \pm 5$ **MeV**
$\mu_B = 24 \pm 4$ **MeV**
$\mu_s = 1.4 \pm 1.6$ **MeV**
[nucl-ex/0412016]

however, also works for $p + p$ and even $e^+ + e^-$ (if we allow for a suppressed yield of strange quarks)

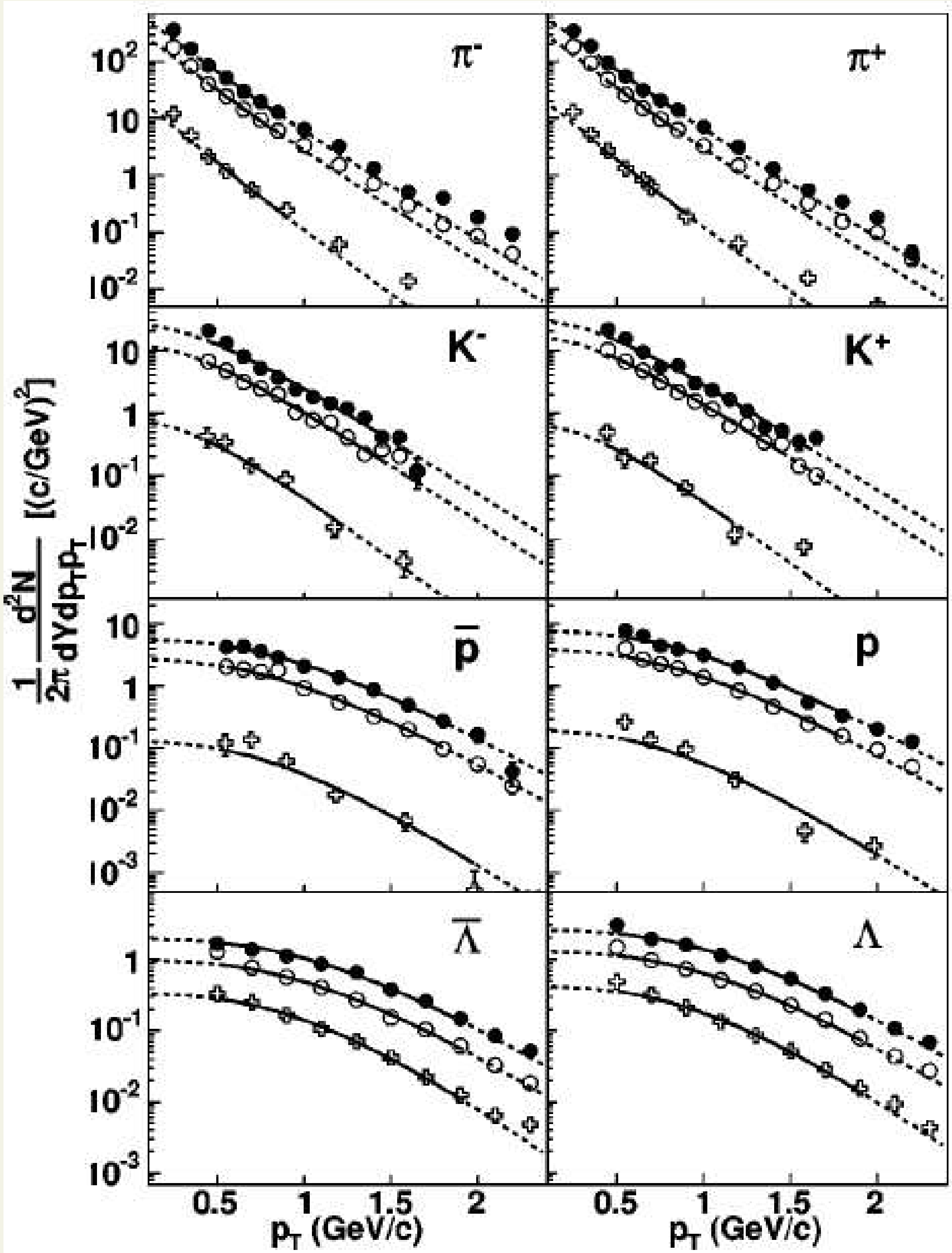
Radial flow

particle spectra $E\frac{dN}{d^3p} \equiv \frac{dN}{d^2p_T\,dy}$ **indicate strong transverse expansion (flow)**





transverse boost flattens → spectra



"Blast-wave" model

- a distribution of moving thermal sources

[Schnedermann, Sollfrank, & Heinz, PRC48, 2462 ('94)]

fits for Au+Au at RHIC

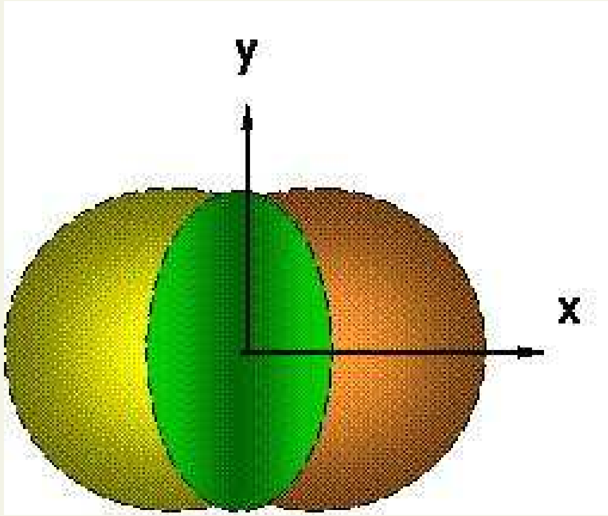
[Retiere & Lisa, PRC 70, 044907 ('04)]

average radial velocity
$\langle v_r \rangle \sim 0.5c$**(!)**

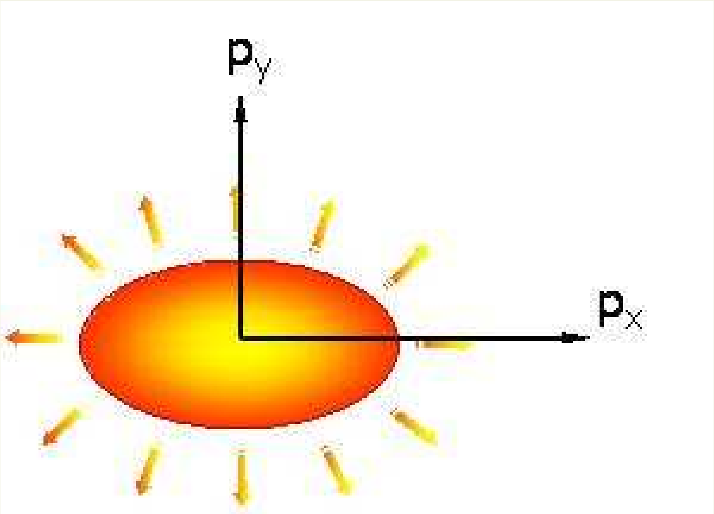
kinetic freezeout temperature
$T_{fo} \sim 110$ **MeV**

Elliptic flow

initial spatial anisotropy converts to final momentum space anisotropy



$\longrightarrow$



$$\varepsilon \equiv \frac{\langle x^2 - y^2 \rangle}{\langle x^2 + y^2 \rangle} \qquad\qquad v_2 \equiv \frac{\langle p_x^2 - p_y^2 \rangle}{\langle p_x^2 + p_y^2 \rangle} \equiv \langle \cos 2\phi_p \rangle$$

hydrodynamic models can generate the large v_2 observed at RHIC

Ideal (Euler) hydrodynamics

for an isotropic system in local equilibrium, in local rest frame (LR) of the fluid, the energy momentum tensor and conserved currents

$$T^{\mu\nu}_{LR} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} , \qquad N^{\mu}_{i\ ,LR} = (n_i, \vec{0}) \quad [i = 1, ..., I]$$

are given by a few, local, macroscopic variables: energy density $\varepsilon(x)$
pressure $p(x)$
charge densities $n_i(x)$

In an arbitrary frame where the fluid could be moving

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p\, g^{\mu\nu} , \qquad N^\mu_i = n_i\, u^\mu$$

where $u^\mu(x) = \frac{1}{\sqrt{1-v^2(x)}}(1, \vec{v}(x))$ is the flow 4-velocity of the fluid ($u^\mu u_\mu = 1$)

Equations of motion

given by energy-momentum and charge conservation

$$\partial_\mu T^{\mu\nu}(x) = 0 \ , \qquad \partial_\mu N_i^\mu(x) = 0 \quad [i = 1, ..., I]$$

4 + I equations for 5 + I unknowns ($\varepsilon, p, u^\mu, \{n_i\}$)

(in heavy ion physics, typically $I \leq 1$, we only follow baryon charge n_B, if at all)

missing ingredient: equation of state $p(\varepsilon, \{n_i\})$

———

the above set of partial differential equations can then be solved for given initial and boundary conditions

Decomposing the 4-gradient as

$$\partial_\mu \equiv u_\mu u_\nu \partial^\nu + (g_{\mu\nu} - u_\mu u_\nu)\partial^\nu \equiv u_\mu D + \nabla_\mu \qquad [D^{LR} = \partial_t, \nabla_\mu^{LR} = (0, \vec{\nabla})]$$

we have

$$\dot{u}^\nu = -\frac{\nabla^\nu p}{\varepsilon + p} \ , \qquad \dot{\varepsilon} = -(\varepsilon + p)\partial u \ , \qquad \dot{n}_i = -n_i \partial u$$

where $\dot{A} \equiv DA$ and $\partial u \equiv \partial_\alpha u^\alpha$.

The first equation tells that the fluid is accelerated by pressure gradients.

Notice (Problem 1) that ∂u is the volume expansion rate $(\partial_t \Delta V)/\Delta V$ in the LR ($u^\mu_{LR} = (1, \vec{0})$). Therefore, in the LR

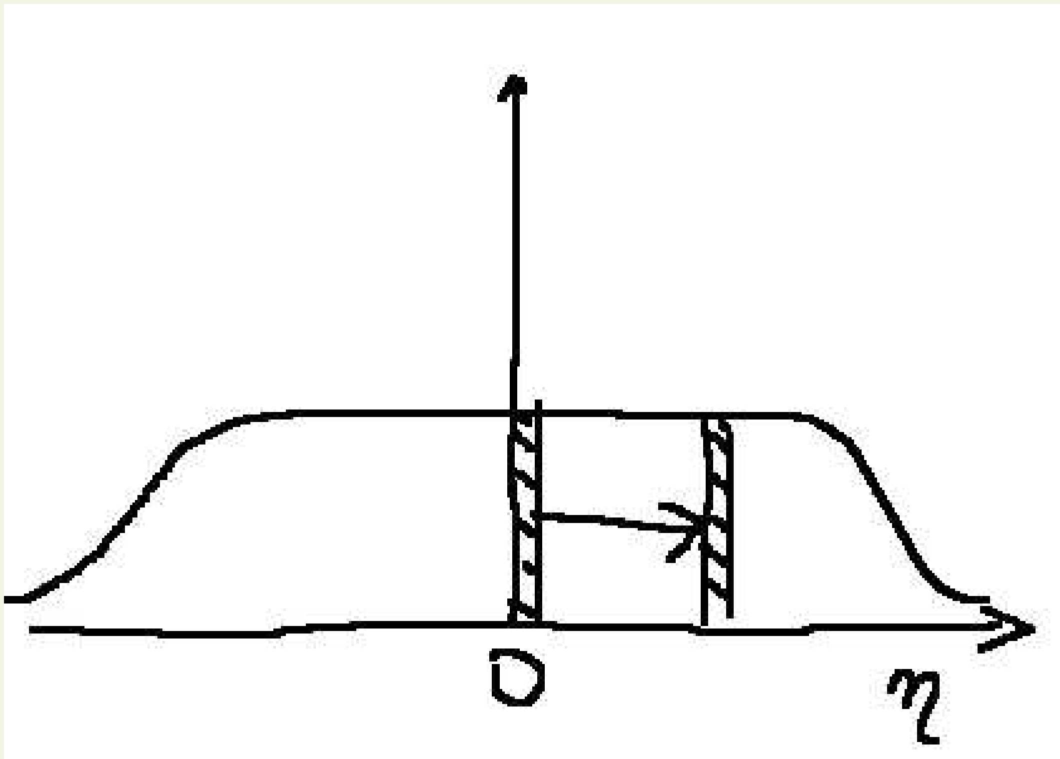
$$\partial_t(\Delta V n_i) = 0 \ , \qquad \partial_t(\Delta V \varepsilon) = -p\Delta V$$

From the thermodynamic identity $TdS = dE + pdV - \mu_i dN_i$ we then see that ideal hydrodynamics conserves entropy $\partial_\mu(s\, u^\mu) = 0$ (no dissipation).

Bjorken hydrodynamics [Bjorken, PRD27, 140 ('83)]

In high-energy collisions, a flat dN/dy was expected over a wide rapidity range (rapidity plateau) $\rightarrow$ Bjorken hypothesized that physics is rapidity independent (a sufficient condition).

Observables at nonzero rapidity can then be obtained from the central rapidity slice $\eta = 0$ via a longitudinal boost



$y \equiv \frac{1}{2} \ln \frac{E+p_z}{E-p_z}$

$m_T \equiv \sqrt{E^2 - p_z^2} = \sqrt{m^2 + p_T^2}$

i.e., $E = m_T \, \mathrm{ch}\, y$, $p_z = m_T \, \mathrm{sh}\, y$

$\eta \equiv \frac{1}{2} \ln \frac{t+z}{t-z}$

$\tau \equiv \sqrt{t^2 - z^2}$

i.e., $t = \tau \, \mathrm{ch}\, \eta$, $z = \tau \, \mathrm{sh}\, \eta$

(hyperbolic coordinates)

Suppose we ignore expansion and inhomogeneities in the transverse directions. Boost invariance then implies a unique flow pattern

$$u^\mu(x) \equiv u^\mu_{BJ} = (\mathrm{ch}\,\eta, 0, 0, \mathrm{sh}\,\eta)$$

(in other words, $v_z = z/t$). It follows (Problem 2) that $D = \partial_\tau$, $\partial u = 1/\tau$ and thus the equations of motion are

$$\dot{\varepsilon}(\tau) = -\frac{\varepsilon(\tau) + p(\tau)}{\tau}\ , \qquad \dot{n}_i(\tau) = -\frac{n_i(\tau)}{\tau}$$

I.e., densities drop in inverse proportion to proper time $n_i(\tau) = n_i(\tau_0)\tau_0/\tau$.

