Lectures on hydrodynamics - Part II: Covariant transport and the hydrodynamic limit

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• Nonrelativistic transport
• Covariant transport
• Connection to hydrodynamics, transport coefficients
• Initial conditions, hadronization
• Heavy-ion collisions - cooling, elliptic flow, hydrodynamic limit
• Radiative transport
Covariant transport

- covariant non-equilibrium description, in terms of phase space densities of (quasi-)particles

- dynamics driven by covariant local scattering rates, capable of equilibration

- in heavy-ion physics: used for the plasma and the hadron gas stages

$\sim 10^{-14} \text{ m} = 10 \text{ fermis}$

$\sim 10^{-23} \text{ sec} = 10 \text{ yocto(!)-secs}$

initial nuclei  parton plasma  hadronization  hadron gas
Nonrelativistic kinetic theory

Lagrangian mechanics

\[ \vec{k}_i \equiv \frac{\partial L}{\partial \dot{x}_i}, \quad \vec{\dot{k}}_i = \frac{\partial L}{\partial x_i}, \quad L = \sum_i \frac{m \dot{x}_i^2}{2} - V(\{x_i\}), \quad (1) \]

Klimontovich phase space distribution

\[ F(\vec{x}, \vec{k}, t) \equiv \sum_i \delta^3 (\vec{x} - \vec{x}_i(t)) \delta^3 (\vec{k} - \vec{k}_i(t)), \quad (2) \]

satisfies the Klimontovich equation (Problem 0)

\[ \frac{d}{dt} F = \frac{\partial}{\partial t} F + \frac{\vec{k}}{m} \frac{\partial}{\partial \vec{x}} F - \vec{\dot{K}} [F, V] \frac{\partial}{\partial \vec{k}} F = 0 \quad (3) \]

E.g., for static two-body potentials

\[ V(\{x_i\}) = \frac{1}{2} \sum_{i \neq j} V(|\vec{x}_i - \vec{x}_j|) \Rightarrow \vec{\dot{K}} = \int d^3x_2 d^3k_2 \nabla_x V(|\vec{x} - \vec{x}_2|) F(\vec{x}_2, \vec{k}_2, t) \quad (4) \]
**BBGKY hierarchy** [Bogoliubov-Born-Green-Kirkwood-Yvon]

Define 1-, 2-, ...(n-) particle phasespace distributions via averaging over an ensemble of particles

\[
f_1(t) \equiv \langle F(1, t) \rangle
\]

\[
f_{12}(t) \equiv \langle F(1, t) F(2, t) \rangle - \delta^6(1-2)f_1(t)
\]

\[
\ldots
\]

where \(1 \equiv (\vec{x}_1, \vec{p}_1)\), etc.

Equations of motions for these can be obtained via averaging the Klimontovich eqn, as needed multiplied by \(F, F \cdot F, \ldots\) terms

For 2-body potentials the eqns are not closed, lowest-order involves \(f_{12}\) on the RHS

\[
(\partial_t + \frac{k}{m} \vec{\nabla}_x \vec{x}_1) f_1(t) = \langle \vec{K}[F] \vec{\nabla}_{k_1} F(1, t) \rangle
\]

IGNORING correlations \(f_{12} = f_1 f_2\) gives the **Vlasov eqn**

\[
\left[ \partial_t + \frac{\vec{p}}{m} \vec{\nabla}_x - \left( \int d^3x_2d^3p_2 f(\vec{x}_2, \vec{p}_2, t) \vec{\nabla}_x V(|\vec{x} - \vec{x}_2|) \right) \vec{\nabla}_p \right] f(\vec{x}, \vec{p}, t) = 0 ,
\]

which gives in a self-consistent field. E.g., for Coulomb \(-(...) = q \vec{E}_{selfc.}\)
With the Vlasov eqn, you can study dielectric properties (Problem 0b) taking a small external field \( \delta \vec{E}_{\text{ext}}(\vec{x}, t) \rightarrow \delta f(\vec{x}, \vec{p}, t) \rightarrow \delta \vec{E}(\vec{x}, t) \) gives

\[
\epsilon_L(\vec{k}, \omega) = \frac{\delta E_{\text{ext}}(\vec{k}, \omega)}{\delta E_{\text{ext}}(\vec{k}, \omega)}
\]  

