Real Time Methods

Plan for the 3 lectures:

- Right way to solve real time dynamics
  and why we can't actually do it
- Semiclassical approximations
  and when we can get away with them
- Vlasov system, Hard Loops
- Numerical method: real time latt. gauge theory
  with added degrees of freedom

Lecture 1: generalities

- Density matrix
- Measurables
- Schwinger-Keldysh path integral
- (Failed) attempt to treat it numerically
- (Partially successful) use of analytic continuation

Density matrix

Most general initial condition for doing field theory:

$$\hat{\rho} \equiv \sum_{mn} \rho_{mn} |m\rangle\langle n|$$

Hermitian $\hat{\rho}^\dagger = \hat{\rho}$; Diagonal in some orthonormal basis:

$$\hat{\rho} = \sum_{m'} \rho_{m'} |m'\rangle\langle m'| \qquad \text{with } \rho_{m'} \geq 0, \quad \sum_{m'} \rho_{m'} = 1$$

prob. interpretation: states m' with prob $\rho_{m'}$

Time evolution:

$$\frac{d\hat{\rho}}{dt} = -i\Big[\,\hat{H}\,,\,\hat{\rho}\,\Big] \quad \rightarrow \quad \hat{\rho}(t) = U(t)\hat{\rho}(0)U^{-1}(t)\,,$$

with $U(t) = \mathrm{Texp}(\int_0^t -i\hat{H}dt')$ as usual.

Values of measurables

expectation vals of ops are given by tracing over $\hat{\rho}$:

$$\langle T^{\mu\nu}(x,t)\rangle = \mathrm{Tr}\hat{\rho}\hat{T}^{\mu\nu}(x,t) = \sum_{mn}\rho_{mn}\langle n|U^{-1}(t)\hat{T}^{\mu\nu}(x)U(t)|m\rangle$$

This exp. val gives time history of stress tensor.

Other interesting op's: Photon production

$$e^2\int d^4x d^4y e^{ik\cdot(x-y)}\mathrm{Tr}\,\hat{\rho}\hat{J}^{\mu}(x)\hat{J}_{\mu}(y)$$

quark numbers and susceptibilities/fluctuations

Correlation functions of $T_{\mu\nu}$ components (transport, etc)

Lightlike separated transverse color fields: jet quenching

Evaluation 1

If operators are of form

$$\langle \overline{\mathrm{T}}(\hat{\mathcal{O}}_1(x_1)\ldots\mathcal{O}_m(x_m))\,\mathrm{T}(\mathcal{O}_{m+1}(x_{m+1})\ldots\mathcal{O}_n(x_n))\rangle$$

Product of (0 or more) time-ordered (T) operators times
product of (0 or more) anti-T ($\overline{\mathrm{T}}$) operators
with (T) before ($\overline{\mathrm{T}}$)

Then use following trick: first modify

$$\mathcal{L} \to \mathcal{L} + \sum_i J_i \mathcal{O}_i$$

external sources J_i for operators $\mathcal{O}_i$ of interest

Evaluation 2

Now path-$\int$ represent

$$U(t)\hat{\rho}U^\dagger(t) = \sum_{mn} \rho_{mn} \langle m|\; U^\dagger(t)\; U(t)\; |n\rangle$$

for large t, using INDEPENDENT sources J in each $U(t)$:

Schwinger-Keldysh time contour:

Evaluation 3

Then evaluate

$$
\begin{aligned}
& \langle\overline{\mathrm{T}}(\hat{\mathcal{O}}_1(x_1)\ldots\mathcal{O}_m(x_m))\;\mathrm{T}(\mathcal{O}_{m+1}(x_{m+1})\ldots\mathcal{O}_n(x_n))\rangle \\
= \quad & i\frac{\delta}{\delta J_{1,-}(x_1)}\ldots(-i)\frac{\delta}{\delta J_{m+1,+}(x_{m+1})}\ldots Z(J_{i-},J_{i+})\Big|_{J=0}
\end{aligned}
$$

with ϕ refers to all fields in theory

$$
\begin{aligned}
Z(J_{i+},J_{i-}) = \int\mathcal{D}(\phi_+)\mathcal{D}(\phi_-)\;\; & \exp\left(i\int d^4x\mathcal{L}(\phi_+,J_+)\right) \\
& \exp\left(-i\int d^4x'\mathcal{L}(\phi_-,J_-)\right)
\end{aligned}
$$

with boundary conditions:

integrate over $\phi_+(0)$, $\phi_-(0)$ with weight $\hat{\rho}_{\phi_+,\phi_-}$

So am I done?

Path integral representation is not quite solving problem. Need method to evaluate path integral. We have a few:

- Perturbation theory
- Resummed versions of perturbation theory
- Other semiclassical methods
- AdS-CFT methods (??)
- Numerical evaluation, eg, lattice gauge theory

Perturbation theory

More involved than usual: each field ϕ has ϕ_+, ϕ_-.

More problems than in vacuum:

- Fails at strong coupling $\alpha_\mathrm{s} \sim 1$ [as in vac]
- Fails for some $\hat{\rho}$ even at weak coupling [more to follow]
- Fails at late times due to secular terms:

$$\langle \mathcal{O}(0)\mathcal{O}(t)\rangle \sim c_1 + c_2\alpha_\mathrm{s} t + c_3\alpha_\mathrm{s}^2 t^2 + \dots$$

Resummation procedures can help with last problem here.