For massless gas equation of state $\varepsilon = 3p$ (applicable for a high-temperature plasma)

$$e(\tau) = e(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{4/3}\ , \qquad T(\tau) = T(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1/3}$$

Other ideal hydro solutions

- **sound waves** **(small-amplitude waves over a static uniform background)**

$$\text{speed of sound}: \quad c_s^2 = \left.\frac{\partial p}{\partial \varepsilon}\right|_{s=const} \qquad (\text{Problem } 3)$$

- **various (semi-)analytic solutions** [Landau... Baym... Csorgo et al ...]

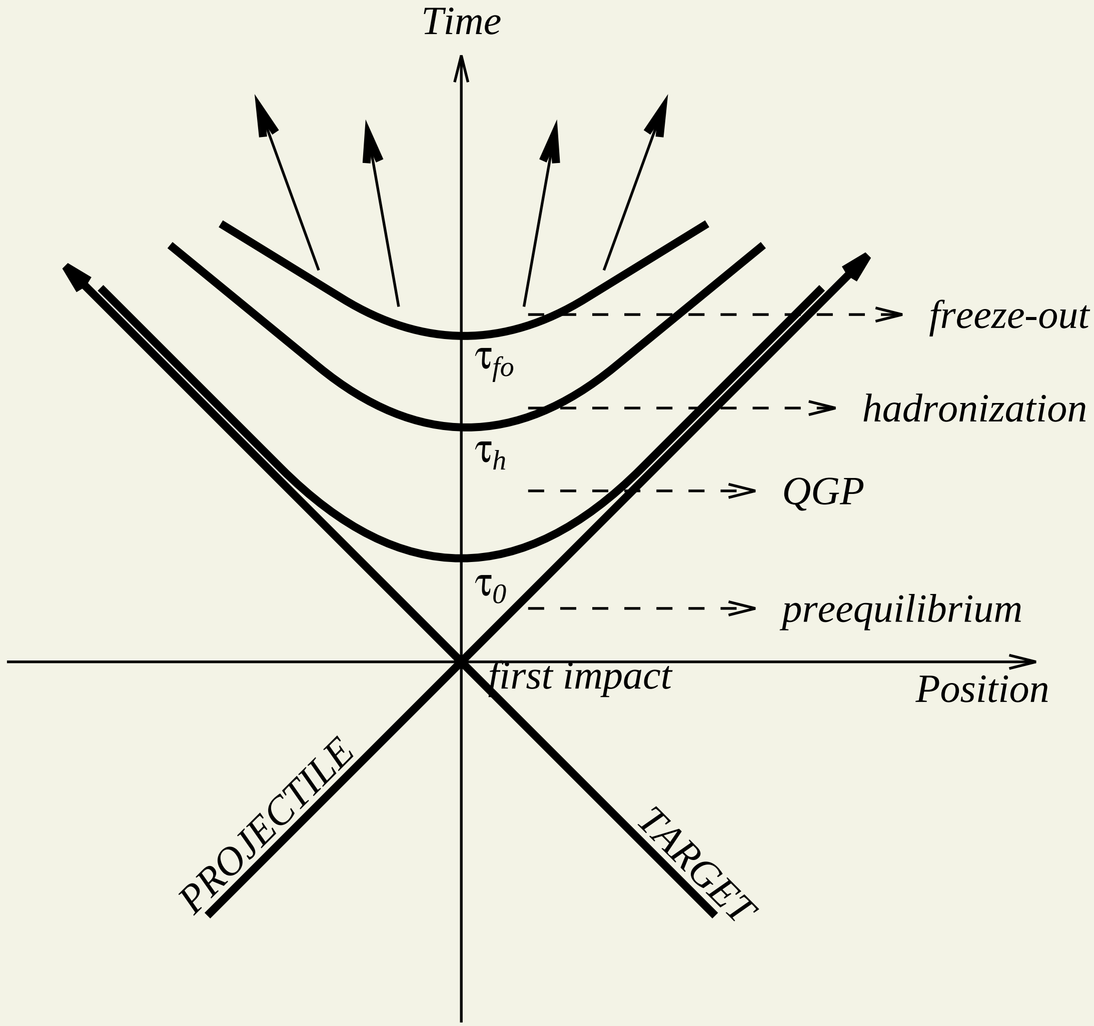
- **numerical solutions (codes) - http://karman.physics.purdue.edu/OSCAR**

boost-invariant 1+1D ($r-\tau$): Dumitru & Rischke BJHYDRO

boost-invariant 2+1D ($x-y-\tau$): Kolb & Froedermann AZHYDRO
also codes by Huovinen, Teaney (private)

3+1D ($x-y-\eta-\tau$): by Hirano, also by Nonaka (private)

3+1D ($x-y-z-t$): at U. of Frankfurt (private)
and SPHERIO by Hama, Kodama et al (private)



we need: **initial conditions**

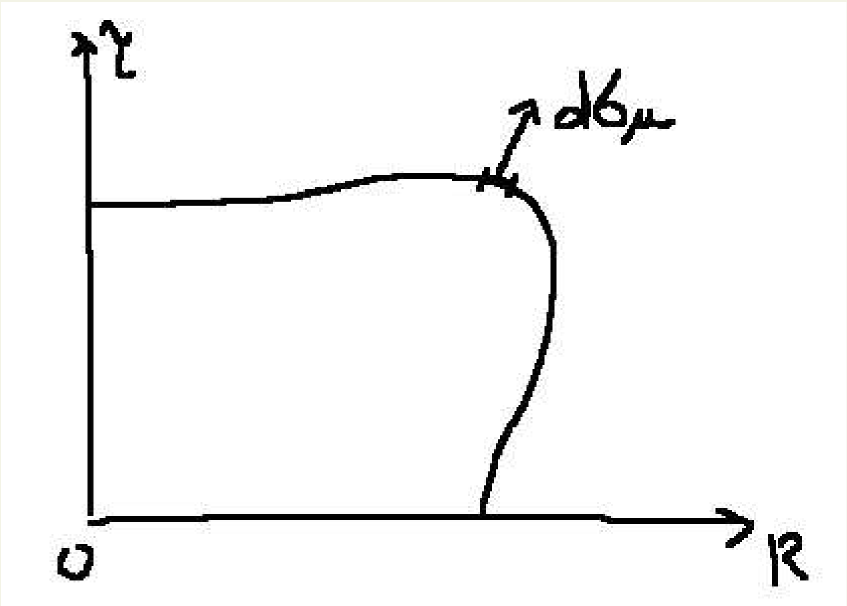
boundary conditions = expansion to vacuum ($\varepsilon = p = 0$ outside)

and a model of freezeout

Freezeout

Search for a really satisfactory solution still on-ongoing

- **Cooper-Frye freezeout: sudden transition to a gas on a 3D hypersurface (typically $T = const$ or $\varepsilon = const$), where f is the phase-space density of a local equilibrium gas boosted with velocity $u^\mu(x)$**



$$E\, dN = p^\mu d\sigma_\mu(x)\, d^3p\, f_{gas}(x, \vec{p})$$

where $d\sigma_\mu$ is the hypersurface normal at x

(covariant analog of $t = const$ freezeout $dN/d^3xd^3p = f(\vec{x}, \vec{p}, t_{fo})$).

Two types of applications:

i) evolve hydro until end of the hadronic stage and then do Cooper-Frye - assume noninteracting gas, decay unstable resonances (T_{fo} or ε_{fo} fit to data)

ii) evolve hydro until end of phase transition, then do Cooper-Frye, and evolve the hadron gas with a hadron transport model (models hadron stage better)

In both cases, conceptual problems:

a) for spacelike normal, $p^\mu d\sigma_\mu < 0$ possible, which gives NEGATIVE yields

ignored in practice because such contributions are small (less than a few percent) - slight violation of energy-momentum and charge conservation

b) determination of the hypersurface requires running hydro until the very end, disregarding that parts of the system have frozen out already

a dynamical interface between hydro and transport is still an open problem

Initial conditions

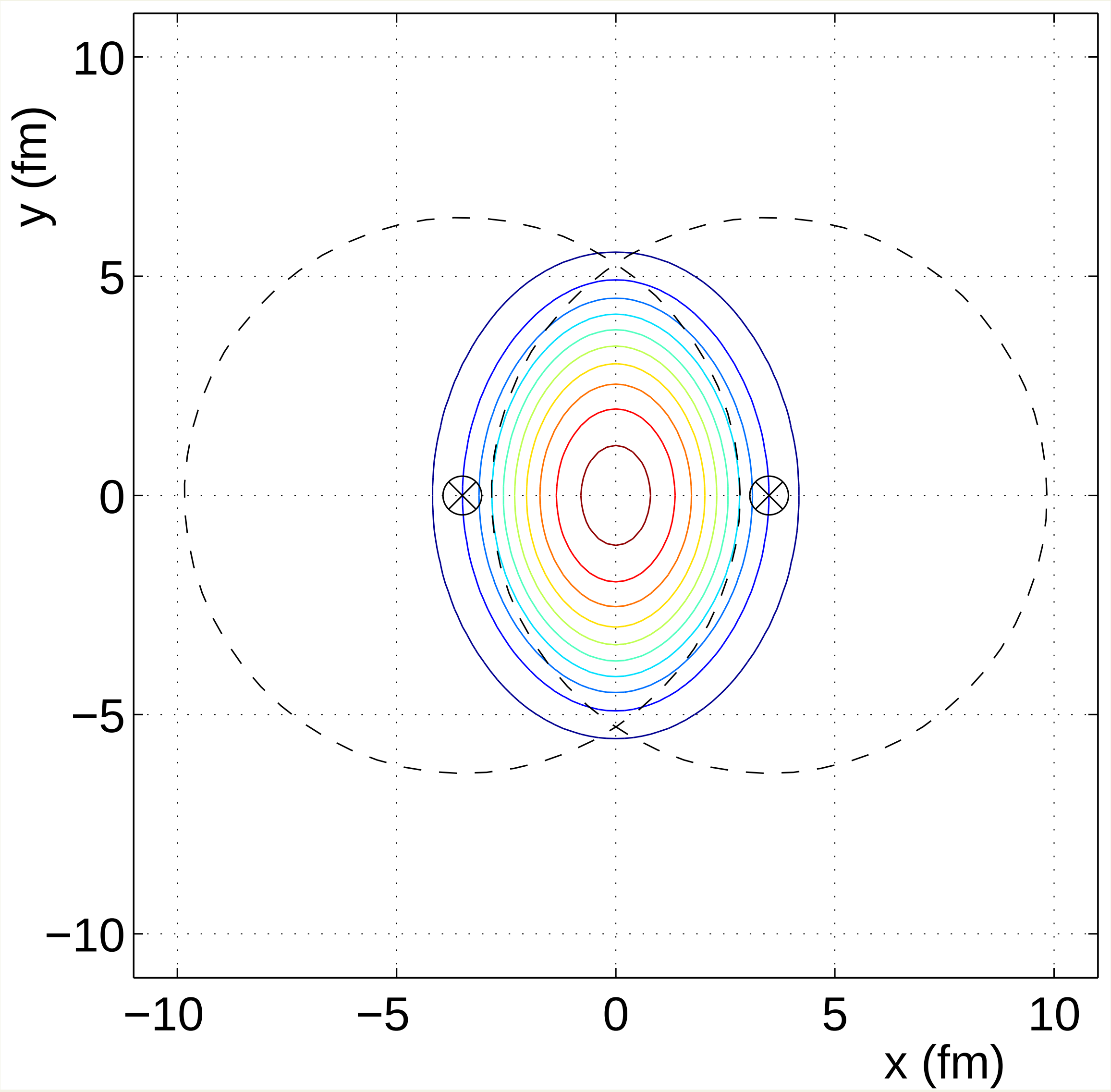
The weakest link - mechanism of thermalization is not yet understood

Typical ingredients:

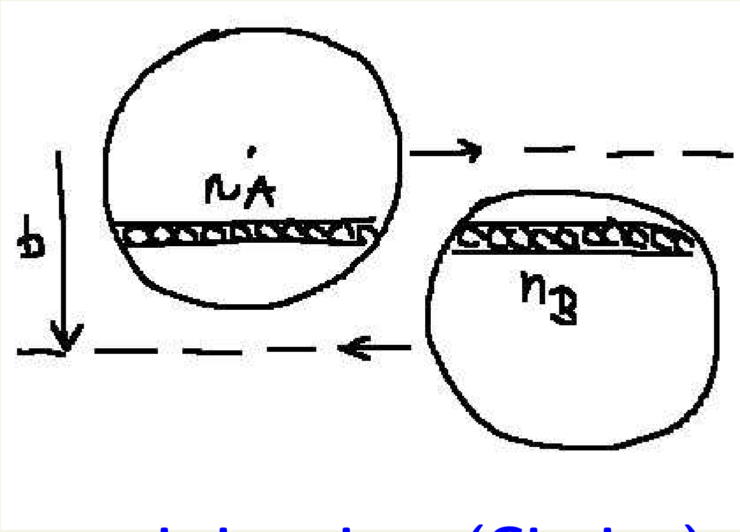
- **thermalization time** τ_0
- **shape of initial entropy (or energy) density profile**
- **maximum entropy (or energy) density** s_0 **(or** ε_0**)**
- **baryon density to entropy ratio** n_B/s **(assumed to be constant)**
- **initial radial flow typically set to zero**

Parameters τ_0, s_0 (ε_0), n_B/s, **and also freezeout parameters** T_{fo} (ε_{fo}), **are fit to data for central collisions. Noncentral collisions can then be predicted.**

profile in transverse plane $\vec{x}_T = (x, y)$



commonly used profiles:



thickness function

$T(r_T) = \int dz\, \rho(\sqrt{r_T^2 + z^2})$

number of nucleons "seen"

$n(r_T) = \sigma_{NN} T(r_T)$

- **wounded nucleon (Glauber)** $\sim n_A + n_B$

$$n_{WN}(\vec{x}_T) \;\propto\; n_A(\vec{x}_T + \vec{b}/2) \left[1 - \left(1 - \frac{n_B(\vec{x}_T - \vec{b}/2)}{B} \right)^B \right] + A \leftrightarrow B$$

- **binary collisions** $n_{BC}(\vec{x}_T) \;\propto\; n_A(\vec{x}_T + \vec{b}/2)\, n_B(\vec{x}_T - \vec{b}/2)$