(8)

E.g., with a static point-charge \( \delta q \)

\[
\epsilon_L(\omega = 0, \vec{k}) = 1 + \frac{4\pi q^2 n}{Tk^2} = 1 + \frac{\mu_D^2}{k^2} \rightarrow \phi(r) = \frac{\delta q}{r} e^{-\mu_D r} .
\]  

(9)

There is more to the story if you include higher correlations (quite involved). Progress can be made only if one assumes (Bogolyubov) that higher-order correlations evolve on progressively much faster scales than lower-order ones.

Next order (Lenard-Balescu-Landau) gives so-called collision terms, for Coulomb interactions these correspond to a screened scattering cross section

\[
u_{\text{rel}} \frac{d^3 \sigma_{\text{eff}}}{d^3 k} = \frac{4e^4 \delta(\vec{k}(\vec{v}_p - \vec{v}_{p1}))}{k^4} \frac{1}{|\epsilon_L(\vec{k}, \vec{k} \cdot \vec{v}_p)|^2}
\]  

(10)
Relativistic generalization

**NO-GO theorems** in relativistic Hamilton dynamics Currie, Jordan, Sudarshan...
- no viable relativistic potential approach

### Alternatives

- **live with Vlasov limit**

\[
p^\mu \left( \partial_\mu + qF_{\mu\nu} \frac{\partial}{\partial p_\nu} \right) f = 0
\]

\[
\partial_\mu F^{\mu\nu} = J^\nu = q \int \frac{d^3p}{E} p^\nu f
\]

- **have local (in space-time) interactions/rates**

- **derive from quantum field theory (involved)** Heinz, Elze, Gyulassy, Thoma, ...
Covariant transport

(on-shell) phase-space density

\[ f(x, \vec{p}) \equiv \frac{dN(\vec{x}, \vec{p}, t)}{d^3x d^3p} \quad (p^\mu p_\mu = m^2) \]

is a Lorentz scalar (Problem 1)

Free streaming \( \vec{x}(t) = \vec{x}(t_0) + \vec{v}(t-t_0) \) implies \( f(\vec{x}_0, \vec{p}, t + \Delta t) = f(\vec{x}_0 - \vec{v} \Delta t, \vec{p}, t) \)

\[ \partial_t f + \vec{v} \nabla f = 0 \quad \Leftrightarrow \quad p^\mu \partial_\mu f = 0 \]

manifestly covariant.

Interactions are incorporated via collision term

\[ p^\mu \partial_\mu f(x, \vec{p}) = C[f](x, \vec{p}) \]
E.g., introduce **two-body** scatterings via a rate

\[
\frac{dN_{sc}(x)}{dt} = \sigma \cdot j_p(x) \, dA \cdot n_t(x) \, dz
\]

\( j_p \) - projectile current density, \( n_t \) - target density, \( \sigma \) - cross section

Substitute \( j_p \equiv n_p v_p \) and obtain the rate per unit volume at \( x \)

\[
\frac{dN_{sc}(x)}{dV \, dt} = \frac{dN_{sc}(x)}{d^4x} = \sigma \frac{n_p(x)}{E_p} \frac{n_t(x)}{E_t} E_p E_t v_p .
\]

LHS is a Lorentz scalar (# of scatterings is frame independent), and \( n/E \) is a scalar. Noticing that in the target rest frame

\[
E_p E_t v_p = \sqrt{(p_p \cdot p_t)^2 - m_p^2 m_t^2} \equiv T(p, t) \quad \leftarrow \text{flux factor}
\]

we **DEFINE** sigma to be a scalar via the manifestly covariant

\[
\frac{dN_{sc}(x)}{d^4x} = \sigma \frac{n_p(x)}{E_p} \frac{n_t(x)}{E_t} T(p, t) .
\]