Direct numerical integration?

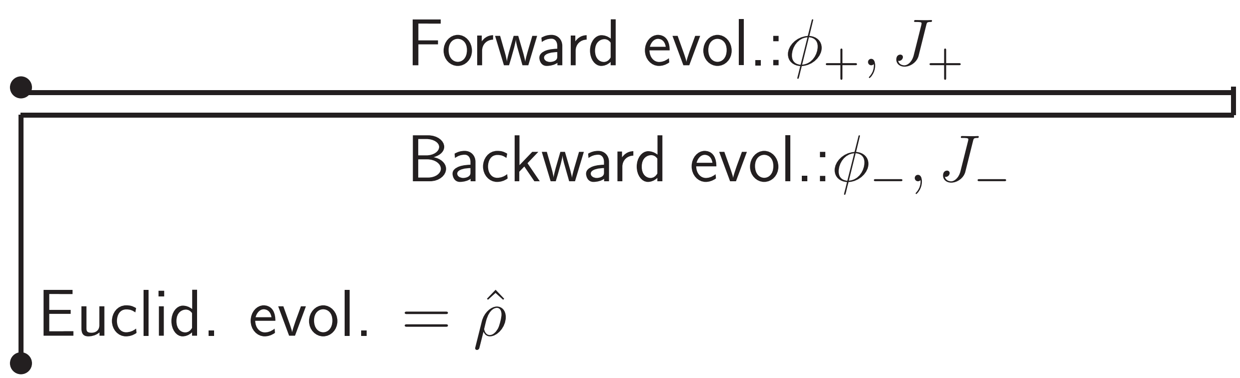
This approach fails for real-time problems.

Failure: importance sampling vs. phase cancellation

Consider equilibrium density matrix

$$\hat{\rho} = \frac{1}{Z}\sum_n e^{-E_n/T}|n\rangle\langle n| = \frac{1}{Z}\sum_n e^{-\hat{H}/T}|n\rangle\langle n| = \frac{1}{Z}e^{-\hat{H}/T}$$

has (Euclidean) path integral representation: complex S-K contour



Euclidean vs. Minkowski

Path integral here is

$$Z(J_+, J_-, J_E) = \int \mathcal{D}(\phi_+, \phi_-, \phi_E) \exp(iS_+ - iS_- - S_E)\,,$$

$$S_E = \int_0^{\beta=1/T} d\tau \int d^3x \mathcal{L}_E(\phi_E, J_E)\,, \qquad \mathcal{L}_E \equiv H$$

The integral

$$\int \mathcal{D}(\phi_E) e^{-S_E}$$

is absolutely convergent and can be done by sampling.

All contrib. add with same sign–no cancellation problems.

Integrals of $S_\pm$ are problematic.

Limited solution

Take no $\delta/\delta_{J_\pm}$. Then

$$\int \mathcal{D}(\phi_+, \phi_-) \exp(iS_+ - iS_-)$$

cancel! Use sources on Euclidean portion.

$$\langle \mathcal{O}_1(0)\mathcal{O}_2(\tau)\rangle \equiv G_E(\tau) = \mathrm{Tr} e^{\tau\hat{H}}\mathcal{O}_2 e^{-\tau\hat{H}}\mathcal{O}_1 e^{-\beta\hat{H}} = \frac{\delta^2 Z(J_E)}{\delta J_1(0)\delta J_2(\tau)}$$

is unphysical but related to

$$\mathcal{O}_1(0)\mathcal{O}_2(t) \equiv G^{>}(t) = \mathrm{Tr}\, e^{-i\hat{H}t}\mathcal{O}_2 e^{i\hat{H}t}\mathcal{O}_1 e^{-\beta\hat{H}}$$

by analytic continuation, $-\tau \to it$!

Limited solution cont'd

After some careful analytic continuation work,

$$\rho_{\mathcal{O}_1,\mathcal{O}_2}(\omega) \equiv \int_{-\infty}^{\infty} dt e^{-i\omega t} \left\langle \left[\mathcal{O}_2(0) \, , \, \mathcal{O}_1(t) \right] \right\rangle$$

related to G_E via

Note, $\omega\rho$ real positive symmetric

$$\begin{aligned} G_{E,\mathcal{O}_1\mathcal{O}_2}(\tau) &= \int \frac{d\omega}{2\pi} \frac{\rho_{\mathcal{O}_1,\mathcal{O}_2}(\omega)}{\omega} K(\omega,\tau) \, , \\ K(\omega,\tau) &= \frac{\omega \cosh(\omega(\tau-\beta/2))}{\sinh(\omega\beta/2)} \end{aligned}$$

This gives some (integral) information about ρ.

Limited solution, limitations

If we determine $G_E(\tau)$ as analytic function:
uniquely determines ρ as analytic function.

If we determine values with error bars, constrains ρ
may allow determination with (strong) priors.

Berges $et\ al$ recently argued that "sloped" path $\int$
can be done, allowing $G(\tau, t)$ for $t \lesssim \tau$
So far, only demonstrated for toy problems arXiv:0708.0779