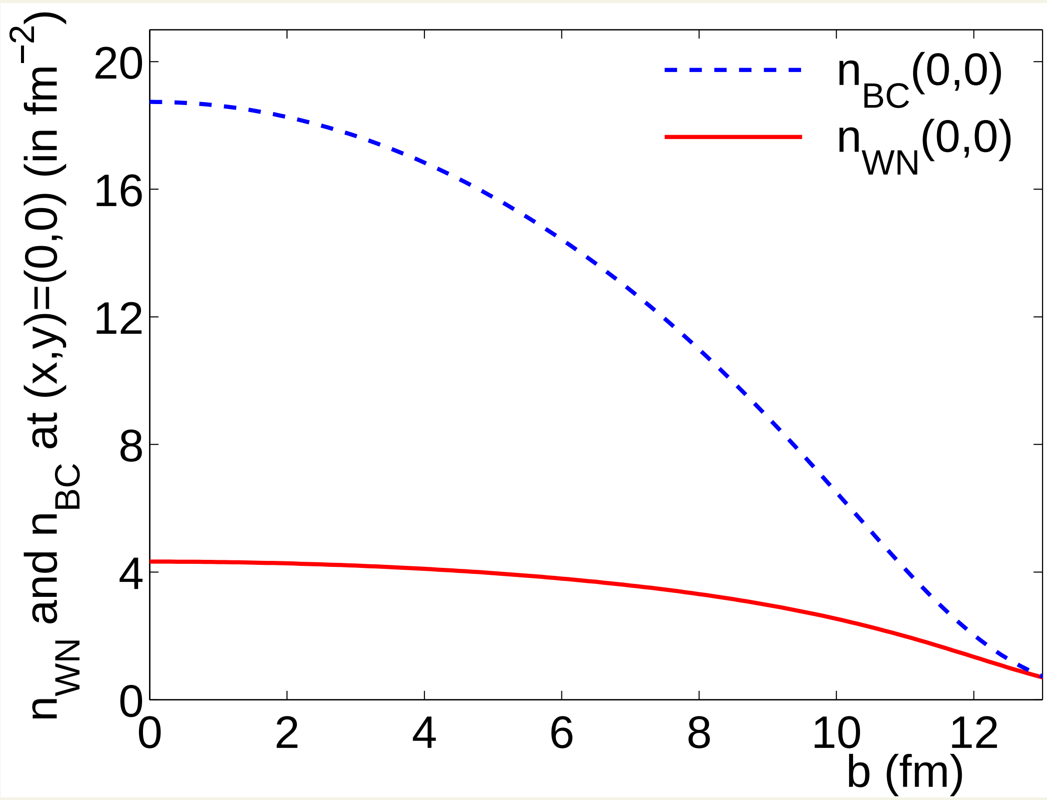
- **saturation (color-glass) model, e.g., Gribov-Levin-Rishkin** $gg \to g$ **formula**

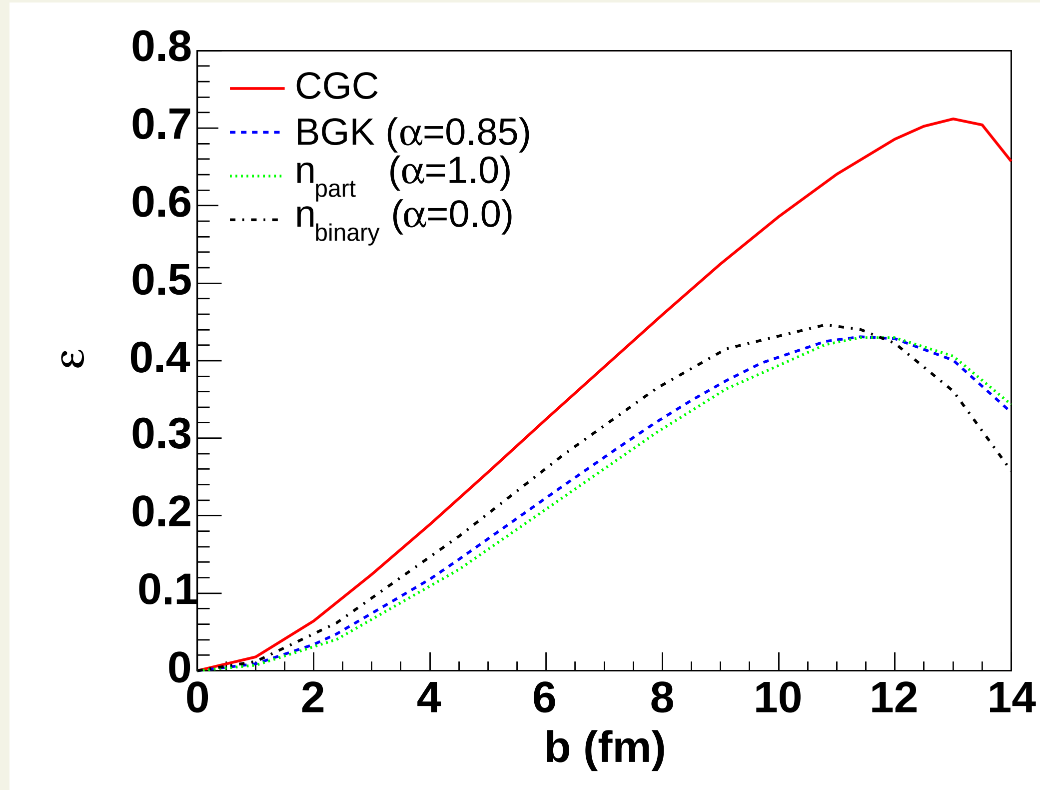
$$\frac{dN_g}{d^2x_T dp_T d\eta} = \frac{4\pi}{C_F} \frac{\alpha_s(p_T^2)}{p_T} \int d^2k_T\, \phi_A(x_1, \tfrac{\vec{p}_T + \vec{k}_T}{2}, \vec{x}_T)\, \phi_B(x_2, \tfrac{\vec{p}_T - \vec{k}_T}{2}, \vec{x}_T)$$

where $\phi(x, k_T, Q_s^2(\vec{x}_T))$ **are unintegrated parton distributions**

and $x_{1,2} = \frac{p_T}{\sqrt{s}} e^{\pm y}$**,** $\quad Q_s^2 = \frac{2\pi^2}{C_F} \alpha_S(Q_s^2)\; xG(x = \frac{Q_s}{\sqrt{s}}, Q_s^2)\; T(\vec{x}_T \pm \vec{b}/2)$

The main difference between the various profiles is in the centrality dependence and the spatial eccentricity:



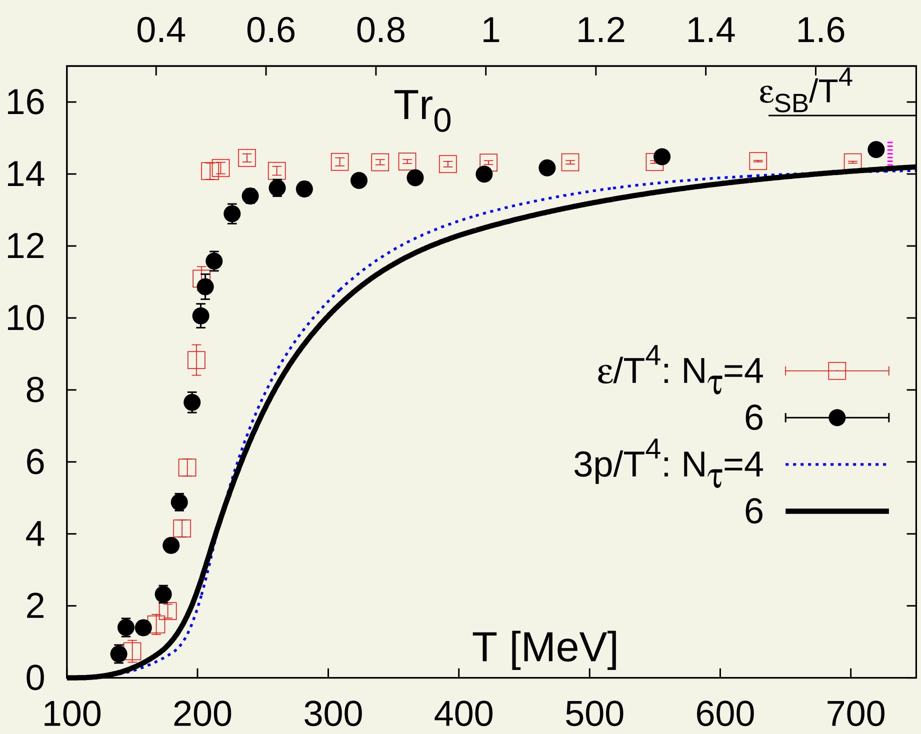


Based on the observed centrality dependence $dN(b)/dy$ of the RHIC Au+Au yields, $\sim$ 0.8 $\times$ wounded + $\sim$ 0.2 $\times$ binary, or the CGC profile are used. (Note, most recent revised CGC calculations give much smaller eccentricities, very similar to the binary case.)

QCD equation of state

Near zero baryon density, calculable from lattice QCD $Z = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}A\, e^{-S_E^{QCD}}$

M. Cheng *et al.*, PRD77, 014511 ('08):



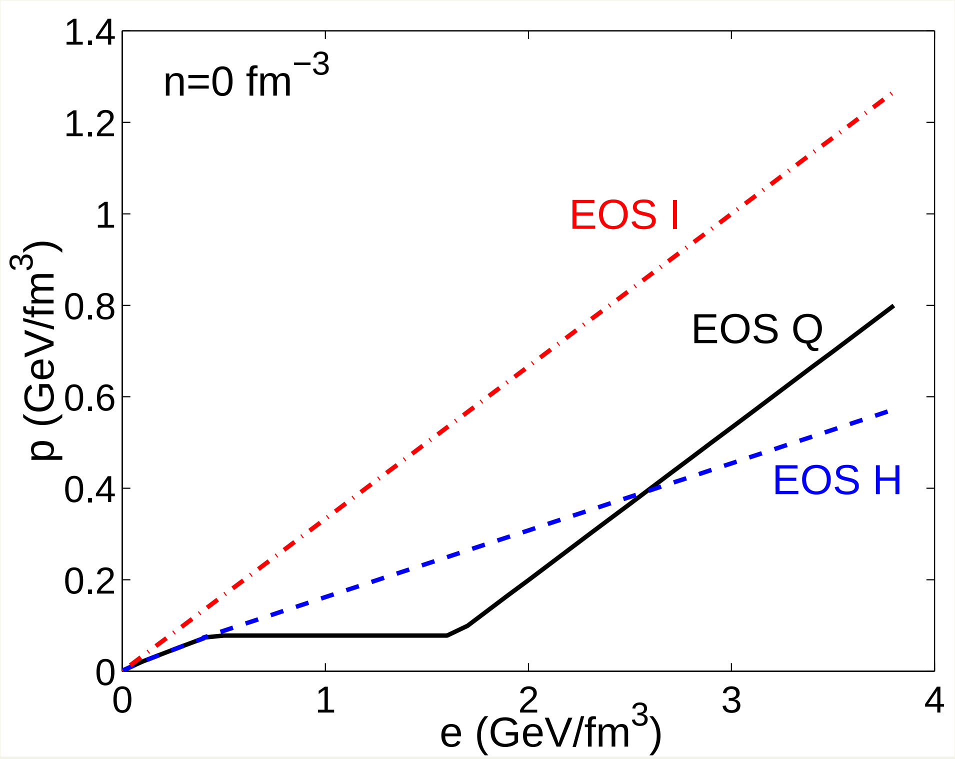
jump in effective degrees of freedom at $T_c \approx 190$ MeV $\sim 2 \cdot 10^{12}$ K

(earlier 2001 results indicated $T_c \approx 170$ MeV)

Stefan-Boltzmann:

$$\frac{\varepsilon_{SB}}{T^4} = \frac{\pi^2}{30}[2 \times (N_c^2 - 1) + \frac{7}{8} \times 2_{spin} \times 2_{anti} \times N_F \times N_c] \sim \frac{d.o.f.}{3}$$

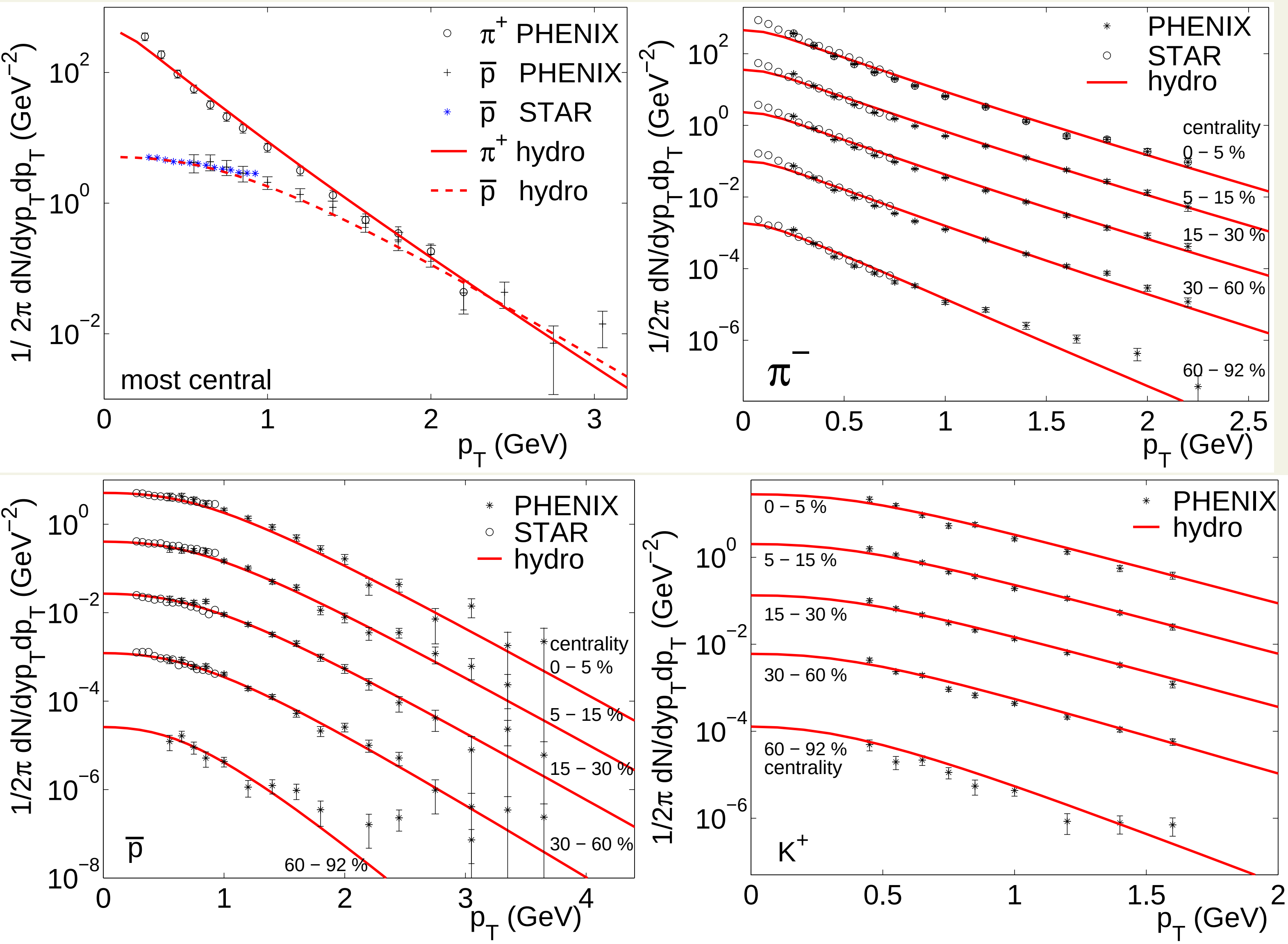
But often a simplified bag-model equation of state (EOS Q) is used instead



where in the hadronic phase a resonance gas equation of state is taken ($c_s^2 \approx 0.15$), while in the plasma phase $p = (\varepsilon - 4B)/3$, where $B \approx (0.23\ GeV)^4$ is the bag constant ($c_s^2 = 1/3$, $T_c \approx 170$ MeV).