[an equivalent alternative form is \( T(1, 2) = E_1 E_2 \sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2} \)]
Differential rate per unit incoming/outgoing momentum cell

\[
\frac{dN_{sc}(x, \vec{p}_1)}{d^4x} = \left( \prod_{i=1}^{4} \frac{d^3p_i}{E_i} \right) \left( E_3E_4 \frac{d\sigma(1, 2)}{d^3p_3 \ d^3p_4} \right) \frac{dn_p(x, \vec{p}_1)}{d^3p_1} \frac{dn_t(x, \vec{p}_2)}{d^3p_2} T(1, 2)
\]

Combine rate of particles leaving from momentum cell \( d^3p_1 - \text{loss term} \)

\[
E_1 \frac{dN_{p}^{\text{loss}}(x, \vec{p}_1)}{d^4xd^3p_1} = -(1 - \frac{1}{2}\delta_{pt}) \int \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3} \frac{d^3p_4}{E_4} \left( E_3E_4 \frac{d\sigma^{p+t}(1, 2)}{d^3p_3 \ d^3p_4} \right) \times f_p(x, \vec{p}_1) f_t(x, \vec{p}_2) T(1, 2)
\]

and those entering cell \( d^3p_1 - \text{gain term} \) to obtain

\[
E_1 \frac{df_1}{dt} = (1 - \frac{1}{2}\delta_{pt}) \int \left( f_3^p f_4^t W_{34\rightarrow12} - f_1^p f_2^t W_{12\rightarrow34} \right) \equiv C_{2\rightarrow2}[f](x, \vec{p}_1)
\]

with shorthands

\[
\int_i \equiv \int \frac{d^3p_i}{E_i} , \quad f_i^a \equiv f_a(x, \vec{p}_i) , \quad W_{12\rightarrow34} \equiv T(1, 2)E_3E_4 \frac{d\sigma^{p+t}(1, 2)}{d^3p_3 \ d^3p_4}
\]
With detailed balance (if time-reversal and parity invariance)

\[ W_{12 \rightarrow 34} = W_{34 \rightarrow 12} \quad (W_{n \rightarrow m} = W_{m \rightarrow n}) \]

we obtain the $2 \rightarrow 2$ transport equation for a one-component system ($p = t$)

\[ p^\mu \partial_\mu f_1 = \frac{1}{2} \int_{234} (f_3 f_4 - f_1 f_2) W_{12 \rightarrow 34} \]

Generalization to multicomponent case is straightforward

\[ p^\mu \partial_\mu f^a_1 = \frac{1}{2} \sum_{bcd} \int_{234} (f^c_3 f^d_4 - f^a_1 f^b_2) W^{ab \rightarrow cd}_{12 \rightarrow 34} \]

and arbitrary $n \rightarrow m$ interactions can also be included

\[ p \partial f = C_{2 \rightarrow 2}[f] + C_{3 \rightarrow 2}[f] + \cdots \]

Connection to perturbation theory:

\[ W_{12 \rightarrow 34} \equiv \frac{s(s - 4m^2)}{4\pi} \frac{d\sigma}{dt} \delta^4(p_1 + p_2 - p_3 - p_4) \]

\[ \equiv \frac{1}{64\pi^2} |M_{12 \rightarrow 34}|^2 \delta^4(12 - 34) \]
Covariant transport solutions

Very challenging to solve, integro-differential eqn. in 6+1D

• analytic solutions for free streaming (linear problem), or linearized transport equation

• approximate 0+1D analytic solutions in relaxation time approximation [e.g., Zhang & Gyulassy (’98)]

• numerical codes (3+1D) - http://karman.physics.purdue.edu/OSCAR

  Cartesian with 2 → 2: Zhang ZPC

  Cartesian with 2 → 2, 3 ↔ 2: Molnar MPC

  \( x-y-\eta-\tau \) with 2 → 2: Cheng, Pratt & Csizmadia GROMIT (private)

  \( x-y-\eta-t \), 2 → 2, 3 ↔ 2: Xu & Greiner BAMS (private)