Success of ideal hydro - spectra

Kolb & Heinz, nucl-th/0305084



Au+Au at 200 GeV

2+1D ideal hydro

bag EOS

75% wounded + 25% binary

$\tau_0 = 0.6$ **fm**

$s_0 = 110$ / **fm**^{3}

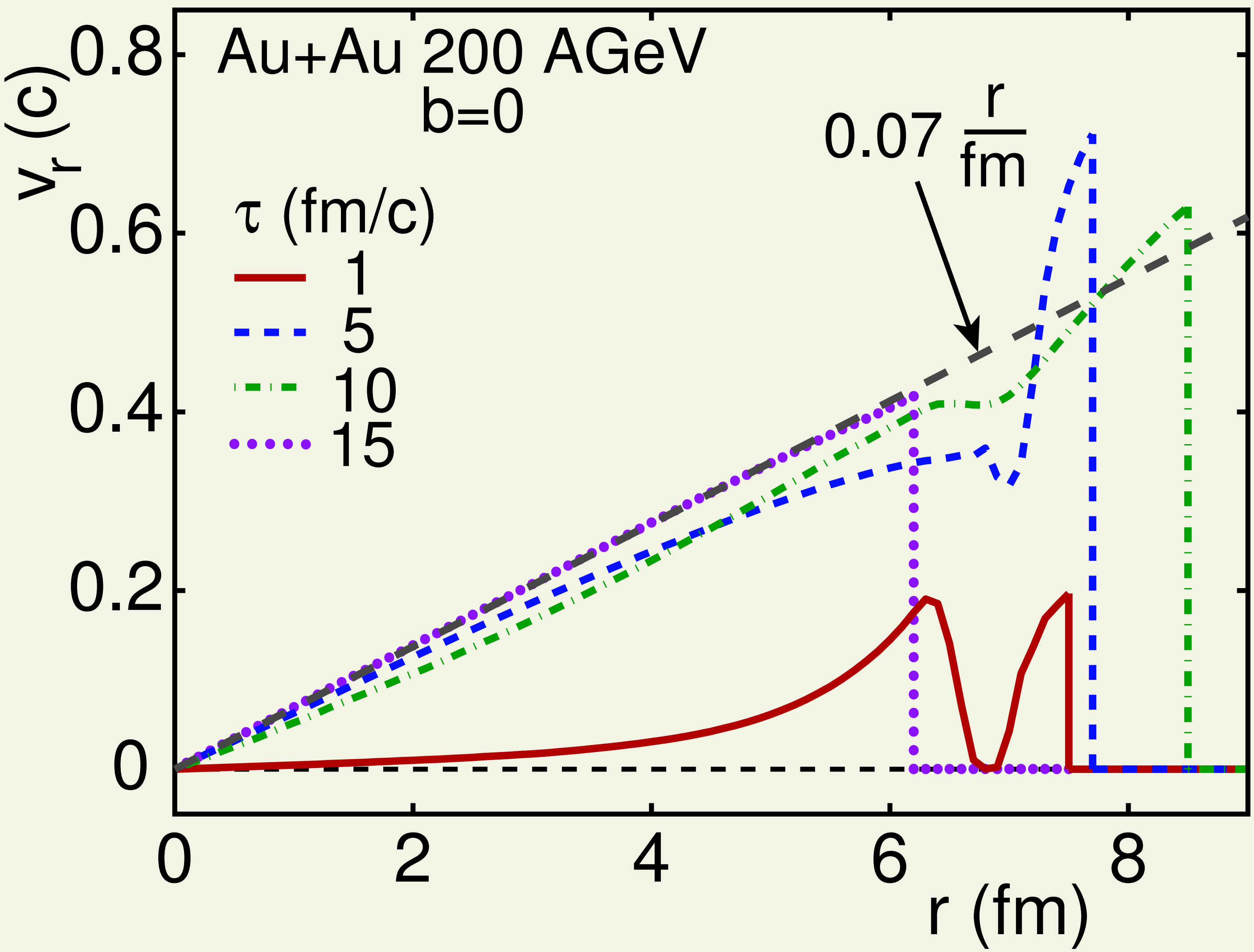
$\varepsilon_0 = 25$ **GeV/fm**^{3}

$T_0 = 360$ **MeV**

$\varepsilon_{fo} = 0.075$ **GeV/fm**^{3}

$T_{fo} \approx 130$ **MeV**

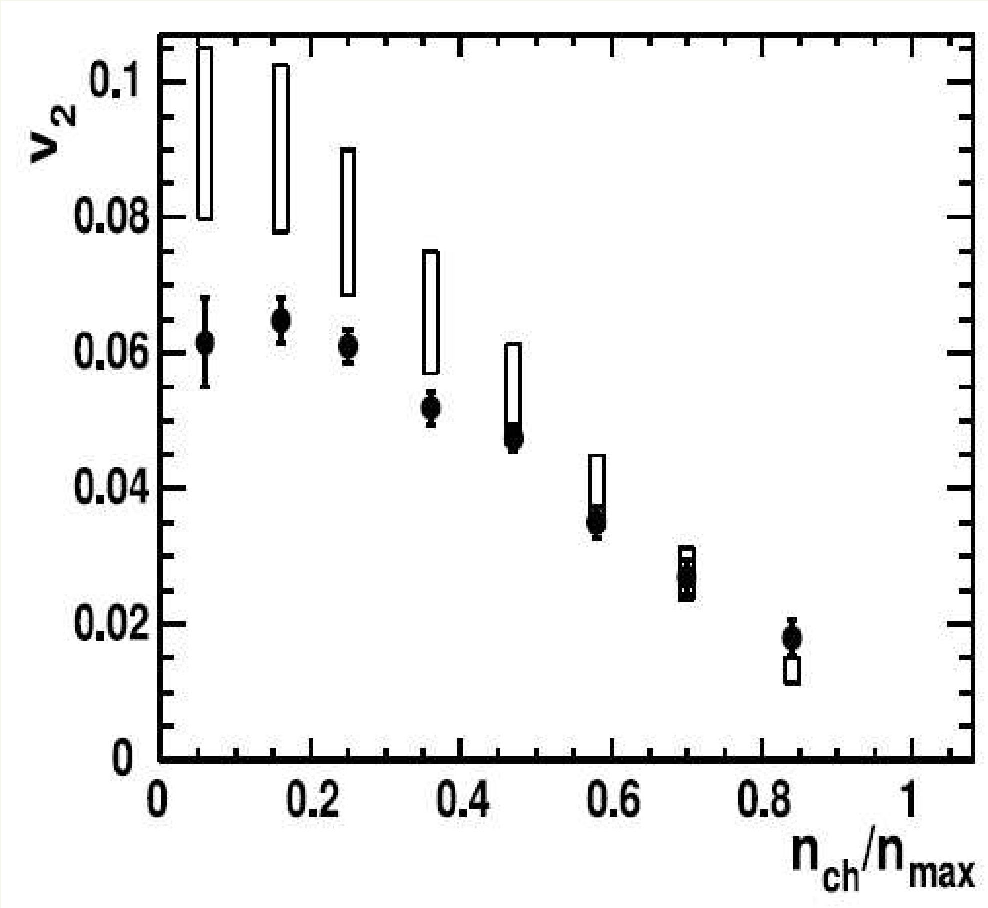
a linear radial flow profile $v_r \sim const \times r$ builds up fairly early



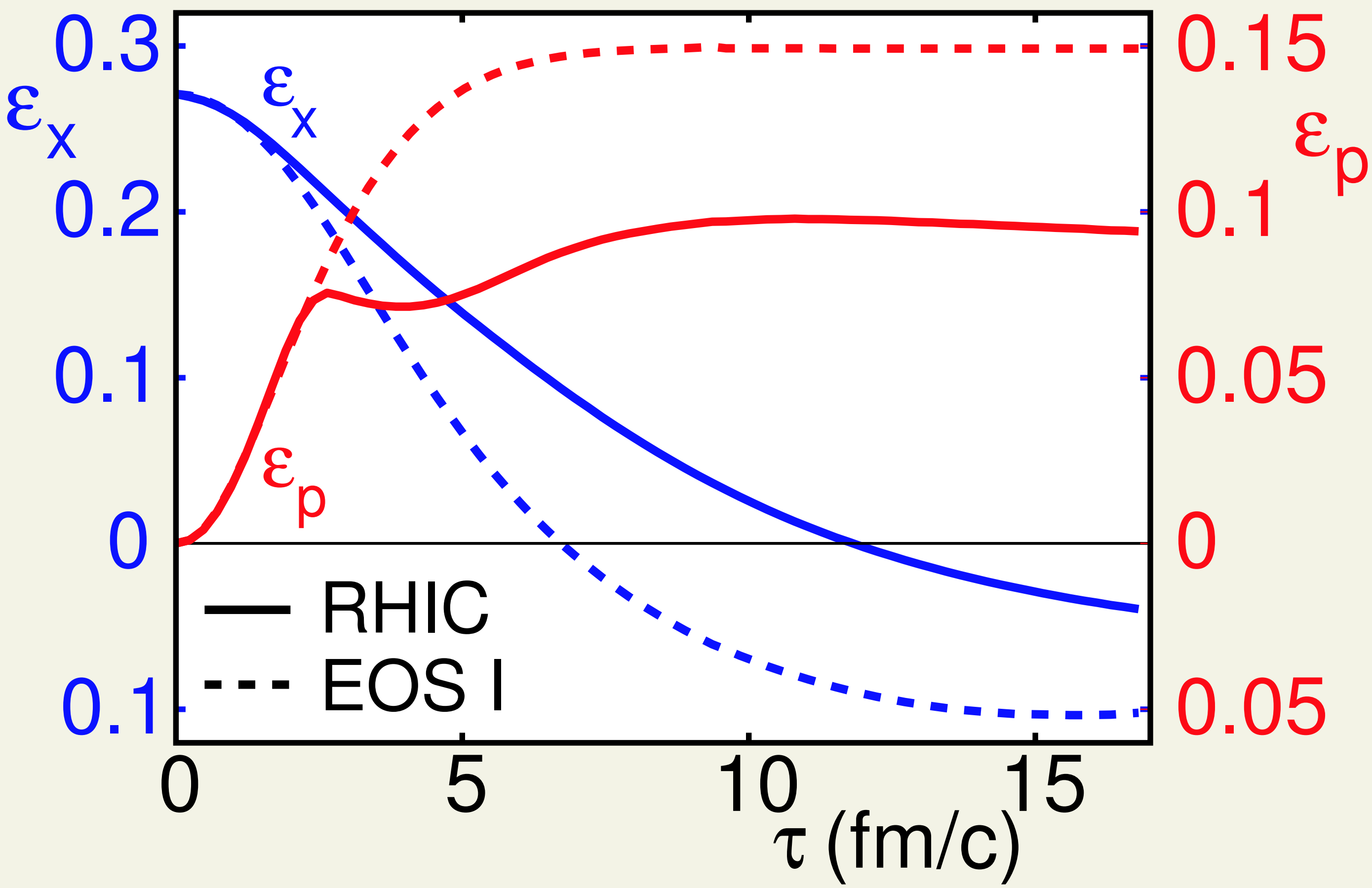
(incorporated in Blast-wave models, but hydro v_r is smaller)

Success of ideal hydro - v_2

Ideal hydro (open boxes) reproduces integrated charged particle v_2 data (black dots) for impact parameters $b \lesssim 7$ fm. In peripheral collisions it overshoots v_2 - the smaller and more dilute the system, the more difficult for it to thermalize.



most of the momentum anisotropy sets in early, by $\tau \sim 3$ fm



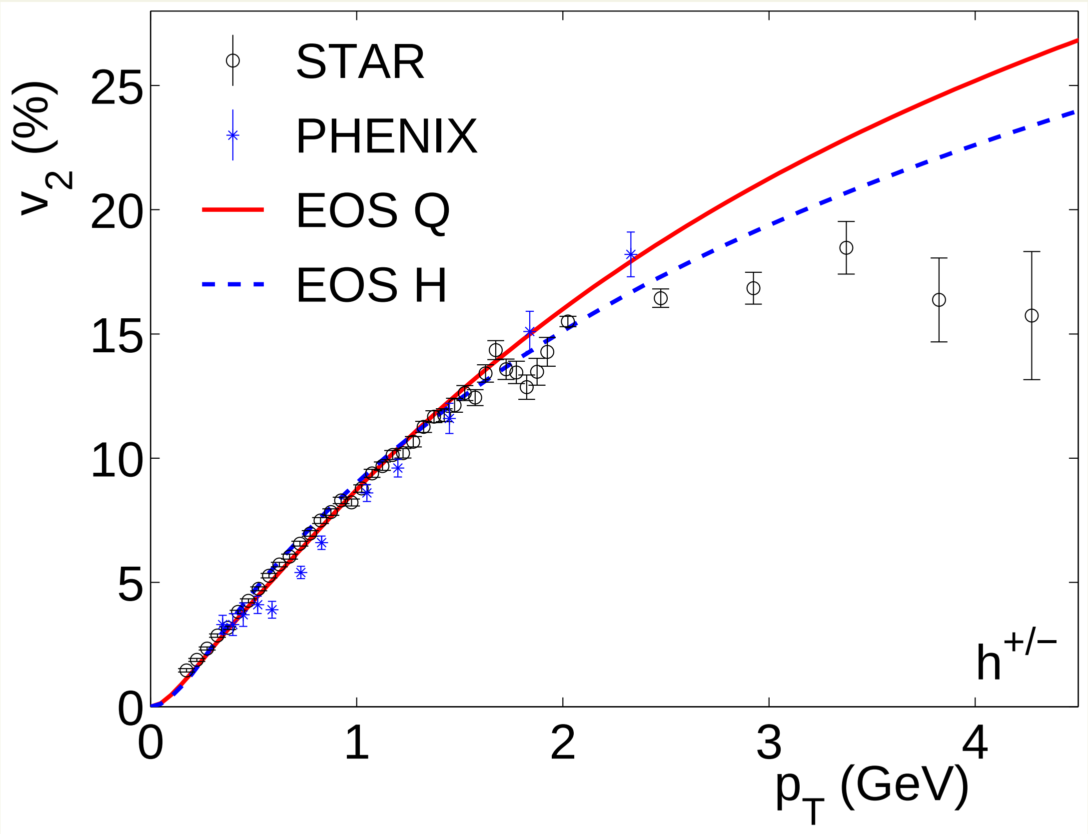
$$\varepsilon_p \equiv \frac{\langle T_{xx} - T_{yy} \rangle}{\langle T_{xx} + T_{yy} \rangle}$$

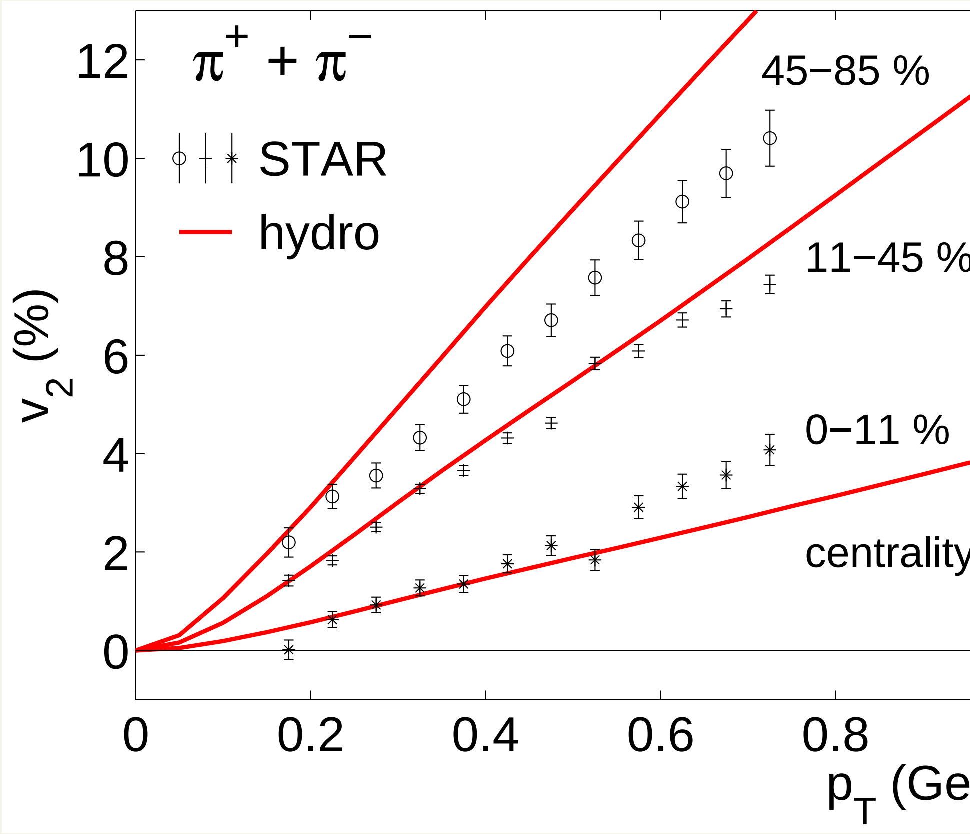
roughly, this means energy-weighted v_2

(EOS I: $\varepsilon = 3p$)

p_T dependence

impact parameter averaged v_2 (left) works much better, up to $p_T \gtrsim 2$ GeV, than detailed centrality dependence (right), $p_T \gtrsim 0.5$ GeV

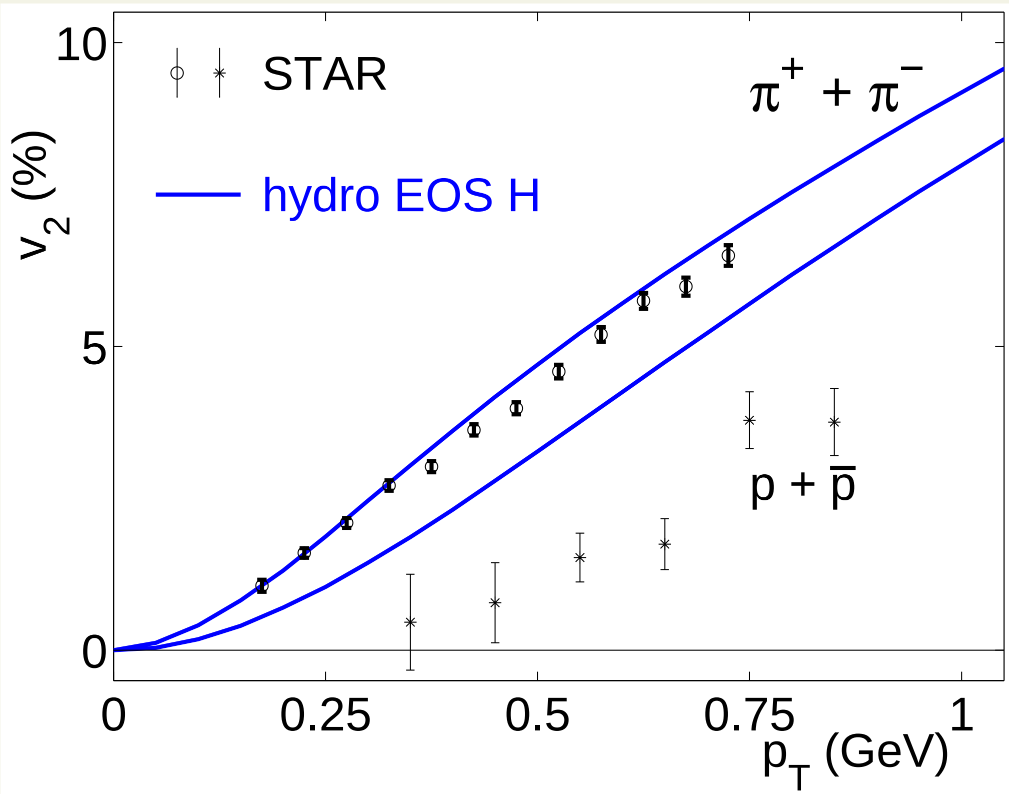


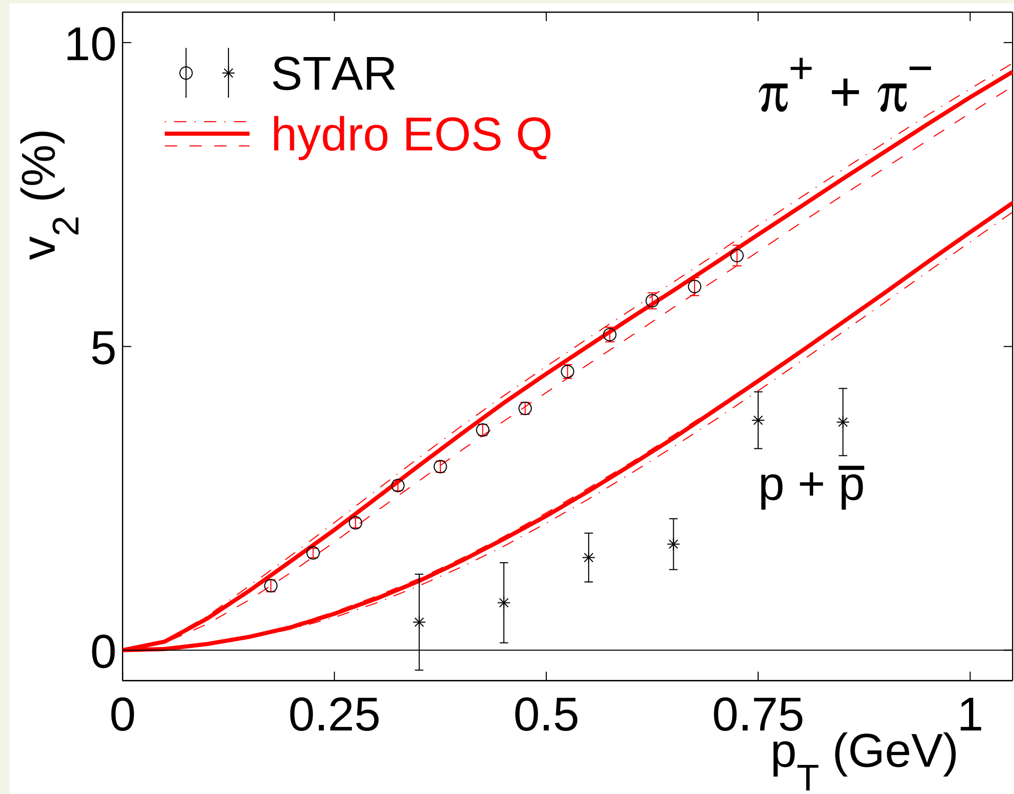


this could be due to the simple initial transverse profiles chosen

Particle species dependence

Kolb, Heinz, Huovinen et al ('01) **minimum-bias Au+Au at RHIC**





pion-proton splitting sensitive to equation of state

plasma EOS favored - but not a perfect agreement for the protons

Separate chemical and thermal freezeout

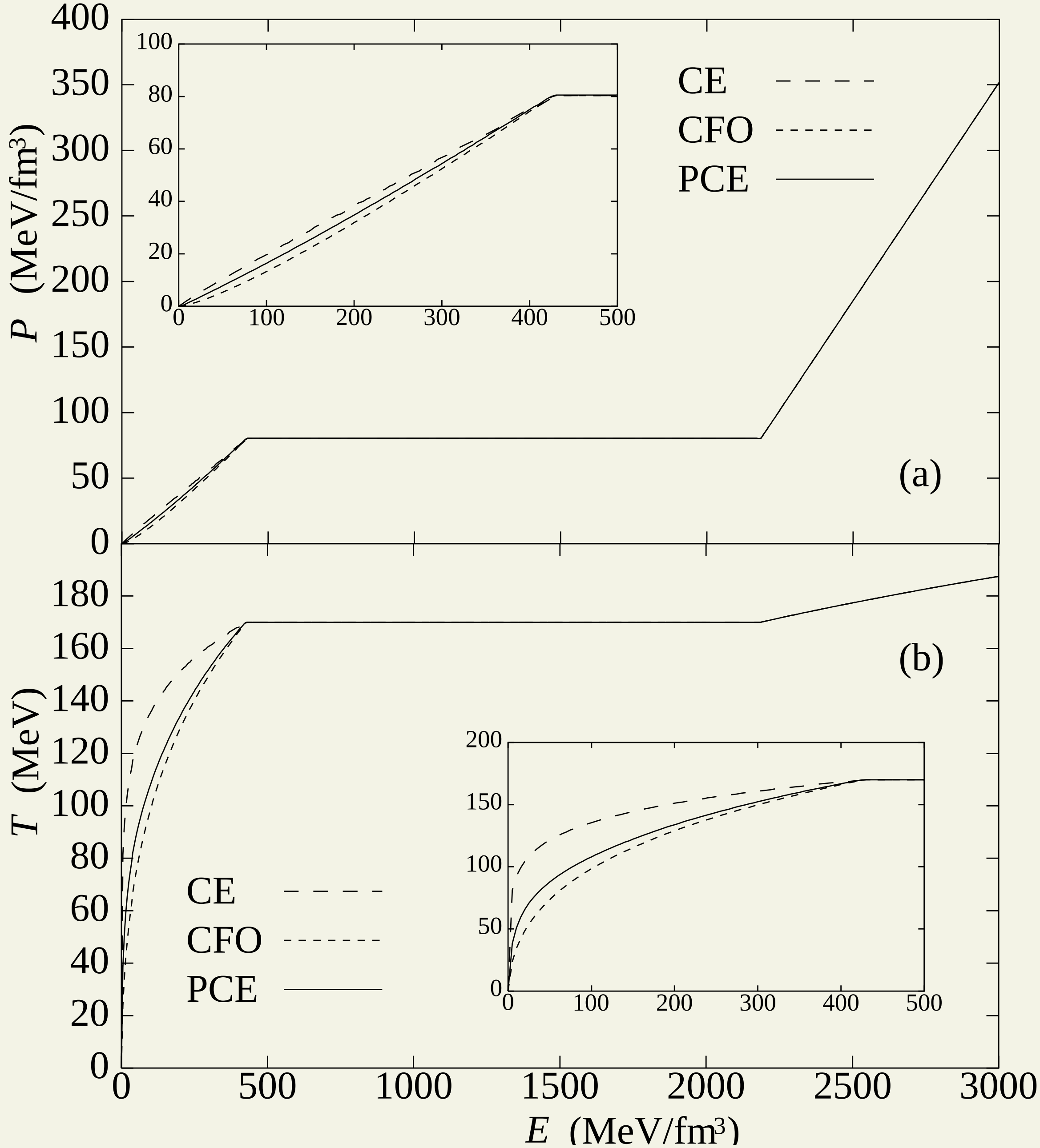
Results shown so far have a caveat - local thermal and chemical equilibrium was assumed until final kinetic freezeout at T_{fo} and therefore chemical composition is off. In particular, final (anti)proton yields are too low compared to experiment (scaled yields were shown).