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**Hadronic codes:**  Bleicher, Bass et al UrQMD
  Ko & Lin & Subrata AMPT
  Nara et al JAM
**mean free path**: characterizes local conditions

\[ \lambda(x) \equiv \frac{1}{\text{cross section} \times \text{density}(x)} \]

\[ \{ \begin{align*} \lambda &= 0 \quad \text{ideal hydrodynamics} \\ \lambda &= \infty \quad \text{free streaming} \end{align*} \]

**transport opacity**: time-integrated, spatially averaged \[ \text{[DM & Gyulassy NPA 697 ('02)]} \]

\[ \chi \equiv \langle n_{coll} \rangle \langle \sin^2 \theta_{CM} \rangle \sim \# \text{ of collisions per parton}\times\text{mom. transfer efficiency} \]

\[ \chi = \int dz \, \rho(z) \sigma_{\text{transp}} = \int dz \, \frac{1}{\lambda_{tr}(z)} \]

near equilibrium: related to **transport coefficients** (viscosity, diffusion constants)

e.g., **shear viscosity** \[ \eta \approx \frac{4}{5} \frac{T}{\sigma_{tr}} \]

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Transport opacity scaling
to good approximation, results (from same initial condition) mainly depend on transport opacity DM & Gyulassy, NPA697 ('02)

**final spectra** normalized to the initial one

\[ \frac{dN_{f,\text{final}}}{dp_{T}\,dy} / \frac{dN_{\text{initial}}}{dp_{T}\,dy} \quad (|y| < 2) \]

- a) \( \chi = 0.52^{C)} \), 0.52^{E)}
- b) \( \chi = 1.14^{D)} \), 1.32^{A)}
- c) \( \chi = 3.40^{F)} \), 3.68^{F)} \), 3.88^{F)} \), 3.90^{C)}
- d) \( \chi = 7.90^{C)} \), 8.22^{C)} \), 8.58^{D)}

**elliptic flow** \( v_2(p_T) \)

\[ v_2 \text{ at midrapidity (}|y| < 2) \]

- red: \( \chi = 1.70^{F)} \), \( \mu/T_0 = \infty \)
- green: \( \chi = 1.95^{C)} \), \( \mu/T_0 = 1 \)
- blue: \( \chi = 1.84^{F)} \), \( \mu/T_0 = 0.65 \)

MPC Au+Au @ 130A GeV, \( b = 8 \text{ fm} \)

\[ [\mu/T_0 \to \infty: \text{ isotropic scattering, } \mu/T_0 \to 0: \text{ forward-peaked}] \]
Exact scalings of solutions

[DM & Gyulassy, PRC62 ('00)]

- **extended subdivision covariance:**

  \[ f_i \rightarrow f'_i \equiv \ell \cdot f_i, \quad W^{n\rightarrow m} \rightarrow W'^{n\rightarrow m} \equiv W^{n\rightarrow m}/\ell^{n-1} \quad (\sigma \rightarrow \sigma' \equiv \sigma/\ell) \]

  - rescaled problem gives same answer, provided final \( f \) is divided by \( \ell \)

- **momentum scaling:**

  \[ f(x, p) \rightarrow f'(x, p') \equiv \ell_p^{-3} f\left(x, \frac{p}{\ell_p}\right), W\{p_i\} \rightarrow W'\{p_i\} \equiv \ell_p^2 W\left(\left\{ \frac{p_i}{\ell_p} \right\}\right) \]

  \[ m \rightarrow m' = m/\ell_p \]

  - rescales momenta and \( W \) such that particle density is unchanged

- **coordinate scaling:**

  \[ f(x, p) \rightarrow f'(x, p') \equiv f\left(\frac{x}{\ell_x}, \frac{p}{\ell_x}\right), \quad W \rightarrow W' \equiv \frac{W}{\ell_x} \]