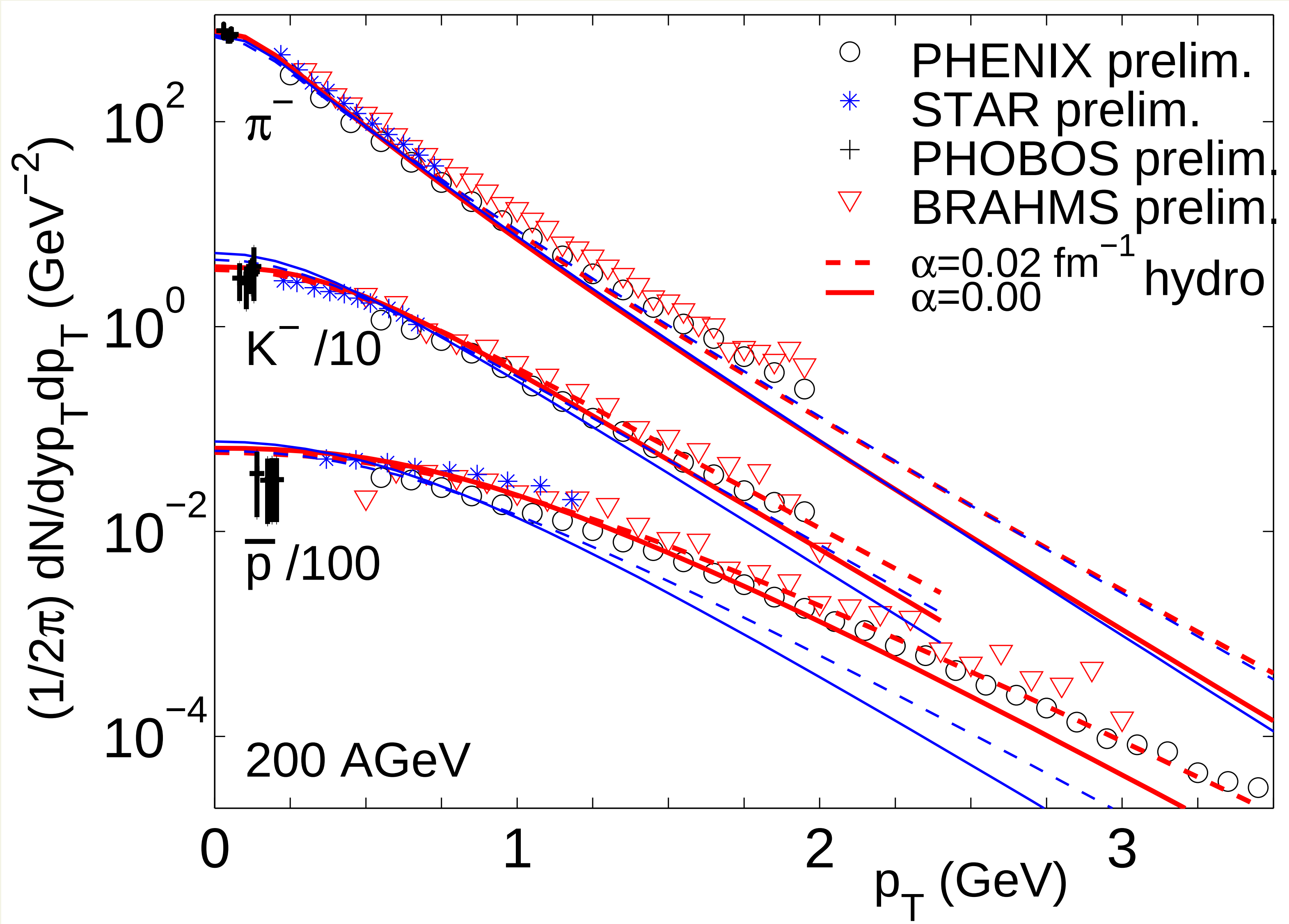
Early chemical freezeout EOS: **maintain the chemical composition at $T = T_{chem}$ via introducing chemical potentials for ALL particle species $\pi, K, p, ...$. This way, local thermal equilibrium dynamics can be followed below $T < T_{chem}$ with particle yields preserved**

Partial chemical freezeout EOS: **even below T_{chem} allow high-cross section elastic processes that go through a resonance equilibrated: $\pi\pi \to \rho \to \pi\pi$, $\pi N \to \Delta \to \pi N$, $\pi K \to K^* \to \pi K$** [Hirano, Kolb & Rapp, Huovinen, ..]

These affects $p(\varepsilon, n_B)$ very little, however, $T(\varepsilon, n_B)$ drops faster with decreasing energy in the hadron phase.

Hirano & Tsuda, PRC66, 054905 ('02)





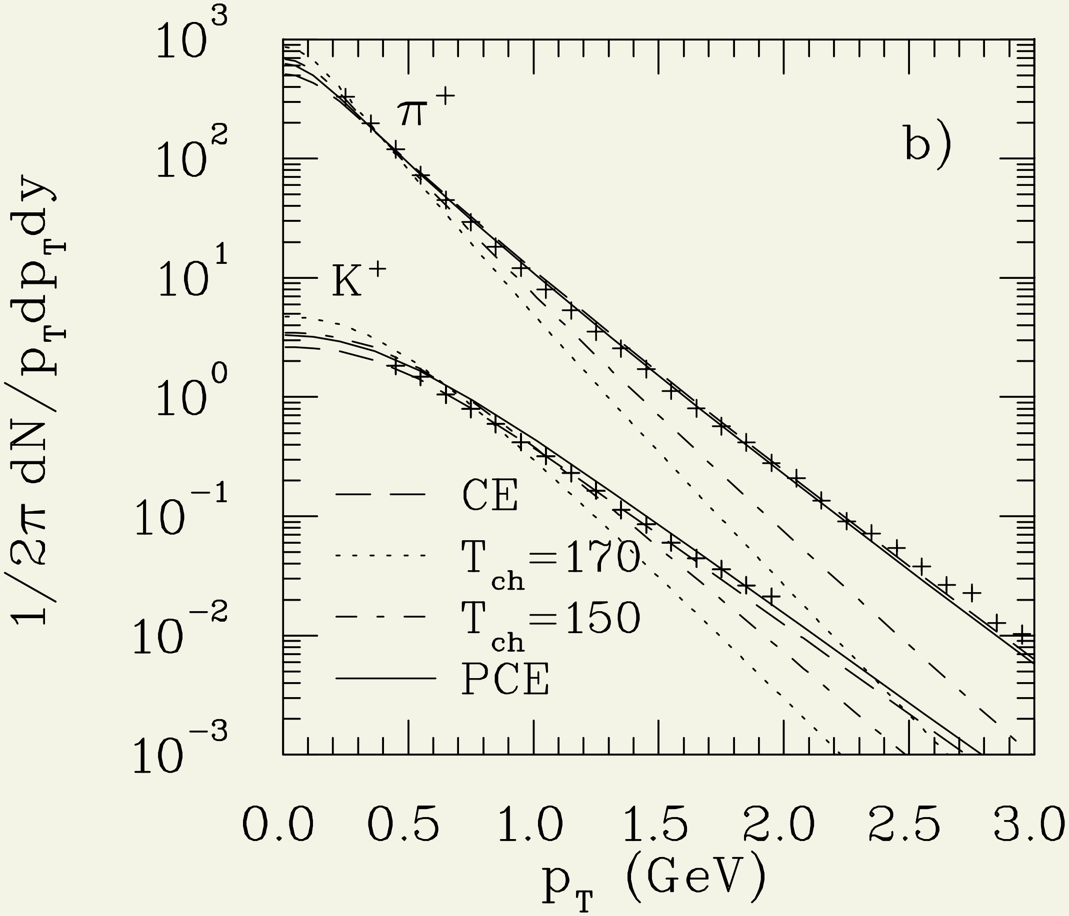
Neither freezeout at $T_{chem} = 165$ **MeV, nor at** $T_{fo} = 100$ **MeV works (pion spectra are insensitive), unless about 1/3 of final transverse flow put in initially at** τ_0 **(dashed) -** $v_r(r, \tau_0) \sim \alpha\, r$ [Kolb & Rapp]

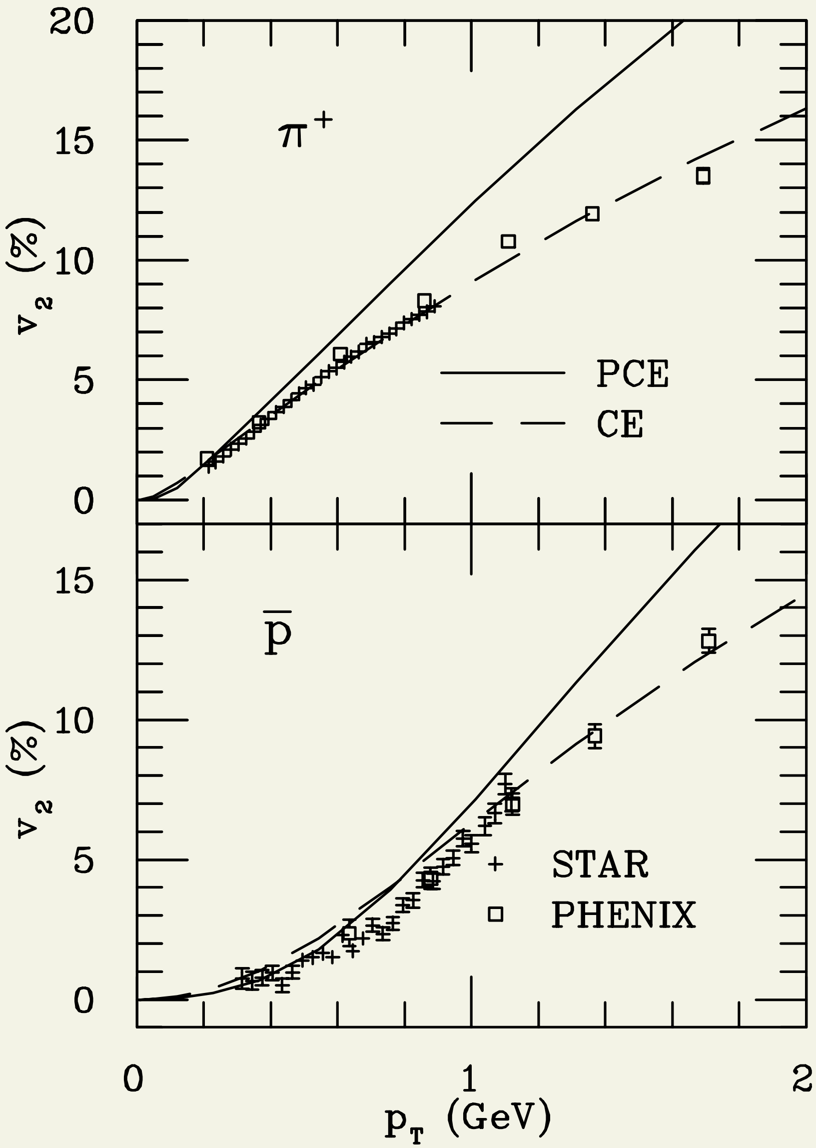
freezing yields at $T_{ch} = 150 MeV$ (misses strange baryons) improves spectra but even pion elliptic flow fails

Huovinen, arxiv/0710.4379

CE: chemical equilibrium until freezeout

PCE: partial chemical freezeout





3+1D ideal hydro + transport

[Hirano et al, PLB636, 299 ('06)]

3+1D hydro:

- can also address rapidity dependence of observables

- rapidity dependent initial conditions can be taken in a factorized form

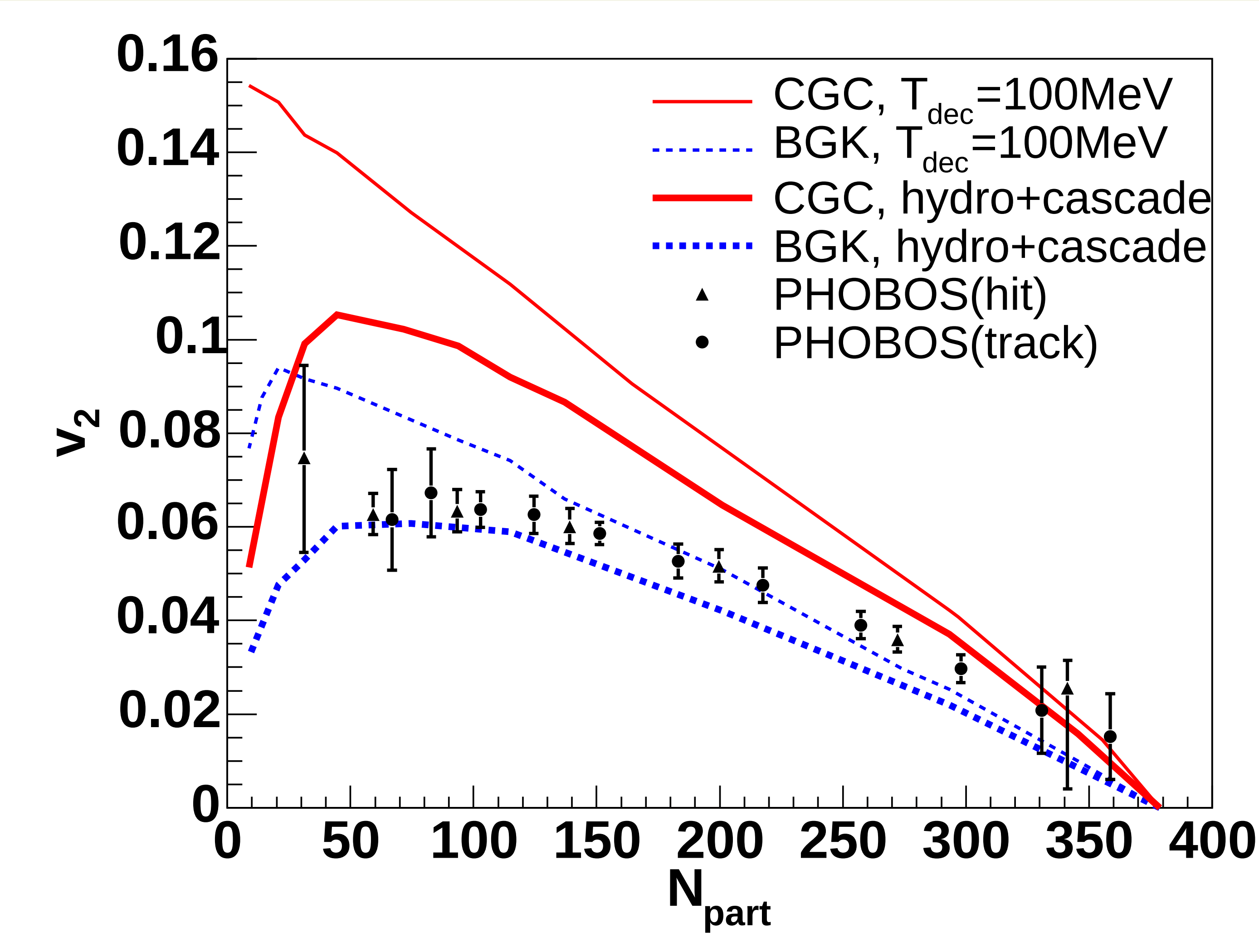
$$s(\vec{x}_T, \eta, \tau_0) = s(\vec{x}_T, \tau_0) \times H(\eta)$$

where $H(\eta)$ is matched to reproduce the observed $dN_{charged}/dy$

OR, can be calculated, e.g., from a saturation model

Switching from the hydro to a hadronic cascade after the phase transition is a big improvement over sudden thermal/chemical freezeout, even if it still has unsolved caveats (as discussed before).