  - rescales space-time and \( W \) such that the particle density stays the same
Suppose you have some initial conditions with timescale $\tau_0$, lengthscale $R_0$, temperature $T_0$, momentum/mass scale $\mu$, cross sections $\sigma$ and rapidity density $dN_0/d\eta$. The scalings imply that

$$\sigma' = l_x^{-1} l^{-1} \sigma, \quad T_0' = l_p T_0, \quad R_0' = l_x R_0,$$

$$\frac{dN_0'}{d\eta} = l_x l \frac{dN_0}{d\eta}, \quad \mu' = l_p \mu, \quad \tau_0' = l_x \tau_0.$$

Therefore, we can scale a solution to others provided that all three ratios

$$\frac{\mu}{T_0}, \quad \frac{R_0}{\tau_0}, \quad \sigma \frac{dN_0}{d\eta} \sim \frac{\tau_0}{\lambda_{MFP}}$$

remain the same (2 → 2 transport)
Macroscopic quantities

charge current: \[ N_c^\mu(x) = \sum_i \int \frac{d^3p}{E} p^\mu c_i f_i(x, \vec{p}) \]

energy-momentum tensor: \[ T^{\mu\nu}(x) = \sum_i \int \frac{d^3p}{E} p^\mu p^\nu f_i(x, \vec{p}) \]

EOM + conservation laws in scatterings imply:

\[ \partial_\mu N_c^\mu = 0 \quad (c_1 + c_2 = c_3 + c_4), \quad \partial_\mu T^{\mu\nu} = 0 \quad (p_1 + p_2 = p_3 + p_4) \]

for a local equilibrium distribution (see next slide)

\[ f_{eq}(p, x) = \frac{g}{(2\pi)^3} \exp \left[ \frac{\mu(x) - p_\mu u^\mu(x)}{T(x)} \right] \]

we recover (Problem 2) the ideal hydrodynamic expressions

\[ T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu - p g^{\mu\nu}, \quad N_c^\mu = n_c u^\mu \]
Hydrodynamic limit

entropy current ($g = 1$):

$$S^\mu(x) = \sum_i \int \frac{d^3 p}{E} p^\mu f_i(x, \vec{p}) \left\{ 1 - \ln[f_i(x, \vec{p}) h^3] \right\}$$

H-theorem:

$$\partial_\mu S^\mu \geq 0 \quad \Rightarrow \quad \text{entropy production in general}$$

$$\begin{align*}
\partial_\mu S^\mu &= - \int \ln(h^3 f_1) C[f_1] \\
&= \frac{1}{2} \int_{1234} (f_3 f_4 - f_1 f_2) W_{12 \to 34} \frac{\ln f_3 + \ln f_4 - \ln f_1 - \ln f_2}{4} \\
&= \frac{1}{8} \int_{1234} f_1 f_2 W_{12 \to 34} (z - 1) \ln z \quad \geq \quad 0
\end{align*}$$

where $z \equiv f_3 f_4 / (f_1 f_2) \geq 0$ and $(z - 1) \ln z \geq 0$.

Equality (entropy maximum) requires $f_3 f_4 = f_1 f_2$ for ANY momenta

$$\Rightarrow \quad f_{eq}(p, x) = e^{p_\mu A^\mu(x) + B(x)} = \frac{g}{(2\pi^3)} \exp \left[ \frac{\mu(x) - p_\mu u^\mu(x)}{T(x)} \right]$$

i.e., entropy production until local equilibrium is reached
Transport coefficients

Hydrodynamic eqns come from expansion in small gradients near local equil

\[ f(x, \vec{p}) = f_{eq}(x, \vec{p})[1 + \phi(x, \vec{p})] \quad (|\phi| \ll 1, \quad |p^\mu \partial_\mu \phi| \ll |p^\mu \partial_\mu f_{eq}|/f_{eq}) \]

AND substitution of the \( N^\mu \) and \( T^{\mu\nu} \) moments of the solution into the conservation laws.

As we saw, the 0-th order \( \phi = 0 \) gives ideal hydrodynamics \((N^\mu_0, T^{\mu\nu}_0)\). The first order solution

\[
p^\mu \partial_\mu f_{eq}(x, \vec{p}) = C[f_{eq}, f_{eq}\phi](x, \vec{p}) + C[f_{eq}\phi, f_{eq}](x, \vec{p})
\]

\[
C[f, g] \equiv \frac{1}{2} \int_{234} (f_3g_4 - f_1g_2)W_{12 \rightarrow 34}
\]

leads to the Navier-Stokes equations. Comparison to

\[
T^{\mu\nu}_1 = T^{\mu\nu}_0 + \eta_s(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} \partial^\alpha u_\alpha) + \zeta \Delta^{\mu\nu} \partial^\alpha u_\alpha
\]

\[
N^\mu_1 = N^\mu_0 + \kappa_q \left( \frac{nT}{\varepsilon + p} \right)^2 \nabla^\mu \left( \frac{\mu}{T} \right) \quad (\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu, \quad \Delta^\mu \equiv \Delta^{\mu\nu} \partial_\nu)
\]

gives the shear \((\eta_s)\) and bulk \((\zeta)\) viscosities, and heat conductivity \((\kappa_q)\).
Computing transport coefficients can be involved \[\text{see De Groot et al, } \textit{Relativistic kinetic theory}, \text{ or Arnold, Moore & Yaffe, JHEP 0011, 001 ('00) ...}\]

Nevertheless, because the product \(\sigma \phi\) appears, it is clear that \(\phi \propto 1/\sigma\) and therefore

\[
\eta_s, \zeta \propto \frac{T}{\sigma} \sim nT \lambda_{MFP}, \quad \kappa_q \propto \frac{1}{\sigma} \sim n\lambda_{MFP}
\]

i.e., in the zero mean free path limit the transport coefficients vanish and we recover ideal hydrodynamics.
Heavy-ion applications (early stage)

- perturbative calculations - Debye-screened perturbative cross sections, e.g.,

\[
\frac{d\sigma^{gg \rightarrow gg}}{dt} = \frac{1}{16\pi s^2} |\tilde{M}_{gg \rightarrow gg}|^2 = \frac{9\pi \alpha_s^2}{2} \frac{1}{t^2} \rightarrow \frac{9\pi \alpha_s^2}{2} \frac{1}{(t-\mu_D^2)^2} \quad (\mu_D \sim gT)
\]

in radiative $ggg \leftrightarrow gg$, account for the LPM effect (need $\tau_{form} \lesssim \lambda_{MFP}$)

\[
|M_{gg \rightarrow ggg}|^2 = \left( \frac{9g^4}{2} \frac{s^2}{(q_T^2 + \mu_D^2)^2} \right) \left( \frac{12g^2q_T^2}{k_T^2[(k_T - q_T)^2 + \mu_D^2]} \right) \Theta(k_T\lambda_{MFP} - chy)
\]

- studies near the hydro limit - match $\sigma$ to required transport coefficients, e.g.,

\[
\eta_s \approx \frac{4T}{5\sigma_{tr}}
\]
we need: initial conditions
boundary conditions = expansion to vacuum ("empty" outside)
and hadronization model
Hadronization

A poorly understood process - significant theory uncertainties.

- **local parton-hadron duality (one to one)** e.g., Eskola et al

assumes quarks and gluons convert to hadrons (mainly pions), preserving momenta, especially useful at low pT

- **independent fragmentation (one to many)** Feymann, Field, ...

- **coalescence/recombination (few to one)** Hwa, Yang, Biró, Zimányi, Lévai, Csizmadia, Ko, Lin, Voloshin, DM, Greco, Fries, Müller, Nonaka, Bass, ...

  relevant in A+A at intermediate $2 \lesssim p_T \lesssim 6$ GeV (needs high phasespace densities)

  to lowest order $q\bar{q} \rightarrow M$, $qqq \rightarrow B$

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Initial conditions

Similarities with hydrodynamics (one needs to specify initial “shapes”), however, transport also requires momentum distributions as input.