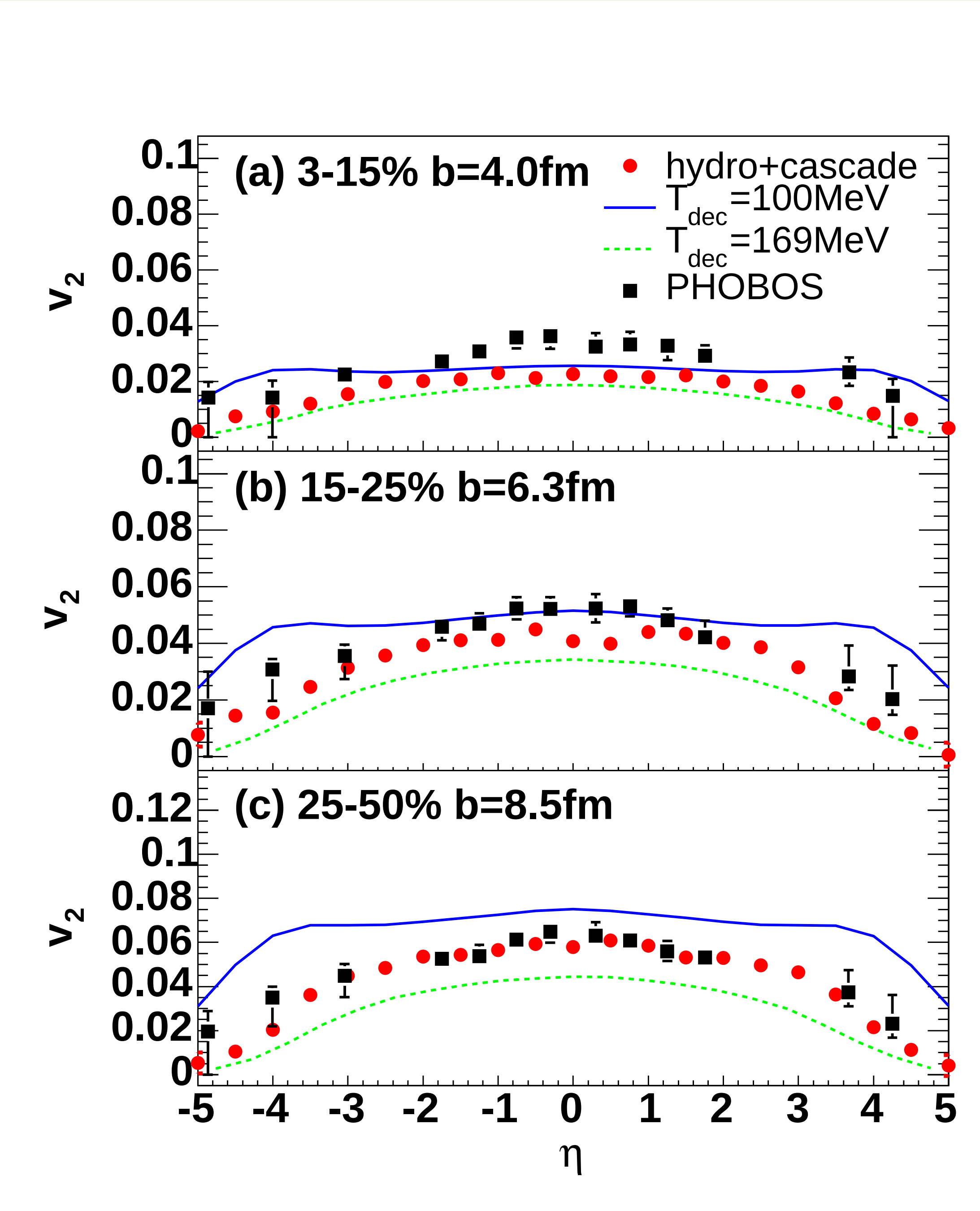
hadron transport improves v_2 description for large impact parameters



BGK: 85% wounded + 15% binary

bag EOS

without dissipation in the plasma phase, saturation initconds overpredict v_2 (too large initial spatial eccentricity)



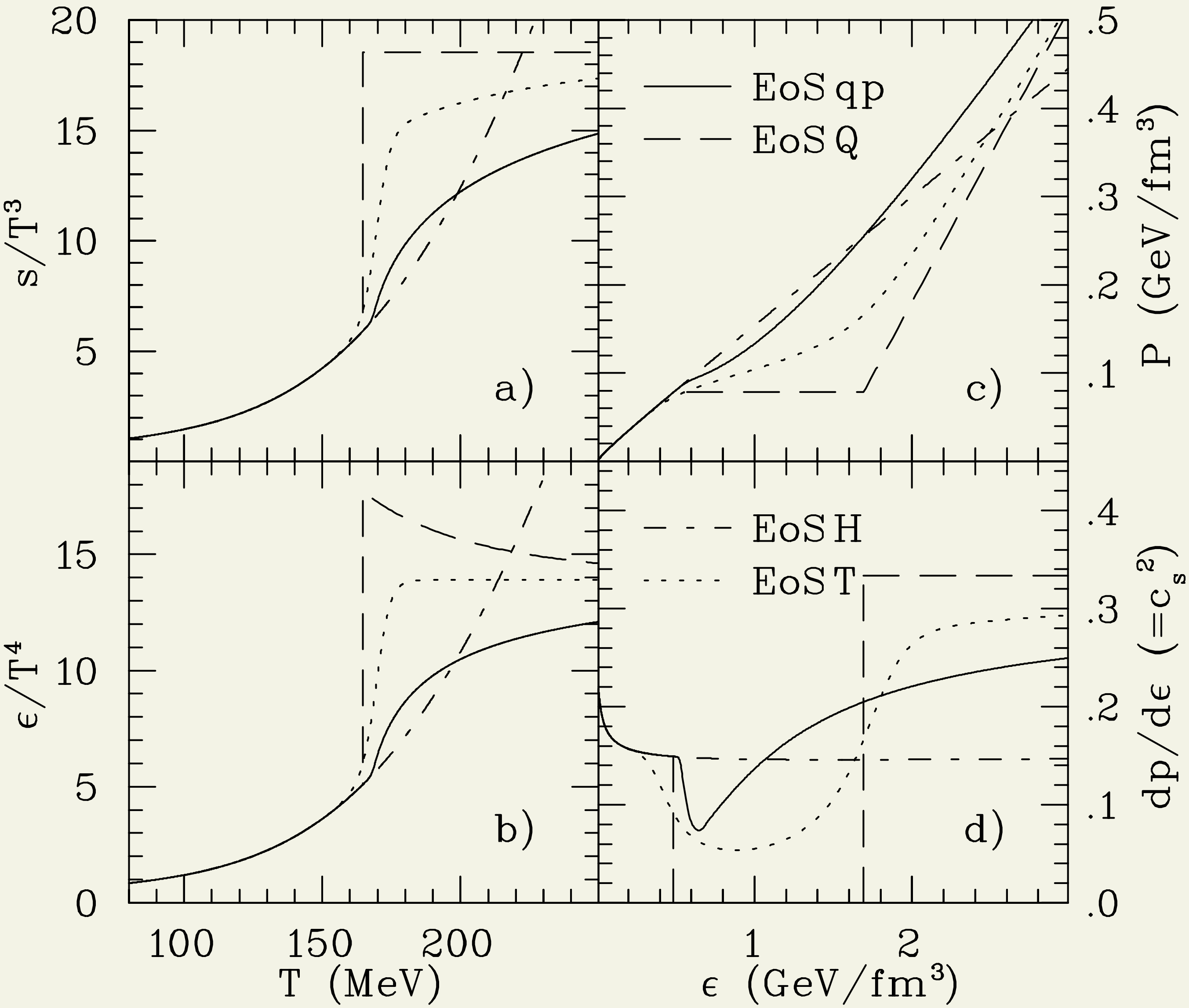
pure hydro overpredicts v_2 at high rapidities

with hydro+transport, the rapidity dependence is much better explained

this is expected, transport is more applicable at low densities, i.e., in peripheral collisions and/or large rapidities

Hydro with realistic QCD EOS

Another caveat - results shown so far were based on the unrealistic bag EOS.



Huovinen, NPA761, 296 ('05)

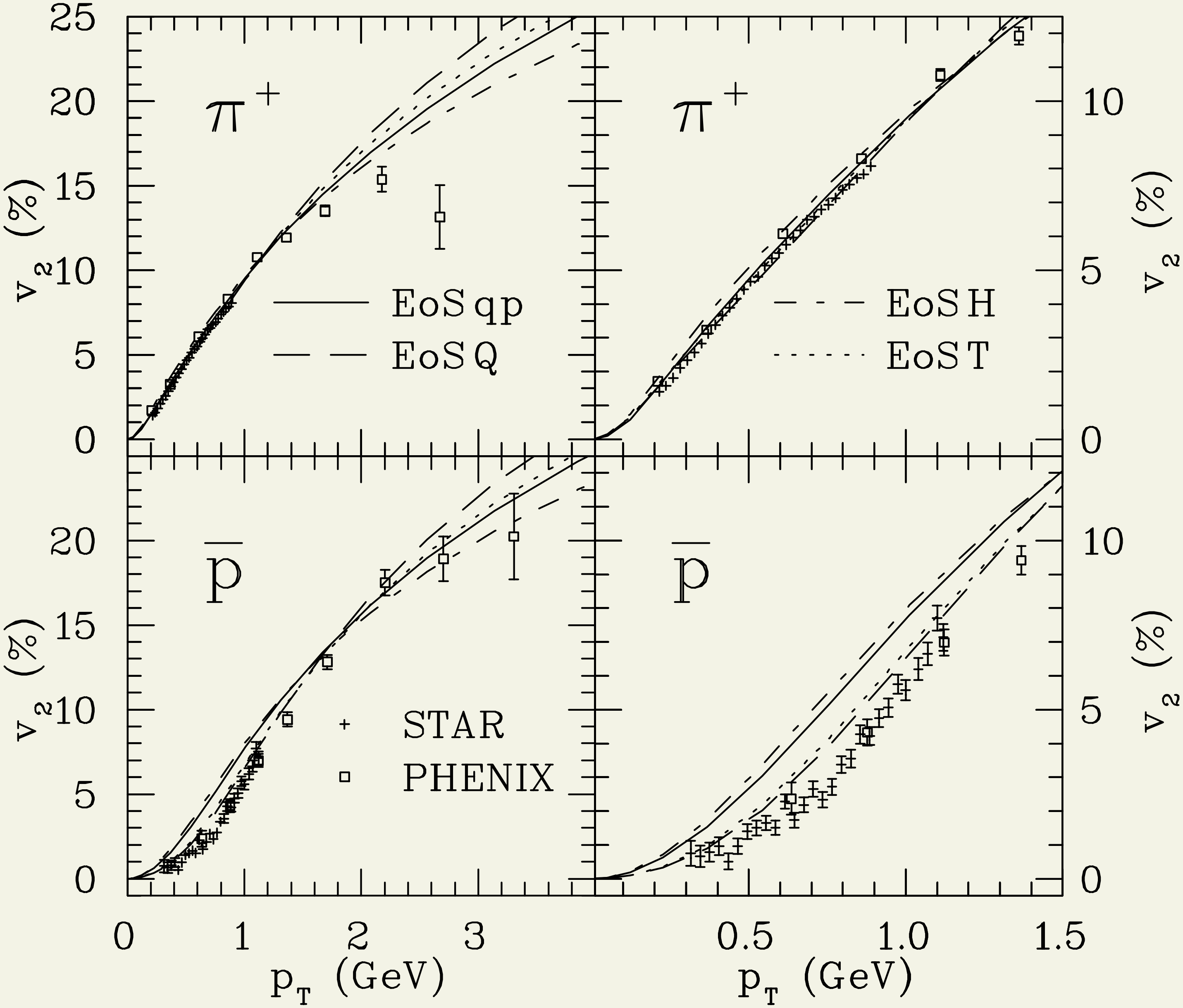
Q: bag model

qp: lattice fit ($T_c = 170$ MeV)

H: hadron gas

T: interpolated $\varepsilon(T)$ between hadron gas and $\varepsilon \propto T^4$ plasma

Huovinen, NPA761, 296 ('05)

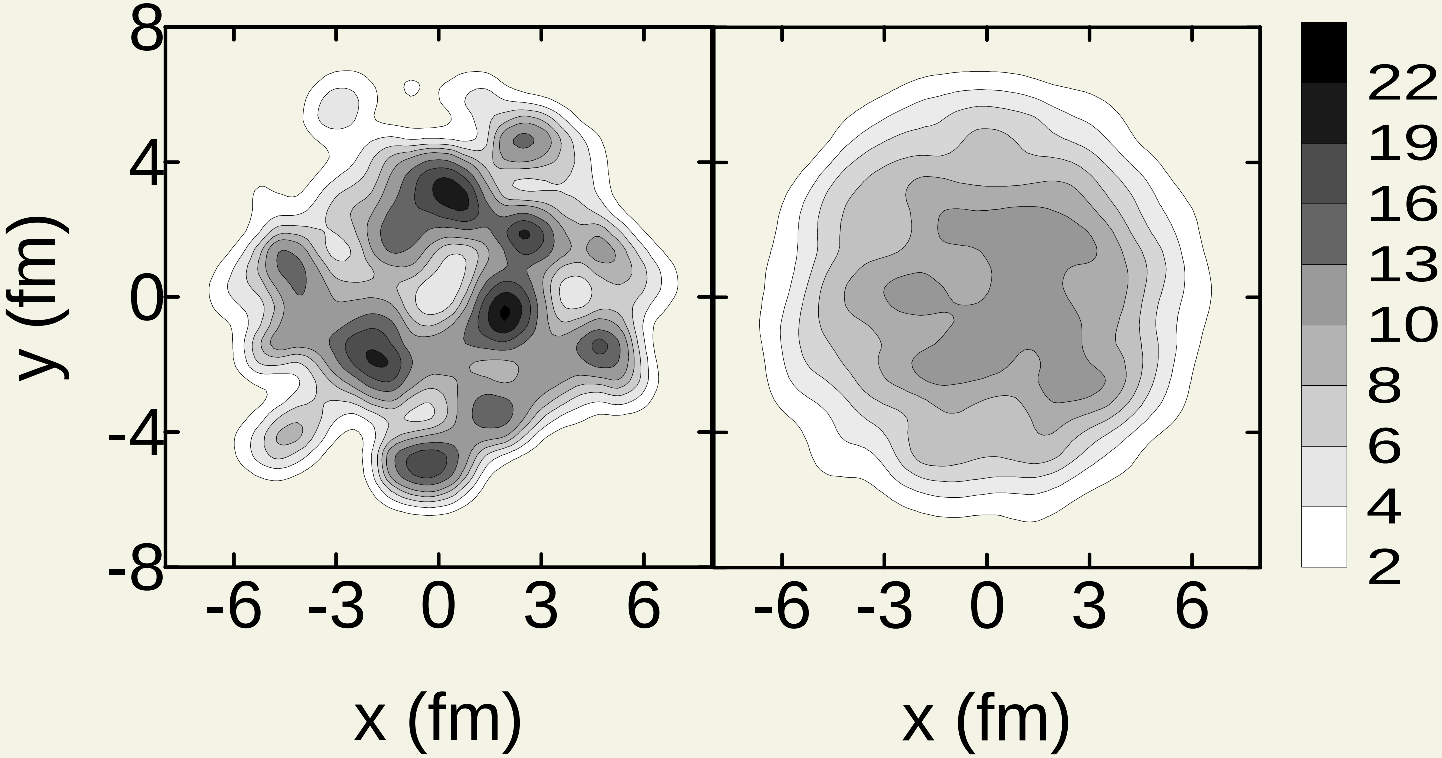


Lattice EOS fit (qp) gives worse agreement than the bag model (protons!)