basic density profiles are similar to hydrodynamic calculations

- wounded nucleon (Glauber)
- binary collisions
- saturation model (e.g., Gribov-Levin-Rishikin approach)

momentum distributions are often based on

- at high $p_T > p_0 \sim 2$ GeV, perturbative QCD jet rates
- at low $p_T < p_0 \sim 2$ GeV, saturation physics or extrapolations (to set a certain total $dN/dy$)

typical initialization times are $\tau_0 \sim 0.1 - 0.2$ fm

in studies at midrapidity, boost invariance can be useful to impose
Cooling

Expanding systems cool due to $p\,dV$ work

Gyulassy, Pang & Zhang ('97): 1+1D

$\sim T_{\text{eff}}$

free streaming ($\sigma = 0$)

Navier-Stokes

Kinetic

Euler Hydro

ideal hydro $T \sim t^{-1/3}$

dissipation in transport slows cooling, especially in 3+1D

DM & Gyulassy ('00): 3+1D $(dN/d\eta = 210)$

MPC vs hydro (1+1D and 3+1D)
Elliptic flow

- spatial anisotropy $\rightarrow$ final azimuthal momentum anisotropy

\[ \mathcal{E} \equiv \frac{\langle x^2 - y^2 \rangle}{\langle x^2 + y^2 \rangle} \rightarrow \mathcal{V}_2 \equiv \frac{\langle p_x^2 - p_y^2 \rangle}{\langle p_x^2 + p_y^2 \rangle} \]

- measures strength of interactions
- self-quenching, develops at early times
macroscopically: pressure gradients

$$\Delta \vec{F}/\Delta V = -\vec{\nabla} p$$

(beam axis view)

⇒ larger acceleration in impact parameter direction

⇒ momentum anisotropy $v_2$

microscopically: transport opacity

smaller momenta more deflection

larger momenta less deflection

variation in pathlength
$v_2$ builds up early

Zhang, Gyulassy & Ko ('99): anisotropy builds up during first $\sim 2 \text{ fm}/c$

![Graph](image)

sharp cylinder $R = 5 \text{ fm}$, $\tau_0 = 0.2 \text{ fm}/c$, $b = 7.5 \text{ fm}$, $dN^{\text{cent}}/dy = 300$
Strong interactions at RHIC

\( \text{Au+Au @ 130 GeV, } b = 8 \text{ fm} \)

DM & Gyulassy, NPA 697 ('02): \( v_2(p_T, \chi) \)

\[
v_2(p_T, \chi) \approx v_{2\text{max}}(\chi) \tanh(p_T/p_0(\chi))
\]

**nonlinear opacity dependence**

need \( 15 \times \) perturbative opacities - \( \sigma_{el} \times dN_g/d\eta \approx 45 \text{ mb} \times 1000 \)

(saturated gluon \( dN_{cent}/d\eta = 1000, T_{eff} \approx 0.7 \text{ GeV, } \tau_0 = 0.1 \text{ fm, 1 parton } \rightarrow 1 \pi \text{ hadronization} \)
Significant randomization

correlation between initial and final momenta

a) deflection angle $\vec{p}_i \angle \vec{p}_f$

\[ b = 8 \text{ fm}, \chi = 3.40^{F)} \]

$\sigma_{el} \approx 7 \text{ mb}$

\[ \cos \theta = \frac{\vec{p}_i \vec{p}_f}{|\vec{p}_i||\vec{p}_f|} \]

b) rapidity shift $y_f - y_i$

$\sigma_{el} \approx 7 \text{ mb}$

light parton momenta randomize to large degree, already for $\sigma \sim 7 \text{ mb}$ ($\chi \sim 7$)

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Not an ideal fluid

dissipation reduces $v_2$ by 30 – 50% even for $\sigma_{gg\rightarrow gg} \sim 50$ mb

DM & Huovinen, PRL94 ('05): final $v_2(p_T)$ $v_2(\tau, p_T)$

$\tau_0 = 0.1$ fm/$c$

$\tau_0 = 0.1$

$\tau=0.65$ fm/$c$
$\tau=1$
$\tau=2$
$\tau=3$

$\tau=0.65$
$\tau=1$
$\tau=2$
$\tau=3$

→ dense, strongly collective system, but still dissipative
Hydrodynamic limit and $v_2$

2+1D calculation (2D space, 2D momenta, no longitudinal expansion)