Fluctuating initial conditions

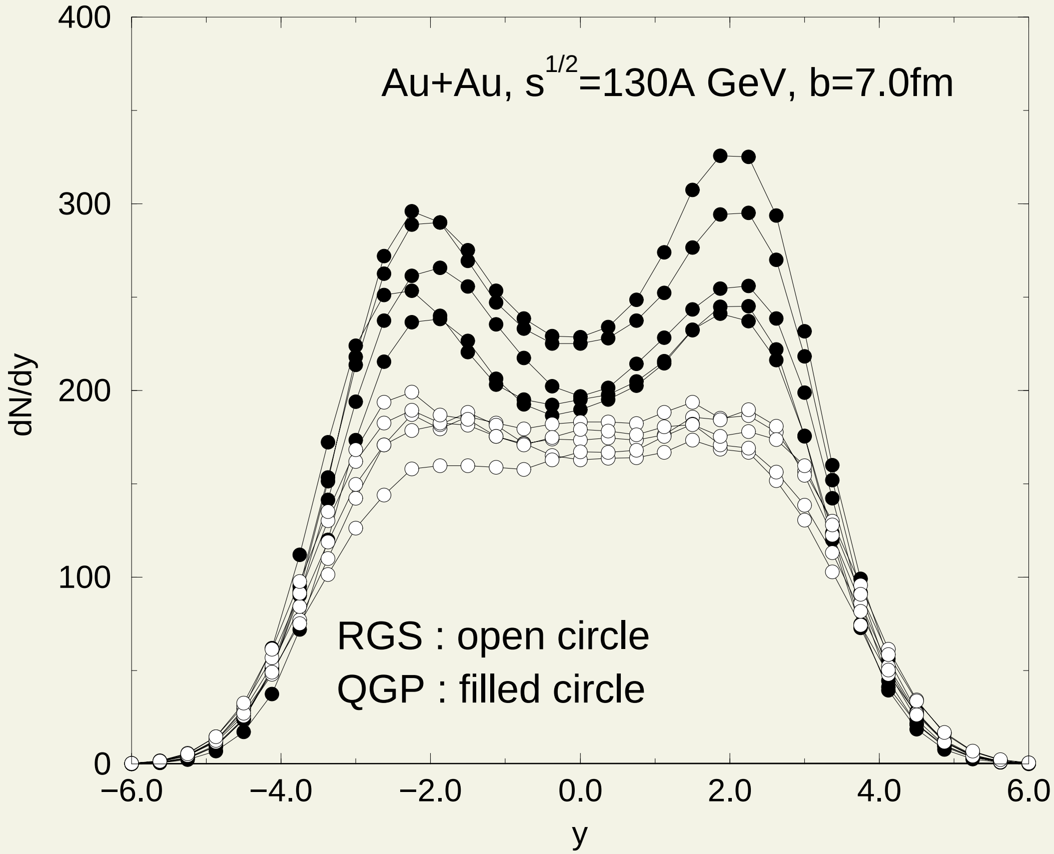
Everything shown so far is for the AVERAGE (smooth) initial condition. However, event-by-event fluctuations are expected and influence hydro observables - hydro is nonlinear, averaging the results is not the same as averaging the initial conditions.

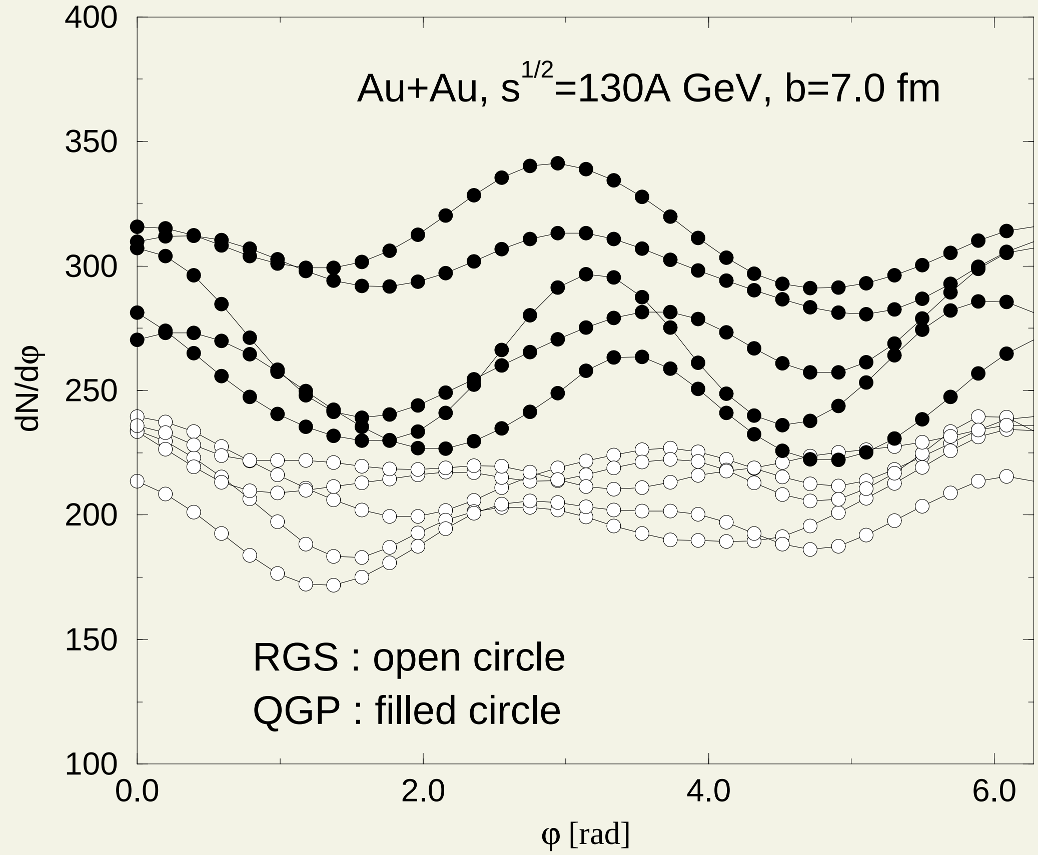
Hama et al, arxiv/0711.4544: **typical event from the NEXUS event generator**



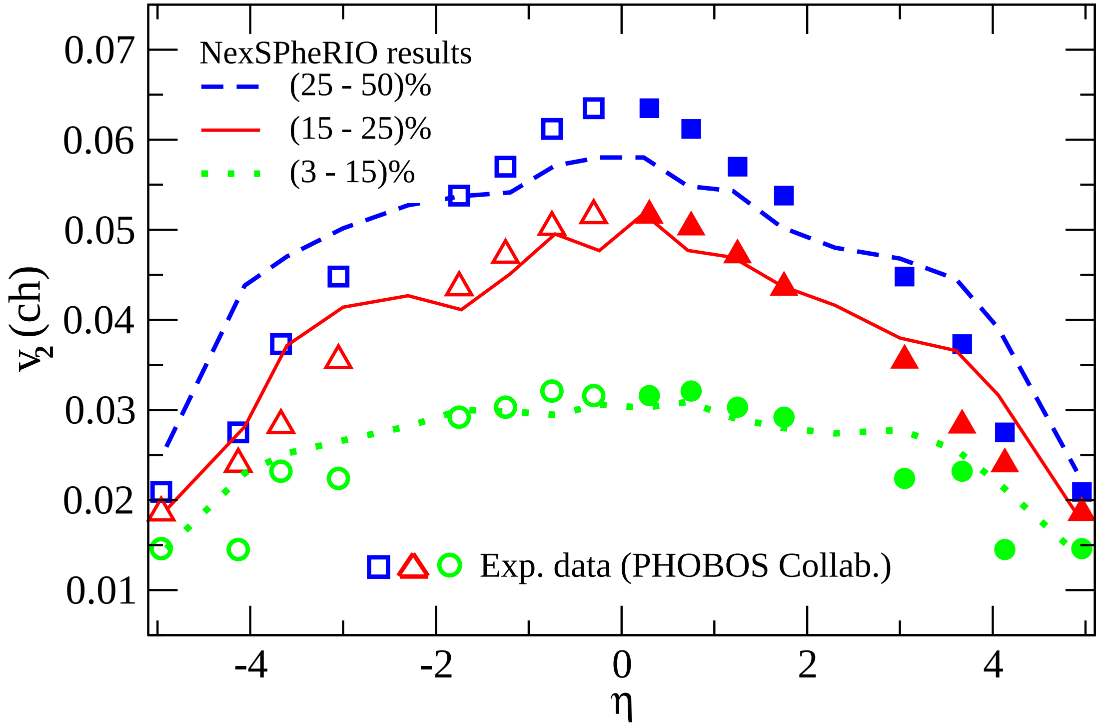
Osada et al, nucl-th/0102011: **5 events**

dN/dy $dN/d\phi$ $(v_2 \equiv \langle \cos 2\phi \rangle)$



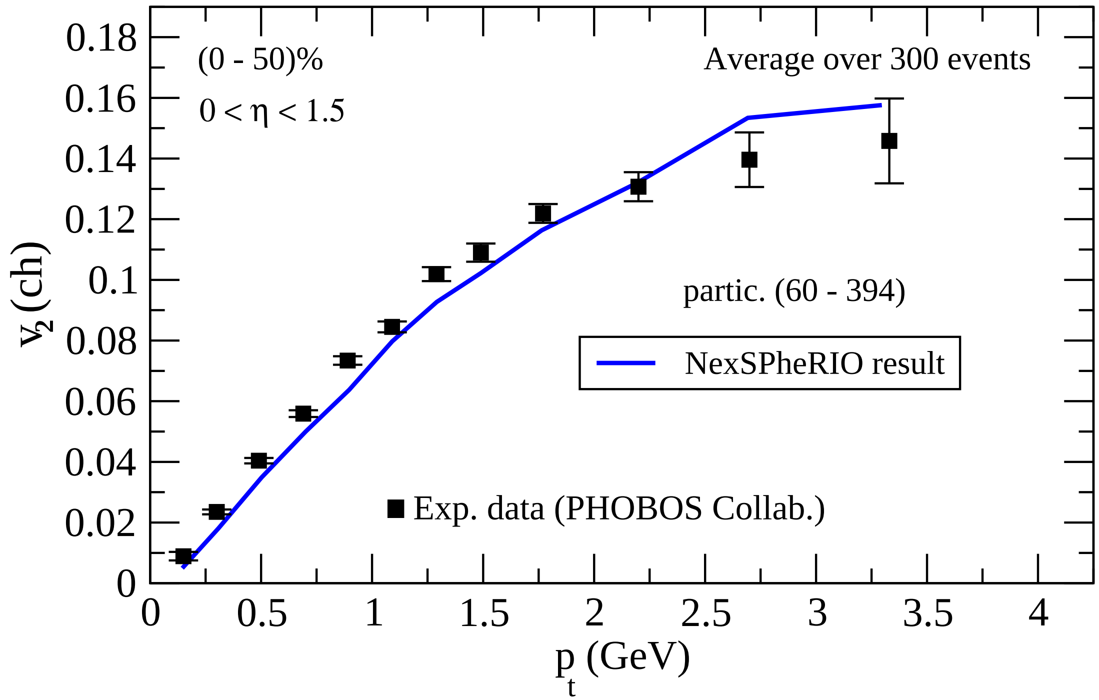


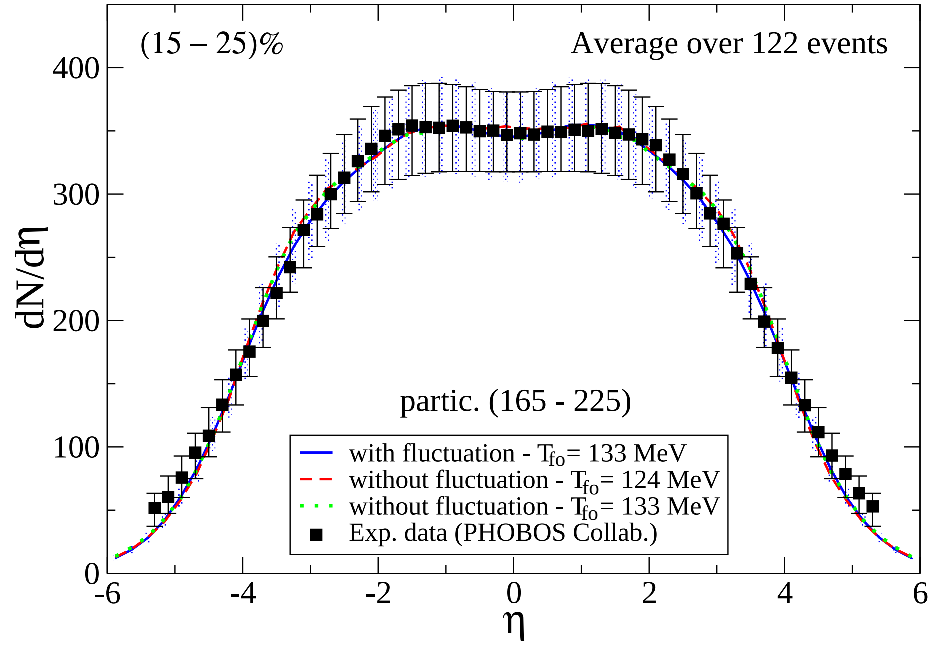
significant event-by-event fluctuations



simultaneous reproduction of $dN/d\eta$ and $v_2(\eta)$

at higher rapidity fluctuations are larger (fewer particles), suppressing v_2





Ideal hydro - summary

• ideal hydro describes the evolution of a system in local equilibrium, in terms of a few macroscopic parameters - no dissipation

• successful reproduction of spectral shapes and elliptic flow data in Au+Au at RHIC up to mid-central $b \lesssim 7$ fm , based on the bag EOS and Cooper-Frye freezeout - but baryon yields are off

• much improved chemical/kinetic freezeout description from hydro+transport approach, correct baryon yields + wider impact parameter and rapidity coverage - on the other hand, early/partial chemical freezeout approaches do not work very well

• saturation initconds overpredict v_2, without dissipation during the plasma stage

• however, attempts to incorporate realistic QCD EOS encounter difficulties

• still conceptual problems with freezeout

• need to constrain initial conditions better - thermalization mechanism needs to be understood, fluctuation models tested, etc