Ollitrault & Gombeaud ('07):

\[
v_2^{\text{integrated}} \sim \frac{\tilde{v}_2}{\sigma_0/\sigma + 1}
\]

(\eta_{\text{shear}} \sim 1/\sigma \sim K_{\text{Knudsen}})

also works for 3+1D transport - $v_2(p_T, \sigma) = v_2^{max} \tanh(p_T/p_0)$ fits to MPC results for Au+Au at RHIC, $b = 8$ fm

\[
v_2^{max}(\sigma) \approx \frac{0.404}{0.554 \text{ mb}/\sigma + 1}, \quad p_0(\sigma) \approx \frac{2.92 \text{ GeV}}{0.187 \text{ mb}/\sigma + 1}
\]
higher-order processes also contribute to thermalization

\[ \frac{dE_t}{dy} \bigg|_{y=0} [\text{GeV}] \]

1+1D cooling (p dV work)

\[ \sim T_{eff} \]

\( \sigma_{tot} = 2\text{mb} \)

\( \sigma_{tot} = 10\text{mb} \)

100% inelastic
50 - 50%
100% elastic
free streaming

DM & Gyulassy, NPA 661, 236 ('99)

**fixed transport cross section**
but vary degree of inelasticity

100% elastic, 100% inelastic, 50-50%

2 \rightarrow 2
3 \leftrightarrow 2
mixed

**isotropic scattering(!)**

\( \Rightarrow \) inelastic 3 \leftrightarrow 2 roughly same as elastic with same transport cross section

more enhanced for pQCD cross sections because 3 \rightarrow 2 allows large angles
Greiner & Xu '04: **find thermalization time-scale** $\tau \sim 2 - 3 \text{ fm/c}$

**2 → 2, 2 → 3 transport cross sections**

**spectra vs. time**

- Inelastic roughly doubles $\sigma_{tr}$
- Rapid cooling via $2 \rightarrow 3$, but may be because low-momentum region is empty
this other extreme also indicates short timescales $\sim 2$ fm
elliptic flow with $ggg \leftrightarrow gg$ (minijet initconds, $p_0 = 1.4$ GeV)

close to the experimental values at RHIC
Other interesting areas

- **heavy quarks** - useful cross-check of dynamics/equilibration

  can also be done in the Fokker-Planck (small-angle) approximation, or in a Langevin approach (many random scatterings - Brownian motion)
  
  Moore & Teaney... Gossiaux...

- **coupling to classical color fields**

  Mrowczynski... Arnold, Moore, Yaffe... Dumitru, Strickland...

**Wong equation: color Vlasov-Boltzmann**  Wong, Heinz

\[
p^\mu \left( \partial_\mu + g t_a F^{a}_{\mu \nu} \partial_{p_\nu} + g f_{abc} A^b_{\mu} t^c_{t_a} \right) f = C[f]
\]

\[
[D_\mu, F^{\mu \nu}_a] = J^\nu = g \int p^\nu t_a (f_q - \bar{f}_q + f_g) dP dQ
\]

thermal plasma at \( T \gg T_c \) would appear neutral - fast color rotations

\[
\tau_{\text{color}}^{-1} \sim g^2 \ln(1/g) T \gg \tau_{\text{mom}}^{-1} \sim g^4 \ln(1/g) T
\]

Selikhov '91

Gyulassy '92

However, anisotropic particle distributions are rapidly isotropised (\( \sim \) two-stream instability)
Covariant transport - summary

- covariant transport is a nonequilibrium framework to study a system of on-shell (quasi-)particles
- also useful to test formulations of hydrodynamics (transport is always causal and stable)
- elliptic flow data in Au+Au at RHIC reproduced suprisingly well with $15 \times$ enhanced perturbative $2 \rightarrow 2$ rates - not enough for ideal fluid behavior
- radiative $3 \leftrightarrow 2$ is very important for thermalization, results challenge the strongly-coupled plasma paradigm (should be verified independently)
- limited equation of state (no phase transition)
- hadronization challenging
- thermalization benefits from new ingredients such as classical fields